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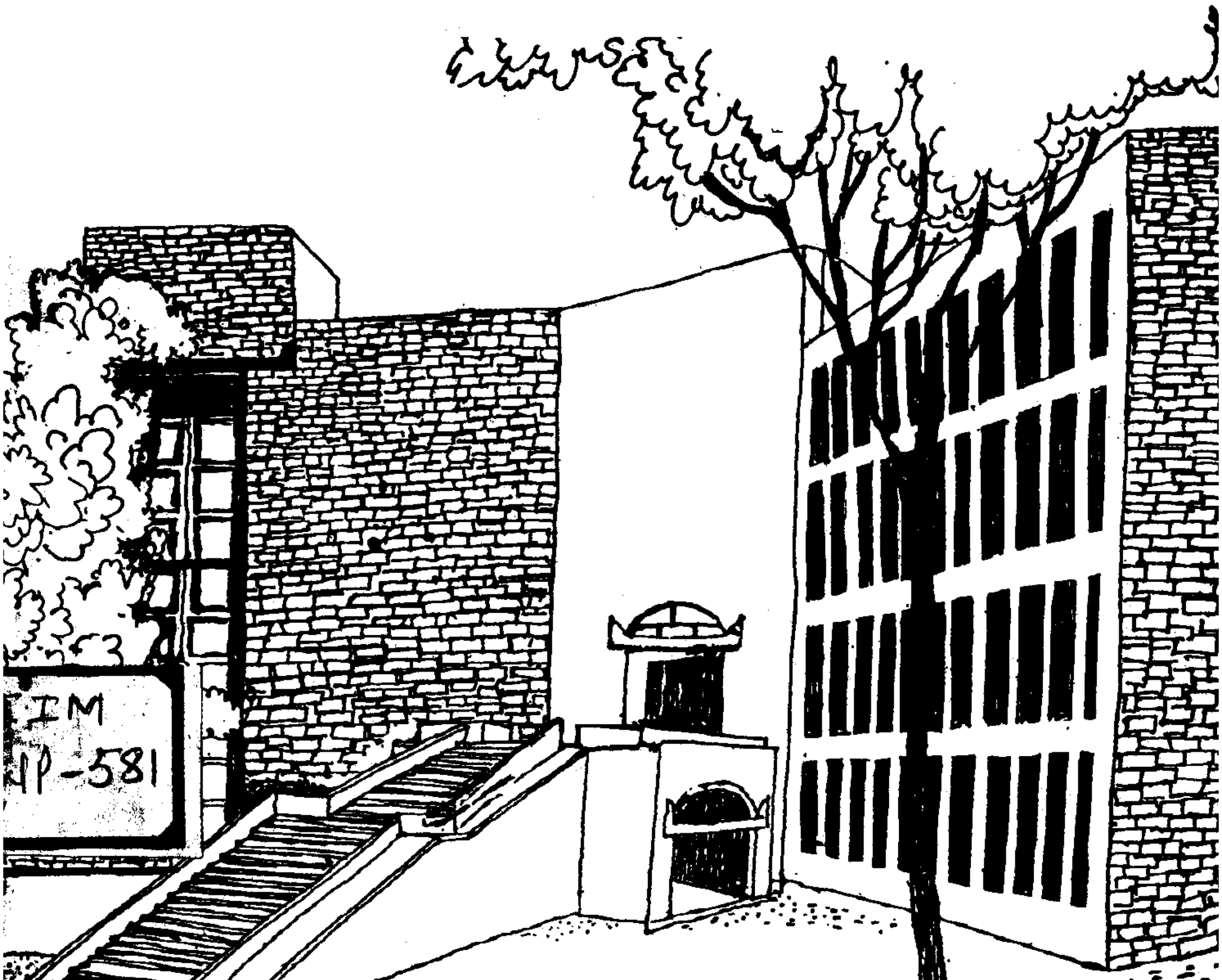

A LOWER BOUND ON FLEET-SIZE IN VARIABLE-SCHEDULE FLEET-SIZE PROBLEM

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Abstract

This paper discusses an approach to compute a lower bound on fleet-size in variable-schedule fleet-size problem. The lower bound is computed in two stages. In stage one, a fixed-schedule fleet-size problem is solved for a relaxed set of trips where each trip is assumed to be departing at its latest permissible departure time and yet arrive at earliest arrival time resulting in reduced elapsed time. In second stage the lower bound is augmented by the minimum additional fleet-size required to make each trip individually restorable to its original elapsed time.

