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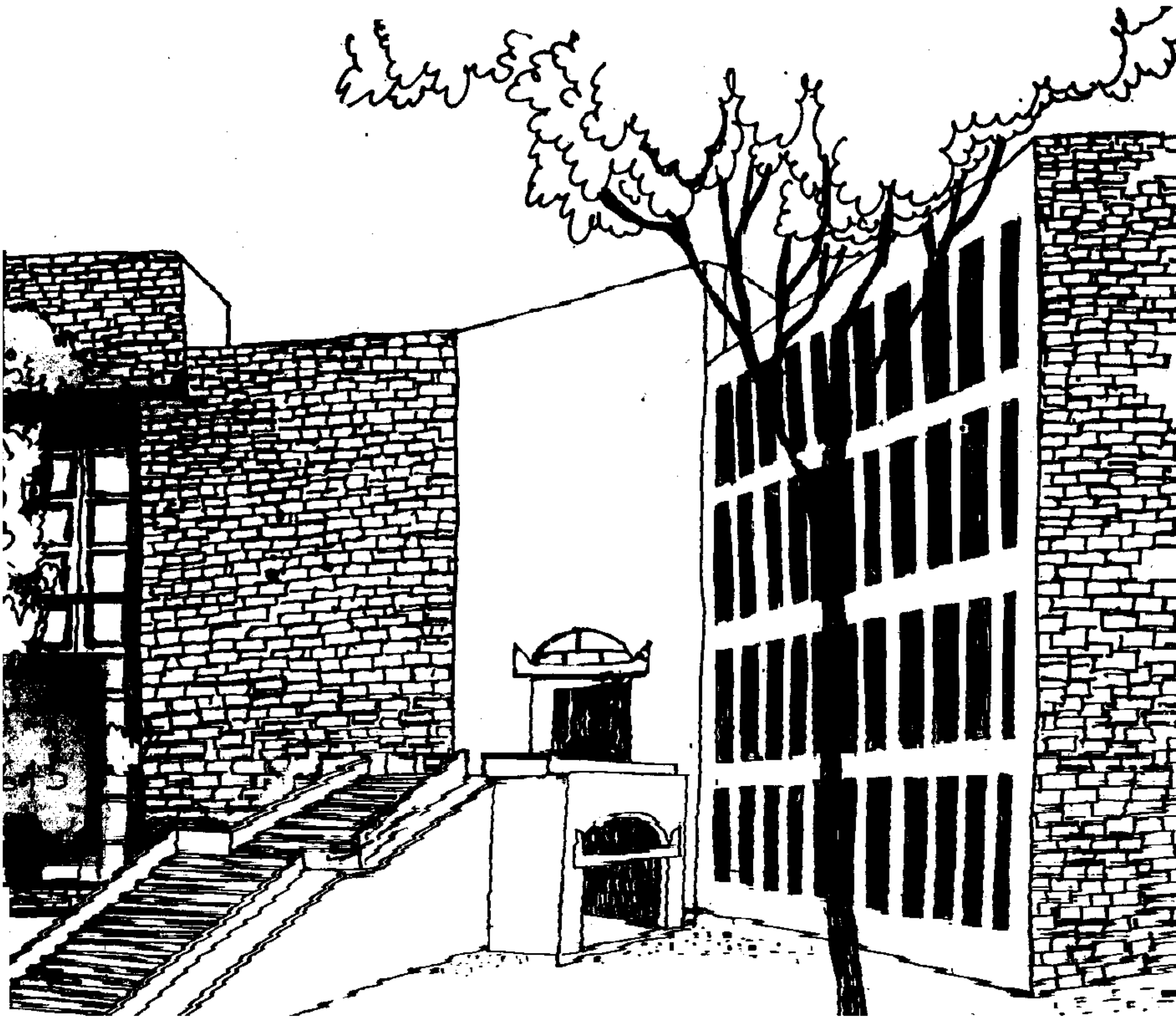

TRANSIENT ANALYSIS OF MULTIPLE UNIT RELIABILITY SYSTEMS

By

O.P. Sharma

N. Ravichandran

&

J. Dass

W P No. 645

December, 1986

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

TRANSIENT ANALYSIS OF MULTIPLE UNIT RELIABILITY SYSTEMS

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D.P. Sharma*, N.Ravichandran** and J.Dass*

ABSTRACT

A general Markovian model representing several multiple unit redudant repairable system is proposed and it's transient behaviour is studied. Specifically, for multiple unit reliability system the reliability and availability functions are derived in an explicit form for the transient case. The stationary availability and mean time to system failure are deduced from the main results as special cases.

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* Deptt. of Mathematics, Indian Institute of Technology,New Delhi

**Indian Institute of Management, Ahmedabad.

Introduction

Extensive research is reported in the area of stochastic analysis of redudant repairable systems. In the recent monograph Srinivasan and Subramanian [1] and Birolini [4] present the state of art of such systems. The successful analysis of such system, depends on the complexity of the underlying stochastic process [4]. In the literature most of the analysis reported so far confines to the measures like stationary availability, mean time to system failure and interval reliability in the stationary case [4]. The measures steady state availability and mean time to system failure are independent of the transform variable and hence are obtainable in easily computable form. However, in many reliability evaluation models the transient behaviour of the systems is important and may even be essential. The purpose of this paper is to provide the transient analysis of several multiple systems. The solution procedure is presented for a general case and results are obtained for the case of cold and warm standby systems as well as parallel redudancy. This paper is organised as follows. Section-1 formulates the problem and section-2 presents the general methodology in obtaining the results. Section-3 presents the results for the special cases n = 2 and 3 and numerical results.

1. Problem formulation

We list below the assumptions of a general system studied in the paper.

i) The system consists of $n(\geq 1)$ units. The system requires $k(\leq n)$ units for its successful operation.

ii) Initially k units are operative and (n-k) units are kept as standbys.

iii) There is a repair facility with r($\geq$ 1) repairmen. The repair policy is first-in and first-out (F|F).

iv) The life time and repair time durations of the units are independent random variables with negative exponential distribution with parameters λ and μ respectively.

v) The life time of the standby is negative exponential random variable with parameter λ_s.

vi) On failure an operating unit is taken to the repair facility instantaneously where it is repaired according to the first in first-out queue discipline.

vii) The standby units are also taken to the repair facility on failure to follow the same course of action as the online operating units.

viii) When the number of operable units is less than k the system is said to be non-operable.

ix) On repair completion unit joins the pile of spares if there are sufficient number of operable units otherwise it is switched operative.

x) Repair is assumed to be perfect.

Several special cases of the present models are obtained by specifying the values of the parameters k and n.

a) Let $k = 1$ and $r = 1$: the resulting system is a n-unit warm standby system.

b) $k = 1$, $r = 1$ and $\lambda_s = 0$:the resulting system is cold standby system

c) $k = 1$, $r = 1$ and $\lambda_s = \lambda$: the resulting system is a parallel redudant system

d) $k = 1$ $(r > 1)$ and $\lambda_s = 0$: the resulting system is a n unit cold standby with multiple repair facility.

Now, we briefly sketch the stochastic behaviour of the system described in the beginning of this section.

Let $X(t)$ be the number of operable units at time 't'. Then $\{X(t), t \geq 0\}$ is an integer valued stochastic process taking value on $\{0,1,2,\ldots,n\}$. $X(t) = 0$ implies that system is down and $X(t) =$ implies the system is operable with all units 'n'. The point of discontinuity of the stochastic process are those epochs at which either there is a failure (online and standby) or a repair completion of a unit. In both the cases either the process $X(t)$ decreas or increases by one unit. In view of the exponential duration of the life times and repair times $X(t)$ is a Markov process and by th physical nature it is a birth and death process. Because of the multiple units in the models the death rates are state dependent. We use λ_i to denote the failure rate when i units are operable and μ be the repair rate.