A LOWER BOUND ON FLEET-SIZE IN VARIABLE-SCHEDULE FLEET-SIZE PROBLEM

The fleet-size problem is concerned with minimising the fleet-size required to operate a given set of trips. There are two versions of the problem depending upon whether departure times of trips are already fixed, or yet to be fixed within a prespecified interval for each trip.

In the fixed-schedule version of the fleet-size problem where the departure times are already fixed, the task of computing fleet-size and achieving it is a simple matter [1] [2]).

In the variable-schedule version, the departure times are not fixed before hand, but are specified in terms of earliest departure time and latest departure time. Fixing the departure time for each trip within the corresponding interval is part of the problem. The variable-schedule fleet-size problem can be formulated as network flow problem with 'bundle' constraints [3], which has been shown as a NP-hard problem [4]. The solution procedures used to tackle the problem are usually branch-and-bound techniques [5] for relatively smaller problems, and heuristic approaches [1] for large scale problems. Hence it is useful to have a good lower bound on the fleet-size requirement in variable-schedule fleet-size problem. In this paper we develop a method to compute the lower bound on fleet-size using extension of fixed-schedule type of analysis and a graph-theoritic result.

Let $\{\langle p_i, q_i, e_i, l_i, r_i \rangle\}$ be the given set of trips, each trip i being specified in terms of a 5-tuple, $\langle p_i, q_i, e_i, l_i, r_i \rangle$, where,

p_i : departure terminal

q_i : arrival terminal

e_i : earliest departure time

l_i : latest departure time

r_i : elapsed time between departure and arrival.

Accordingly,

$e_i + r_i$: earliest arrival time

$l_i + r_i$: latest arrival time.

The lower bound is computed in two phases. In phase one, minimum fleet-size is computed for the fixed-schedule problem with a set of modified trips, where trips are assumed to be departing at the latest departure time and arriving at earliest arrival time resulting in reduced elapsed time compared to the original elapsed time. Intuitively, it is clear that the minimum fleet-size required for the <u>reduced</u> task would be less than the minimum fleet-size required for the original task. Although, for narrow intervals of departure times, this may be a reasonably good lower bound, it would deteriorate with widening of the intervals. Therefore, in phase two, we enhance this by computing minimum number of <u>additional</u> vehicles required to operate trips with original elapsed time.

Fleet-size Requirement in Fixed-Schedule Problem

In the fixed-schedule problem, since the departure time for each trip is fixed, the trip can be specified by a 4-tuple $\langle p_i, q_i, d_i, r_i \rangle$, where

d_i : departure time

A vehicle would be either active performing a trip, or idle at some terminal after performing a trip and before taking up next trip from that terminal. We assume that dead-heading is not allowed, and hence a vehicle after completing a trip can take up next trip departing from the same terminal as the arrival terminal of the trip just completed.

Let

F	:	minimum fleet-size
$m(t)$	:	number of trips active at time t
$s_a(t)$	:	number of vehicles idle at time t at terminal a
$b_a(t)$	:	number of arrivals at time t minus number of departures at time t occuring at terminal a
T	:	planning period.

We have

Total Fleet-size = Active fleet-size at time t + total idle fleet-size at time t.

Of the above, active fleet-size at time t (i.e. $m(t)$) is determined by the set of trips. Therefore, fleet-size can be minimised only by minimising total idle fleet-size at time t.

Obviously, for feasible operations $s_a(t) \geq \emptyset$ must be satisfied for all t at all the terminals. Fleet-size idle at time t will depend upon net inflow of fleet-size at time t and idle fleet-size at time t and idle fleet-size at time (t-1). Thus,

$$s_a(t) = s_a(t-1) + b_a(t) \qquad \ldots\ldots\ (1)$$

Again, in the above expression $b_a(t)$ is solely determined by the set of trips. In order that $s_a(t)$ is at minimum level, there must exist at least one time-point t* such that $s_a(t^*) = \emptyset$. Otherwise, if $\min_t \{s_a(t)\} > \emptyset$ then clearly $s_a(t)$ can be minimised simply by substracting $x = \min_t \{s_a(t)\}$ from all $s_a(t)$, still retaining the feasibility condition namely $s_a(t) \geq \emptyset$. This gives us a method to compute $s_a(t)$ as follows:

```
procedure s_a
   initialise s_a(-1) <- Ø
   for t = Ø to T do
      s_a(t) <- s_a(t-1) + b_a(t)
   endfor
   x <- min_t {s_a(t)}
   for t = Ø to T do
      s_a(t) <- s_a(t) - x
   endfor
endprocedure
```