Let $P_{ij}(t) = Pr\{X(t) = j/X(0) = i\}$, $i,j \in \{0,1,2,\ldots,n\}$

set $P_j(t) = P_{nj}(t)$, $j \in \{0,1,2,\ldots,n\}$

The function $P_j(t)$ satisfy the following Chapman-Kolmogrov equation

$$P_j'(t) = -(\lambda_j+\mu)\ P_j(t) + \lambda_{j+1}\ P_{j+1}(t) + \mu P_{j-1}(t) \qquad 1 \leq j < n \qquad \text{(a)}$$

$$P_n'(t) = -\ \lambda_n\ P_n(t) + \mu P_{n-1}(t)$$

$$P_o'(t) = -\ \mu P_o(t) + \lambda_1\ P_1(t)$$

and $P_j(0) = \delta_{jn}$

and the equations satisfied by $q_j(t)$ are

$$q_j'(t) = -(\lambda_j+\mu)\ q_j(t) + \lambda_{j+1}\ q_{j+1}(t) + \mu q_{j-1}(t) \qquad 2 \leq j < n \qquad \text{(b)}$$

$$q_n'(t) = -\ \lambda_n\ q_n(t) + \mu q_{n-1}(t)$$

$$q_1'(t) = -(\lambda_1+\mu)\ q_1(t) + \lambda_2\ q_2(t)$$

with $q_j(0) = \delta_{jn}$

The stationary distributions of the stochastic process $\{X(t),\ t \geq$ is obtained by solving the system of equations obtained from (a) by setting the derivatives equal to zero.

Let $\pi_j = \lim_{t\to\infty} P_j(t)$, we get,

$$\pi_j = \frac{1}{\mu^{n-j}} \prod_{k=j+1}^{n} \lambda_k\ \pi_n \qquad 0 \leq j < n \qquad (1.1)$$

$$\pi_n = \{1 + \sum_{k=0}^{n-1} \frac{1}{\mu^{n-k}} \prod_{i=k+1}^{n} \lambda_i\ \}^{-1} \qquad (1.2)$$

and $q_j(t) = Pr\{X(t) = j/X(0) = n, \text{ and } X(u) \neq 0, \; 0 \leq u \leq t\}$
$0 < j \leq n$

The functions $P_{ij}(t)$ represent the probability that the system is in state j at time t given that the system is at t = 0 in state i. The function $q_j(t)$ represents the probability that system is in state j at time t, given the information that state 0 has not been initiated in (0,t). The functions $P_j(t)$ and $q_j(t)$ adequately describe the important operating characteristics of the system.

For example, the availability function of the system is

$$A(t) = 1 - P_o(t) \tag{1.3}$$

and the stationary distribution of stochastic process

$$\pi_i = \lim_{t\to\infty} P_i(t) \tag{1.4}$$

The stationary availability of the system is denoted as

$$\beta = 1-\pi_o = \sum_{j=1}^{n} \pi_j \tag{1.5}$$

The reliability R(t) of the system is given by

$$R(t) = \sum_{j=1}^{n} q_j(t) \tag{1.6}$$

and

$$MTSF = \int_0^\infty R(u)du = \sum_{j=1}^{n} \int_0^\infty q_j(u)du \tag{1.7}$$

Thus, it is clear that the evaluation of the operating characteristics of the system requires an evaluation of the functions $P_j(t)$ and $q_j(t)$

2. General Methodology:

In this section we solve the set of equation (a) and (b) in an explicit form. On taking Laplace transform of (a) and (b) and denoting the Laplace transform of the function f(t) by f(s) the system of equations (a) and (b) reduce to

$$AP = I_{n+1} \tag{2.1}$$

where A is a (n+1)x(n+1) real symmetric tridiagonal matrix and P is a column vector and I_{n+1} is a column vector. Specifically,

$$A = \begin{bmatrix} s+\mu & -\lambda_1 & & & & & O \\ -\mu & s+\lambda_1+\mu & -\lambda_2 & & & & \\ & -\mu & s+\lambda_2+ & -\lambda_3 & & & \\ & & & \ddots & & & \\ & O & & & -\mu & s+\lambda_{n-1}+\mu & -\lambda_n \\ & & & & & -\mu & s+\lambda_n \end{bmatrix}$$

$$P = \begin{bmatrix} P_o(s) \\ P_1(s) \\ \vdots \\ P_n(s) \end{bmatrix}_{m+1} \quad \text{and} \quad I_{n+1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Similarly the system of equations (b) reduces

$$BQ = I_n \tag{2.2}$$

where B is a (nxn) real symmetric tridiagonal matrix obtained by supressing the 1st row and first column of A. I_n is a unit column vector with unity in the nth place and Q is a column vector

$$Q = \begin{bmatrix} q_o(s) \\ q_1(s) \\ \vdots \\ q_n(s) \end{bmatrix}$$

From equations (2.1) and (2.2) by using Crammer's rule the Laplace transform of the function $P_j(t)$ and $q_j(t)$ are explicitely determined as

$$p_j(s) = \frac{|A_j(s)|}{|A(s)|} \quad , \; 0 \leq j \leq n \tag{2.3}$$

and

$$q_j(s) = \frac{|B_j(s)|}{|B(s)|} \quad , \; 1 \leq j \leq n$$

where $A_j(s)$ and $B_j(s)$ are obtained from A(s) and B(s) by the jth column by the unit vector in the R.H.S. of (2.