Having computed $s_a(t)$, it is now a simple matter to compute F. Thus,

$$F = \sum_a s_a(t) + m(t) \quad \ldots\ldots\ldots\ldots \quad (2)$$

where,

$$s_a(t) = s_a(t-1) + b_a(t) \quad \text{such that } \min_t \left\{ s_a(t) \right\} = 0 \ \ldots \ (3)$$

$b_a(t)$ can be computed from trip data as follows

$$\text{Let } D_a(t) = \left\{ i : p_i = a \wedge d_i = t \right\} \quad \ldots\ldots\ldots\ldots \quad (4)$$

$$A_a(t) = \left\{ j : a_j = a \wedge d_i = r_i = t \right\} \quad \ldots\ldots\ldots \quad (5)$$

Then

$$b_a(t) = \left| A_a(t) \right| - \left| D_a(t) \right| \quad \ldots\ldots\ldots\ldots \quad (6)$$

Similarly, m(t) can be directly computed from trip data. At time t, the set of trips that are active is given by

$$m(t) = \left\{ i : d_i \leqslant t < d_i + r_i \right\} \quad \ldots\ldots\ldots \quad (7)$$

A trip is considered to be inactive at the arrival since it is already accounted for in $A_a(t)$.

Suppose that each trip i of the trip set is relaxed to from a modified trip set as follows.

$$N^* = \left\{ \langle p_i, q_i, l_i, r_i - l_i + e_i \rangle \right\}$$

Let F, $s_a(t)$, $b_a(t)$ and m(t) correspond to the trip set

$$N = \left\{ \langle p_i, q_i, d_i, r_i \rangle : e_i \leqslant d_i \leqslant l_i \right\}$$

With respect to time t, the two trip sets are related as follows:

1. Some departures occuring at or before t in N would occur after t in N*, whereas departures occuring after t in N would continue to do so in N*.

2. Some arrivals occuring after t in N would occur at or before t in N*, and arrivals occuring at or before t in N would continue to do so in N*.

The set of departures and arrivals crossing the time instant t can determined as follows.

Let
$$U_a(t) = \{ i:\ p_i = a \ \text{and}\ d_i \leqslant t < l_i \}$$
$$V_a(t) = \{ j:\ q_j = a \ \text{and}\ e_i + r_i \leqslant t < d_i + r_i \}$$
$$u_a(t) = |U_a(t)|$$
$$v_a(t) = |V_a(t)|$$

The trips $U_a(t)$ will now contribute in computation of $s_a(t)$ and $m(t)$ only after t, and the trips $V_a(t)$ will now contribute at and before t. If we continue to operate with fleet-size F for trip set N*, we have

$$F = \sum_a s'_a(t) + m'(t)$$

where
$$s'_a(t) = s_a(t) + u_a(t)$$
$$m'(t) = m(t) - \sum_a u_a(t) - \sum_a v_a(t)$$

Let
$$x^*_a = \min_t \{ s'_a(t) \}$$

We have, $x^*_a \geqslant 0$, since $u_a(t) \geqslant 0$, $v_a(t) \geqslant 0$, $\min_t \{s_a(t)\} = 0$

Accordingly we can compute F^*, $s^*_a(t)$, $m^*(t)$ as follows.

$$F^* = F - \sum_a x^*_a$$

$$s^*_a(t) = s_a(t) + u_a(t) + v_a(t) - x_a$$

$$m^*(t) = m(t) - \sum_a u_a(t) - \sum_a v_a(t)$$

All the values related to trip set N^* can also be computed directly without referring to corresponding values related to trip set N as follows.