The MTSF of the system is obtained by setting s = 0, in equation (2.4) and summing over all j = 1 to n.

Thus

$$MTSF = \sum_{j=1}^{n} q_j(0) \tag{2.5}$$

We next show that the polynomials $|A(s)|$ and $|B(s)|$ have real and distinct roots and hence a partial fraction expression of $p_j(s)$ and $q_j(s)$ is feasible. This is achieved by using some elementary rows and columns transformation on $|A(s)|$ and $|B(s)|$. Now $|A(s)|$ can be expressed as $|A(s)| = s|D(s)|$. Then we get $|D(s)|$ and $|B(s)|$ as real symmetric, tridiagonal determinants with non-zero and non-positive off-diagonal elements i.e.

$$D(s)=\begin{bmatrix} s+\lambda_1+\mu & -\sqrt{\lambda_1\mu} & & & & \bigcirc \\ -\sqrt{\lambda_1\mu} & s+\lambda_2+\mu & -\sqrt{\lambda_2\mu} & & & \\ & -\sqrt{\lambda_2\mu} & s+\lambda_3+\mu & -\sqrt{\lambda_3\mu} & & \\ & & \vdots \; \vdots \; \vdots \; \vdots & & & \\ & \bigcirc & -\sqrt{\lambda_{n-2}\mu} & s+\lambda_{n-1}+\mu & -\sqrt{\lambda_{n-1}\mu} \\ & & & -\sqrt{\lambda_{n-1}\mu} & s+\lambda_n+\mu \end{bmatrix}_n \quad (2.6a)$$

and

$$B(s)=\begin{bmatrix} s+\lambda_1\mu & -\sqrt{\lambda_1\mu} & & & & \bigcirc \\ -\sqrt{\lambda_1\mu} & s+\lambda_2+\mu & -\sqrt{\lambda_2\mu} & & & \\ & -\sqrt{\lambda_2\mu} & s+\lambda_3+\mu & -\sqrt{\lambda_3\mu} & & \\ & & \vdots \; \vdots \; \vdots \; \vdots & & & \\ & \bigcirc & -\sqrt{\lambda_{n-2}\mu} & s+\lambda_{n-1}+\mu & -\sqrt{\lambda_{n-1}\mu} \\ & & & -\sqrt{\lambda_{n-1}\mu} & s+\lambda_n+\mu \end{bmatrix} \quad (2.6b)$$

Now the roots of the polynomials $|D(s)|$ and $|B(s)|$ are the negatives of the eigen values of the matrices D and C given by

$$D = \begin{bmatrix} \lambda_1+\mu & -\sqrt{\lambda_1\mu} & & & \bigcirc \\ -\sqrt{\lambda_1\mu} & \lambda_2+\mu & -\sqrt{\lambda_2\mu} & & \\ & -\sqrt{\lambda_2+\mu} & \lambda_3+\mu & -\sqrt{\lambda_3\mu} & \\ & & \vdots & \vdots & \vdots \\ & \bigcirc & -\sqrt{\lambda_{n-2}\mu} & \lambda_{n-1}+\mu & -\sqrt{\lambda_{n-1}\mu} \\ & & & -\sqrt{\lambda_{n-1}\mu} & \lambda_n+\mu \end{bmatrix} \quad (2.7a)$$

and

$$C = \begin{bmatrix} \lambda_1 & -\sqrt{\lambda_1\mu} & & & \bigcirc \\ -\sqrt{\lambda_1\mu} & \lambda_2\mu & -\sqrt{\lambda_2\mu} & & \\ & -\sqrt{\lambda_2\mu} & \lambda_3+\mu & -\sqrt{\lambda_3\mu} & \\ & & \vdots & \vdots & \vdots \\ & \bigcirc & -\sqrt{\lambda_{n-2}\mu} & \lambda_{n-1}+\mu & -\sqrt{\lambda_{n-1}\mu} \\ & & & -\sqrt{\lambda_{n-1}\mu} & \lambda_n+\mu \end{bmatrix} \quad (2.7b)$$