Let

$$D^*_a(t) = \{i : p_a = a \wedge l_i = t\}$$

$$A^*_a(t) = \{j : q_j = a \wedge l_i + r_i = t\}$$

$$b^*_a(t) = |A^*_a(t)| - |D^*_a(t)|$$

Then

$$s^*_a(t) = s^*_a(t-1) + b^*_a(t) \quad \text{such that} \min_t s^*_a(t) = 0$$

$$F^* = \sum_a s^*_a(t) + m^*(t)$$

Since F^* is a constant $\sum_a s^*_a(t) + m^*(t)$ must remain invariant for all t. Therefore,

$$F^* = \sum_a s^*_a(t) + m^*(t) = \sum_a s^*_a(t-1) + m^*(t-1)$$

giving

$$m^*(t) = m^*(t-1) - (\sum_a s^*_a(t) - \sum_a s^*_a(t-1))$$

$$= m^*(t-1) - \sum_a b^*_a(t)$$

$$= m^*(t-1) + \sum_a |D^*_a(t)| - \sum_a |A^*_a(t)|$$

The above implies that a particular trip contributes positively through the departure event and negatively through the arrival event. This is quite intuitive for 'realistic' trips with departure event chronologically preceding the arrival event. However, in trip set N* we are quite likely to have some 'negative' trips with arrival event chronologically preceding the departure event. This happens when the elapsed time r_i is less than the tolerance available to fix the departure time, i.e.

$$r_i < l_i - e_i$$

We consider the 'positive' and 'negative' trips separately to compute m*(t) directly from the trip data.

Let

$$M^*_P(t) = \{i : l_i \leq t < s_i + r_i\}$$

$$M^*_N(t) = \{i : e_i + r_i \leq t < l_i\}$$

Then,

$$m^*(t) = |M^*_P(t)| - |M^*_N(t)|$$

It can be easily shown that above method of computation would maintain the required invariance of $\sum_a s^*_a(t) + m^*(t)$ as follows.

$$M^*_P(t-1) = \{i : l_i \leq t-1 < s_i + r_i\}$$

$$\{i : l_i < t \leq e_i + r_i\}$$

Similarly, $M^*_N(t-1) = \{i : e_i + r_i < t \leq l_i\}$

leading to

$$m^*(t) - m^*9t-1) = |\{i : (t = l_i \wedge t < e_i + r_i) \vee (t = l_i \wedge e_i + r_i < t)|$$

$$-|\{i: (t = l_i + r_i \wedge t < l_i) \vee (t = l_i + r_i \wedge l_i < t)$$

By adding and subtracting $\left|\left\{i:\ l_i = t \wedge t = e_i + r_i\right\}\right|$ to RHS

$$m^*(t) - m^*(t-1) = \left|\left\{i:\ t = l_i\right\}\right| - \left|\left\{i:\ t = e_i + r_i\right\}\right|$$

Therefore, $m^*(t) = m^*(t-1) - \sum_a b^*_a(t)$

Thus, computation of F* can be summarised as follows.

$$F^* = \sum_a s^*_a(t) + m^*(t)$$

where $s^*_a(t) = s^*_a(t-1) + b^*_a(t)$ such that $\min_t \left\{s^*_a(t)\right\} = 0$

$$b^*_a(t) = \left|\left\{i:\ t = e_i + r_i\right\}\right| - \left|\left\{i:\ t = l_i\right\}\right|$$

$$m^*(t) = \left|M^*_1(t)\right| - \left|M^*_2(t)\right|$$

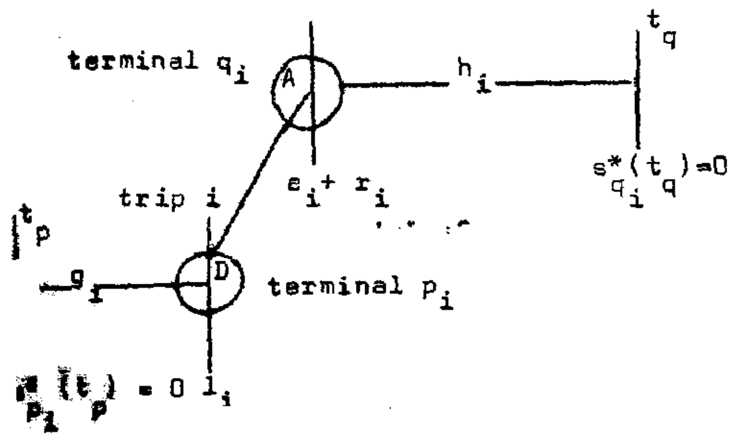
where $M^*_1(t) = \left\{i:\ l_i \leqslant t < e_i + r_i\right\}$

$$M^*_2(t) = \left\{i:\ e_i + r_i \leqslant t < l_i\right\}$$

F* being the fleet-size requirement for the relaxed set of trip forms only a part of the lower bound. All the relaxed trips have reduced elapsed time $r_i - l_i + e_i$ instead of r_i. One way to improve the lower bound is to force the original elapsed time r_i on each trip, and compute the additional minimum fleet-size required to maintain the feasibility condition $s_a(t) \geqslant 0$. The most ideal situation would be if we could compute the additional minimum fleet-size when we force all the trips jointly to have the required elapsed time. But this is likely to be quite difficult as it almost amounts to solving the variable-schedule problem itself. However, we may still improve the lower bound by exploring the recovery of elapsed time r_i of one trip at a time keeping other trips still relaxed at $r_i - l_i + e_i$.

The amount of enhancement $l_i - e_i$ in recovery time can be fully effected either at the departure end or the arrival end or partly at both the ends, as long as the feasibility $s^*_a(t) \geqslant 0$ is not affected.

Suppose the trip i is recovered to $\langle p_i, q_i, d_i, r_i \rangle$ such that $e_i \leqslant d_i \leqslant l_i$. The effect of this would be to reduce $s^*_p(t)$ by 1 in the interval $d_i \leqslant t < l_i$ and reduce $s^*_q(t)$ by 1 in the interval $e_i + r_i \leqslant t < d_i + r_i$, because, the trip i would now negatively contribute in the interval $d_i \leqslant t < l_i$ and <u>fail</u> to contribute positively in the interval $e_i + r_i \leqslant t < d_i + r_i$. Therefore, if there exist any time instant t^*_p at p_i such that $s^*_{p_i}(t^*_p) = 0$ and/or t^*_q at q_i such that $s_{q_i}(t^*_q) = 0$ then the feasibility condition can be restored only by injecting additional vehicles at these terminals. Thus, freedom for relaxation of a trip is restricted by the critical time points with $s(t) = \emptyset$ at either end of the trip.



Let

g_i = time gap between nearest critical time-point before e_i and l_i

h_i = time gap between nearest critical time point after $e_i + r_i$, and $e_i + r_i$.

The trip i can be singly restored without affecting the feasibility as long as $l_i - l_i \leqslant g_i + h_i$.

If $l_i - l_i > g_i + h_i$ for some trip, it can be restored to original elapsed time only by injecting <u>additional</u> vehicle either at p_i or q_i. The effect of additional vehicle is to drive one of the value, g_i or h_i to ∞ for trips either departing from or arriving at the terminal where the additional vehicle is injected. We compute the minimum number of additional fleet-size as follows.

Let

$$R = \left\{ i : l_i - l_i > g_i + h_i \right\}$$

be the set of trips that cannot be restored without sacrificing additional vehicles.

Let $W = \left\{ p_i : i \in R \right\} \cup \left\{ q_i : i \in R \right\}$

be the set of terminals spanned by the set of trips R.

Let $G = \left\{ W, Y \right\}$ be a graph with set of nodes W and set of undirected arcs Y such that

$$Y = \left\{ (p_i, q_i) : i \in R \right\}$$

Each are indicates that there is at least one trip operating between p_i and q_i that cannot be restored unless we inject one additional vehicle either at p_i or at q_i. Injecting a vehicle at any node would result in removal of 'interfering' arcs incident on that node since g_i or h_i of all trips either departing or arriving at that terminals would become non-binding.

Thus we have a node-covering problem of finding the smallest subset of Z* nodes such that each arc of G is covered by at least one node in the subset.

The lower bound on fleet-size for the variable schedule problem is given by F* + Z*.

The node-covering problem or related maximum independent set problem can be solved using well-known methods (6). Although the problem is NP-complete, in the context of lower bound computation, the dimension of 'interference graph' G would be negligibly small compared to the network flow graph of original problem with 'bundle' constraints.

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