Thus from [2] we observe that the eigen values of C and D are real and distinct and from [3] we find that these matrices are positive definite and hence all the eigen values are positive. Denoting $\alpha_{n,k}$ and $\beta_{n,k}$ $(k = 1,2,\ldots,n)$ as the eigen values of D and C, respectively we can write

$$|A(s)| = \delta \prod_{k=1}^{n} (s+\alpha_{n,k})$$

and

$$|B(s)| = \prod_{k=1}^{n} (s+ \beta_{n,k}) \tag{2.8}$$

so that

$$p_j(s) = \frac{|A_j(s)|}{|A(s)|} = \frac{|A_j(s)|}{s \prod_{k=1}^{n} (s+\alpha_{n,k})}$$

and

$$q_j(s) = \frac{|B_j(s)|}{\prod_{k=1}^{n} (s+\beta_{n,k})} \tag{2.9}$$

and using partial fractions, we get

$$p_j(s) = \frac{a_{oj}}{s} + \sum_{i=1}^{n} \frac{a_{ij}}{(s+ \alpha_{n,i})}$$

and

$$q_j(s) = \sum_{i=1}^{n} \frac{b_{ij}}{(s+ \beta_{n,i})} \tag{2.10}$$

After some algebra we obtain

$$a_{oj} = \frac{|A_j(0)|}{\prod_{k=1}^{n} (\alpha_{n,k})} \tag{2.11}$$

$$a_{ij} = \frac{-|A_j(-\alpha_{n,j})|}{\alpha_{n,j} \prod_{\substack{k=1 \\ j\neq k}}^{n} (\alpha_{n,k} - \alpha_{n,j})} \tag{2.12}$$

and

$$b_{ij} = \frac{|B_j(-\beta_{n,j})|}{\prod_{\substack{k=1\\ j\neq k}}^{n} (\beta_{n,k} - \beta_{n,j})} \tag{2.13}$$

Taking inverse Laplace transformation of (2.10) we have

$$p_j(t) = a_{0j} + \sum_{i=1}^{n} a_{ij}\, e^{-\alpha_{n,i} t}$$

$$= \frac{|A_j(0)|}{\prod_{k=1}^{n} (\alpha_{n,k})} - \sum_{i=1}^{n} \frac{|A_j(-\alpha_{n,i})|}{\alpha_{n,i} \prod_{\substack{k=1\\ i\neq k}}^{n} (\alpha_{n,k} - \alpha_{n,i})}\, e^{-\alpha_{n,i}\, t}$$

$$0 \leq j \leq n$$
$$1 \leq i \leq n$$

and

$$q_j(t) = \sum_{i=1}^{n} b_{ij}\, e^{-\beta_{n,i} t}$$

$$= \sum_{i=1}^{n} \frac{|B_j(-\beta_{n,i})|}{\prod_{\substack{k=1\\ i\neq k}}^{n} (\beta_{n,k} - \beta_{n,i})}\, e^{-\beta_{n,i} t} \qquad 1 \leq j \leq n \tag{2.14}$$

Again as $\alpha_{n,k}$ are all positive, stationary distribution for $p_j(t)$ exists and is given by

$$\pi_j = a_{0j} \tag{2.15}$$

which is the same as shown in (1.2)

3. Speical Cases

Taking n = 2, we find that $\alpha_{2,i}$ (i = 1,2) and $\beta_{2,i}$ (i = 1,2 are the eigen values of the matrices

$$D' = \begin{bmatrix} \lambda_1+\mu & -\sqrt{\lambda_1\mu} \\ -\sqrt{\lambda_1\mu} & \lambda_2+\mu \end{bmatrix}$$

$$C' = \begin{bmatrix} \lambda_1 & -\sqrt{\lambda_1\mu} \\ -\sqrt{\lambda_1\mu} & \lambda_2+\mu \end{bmatrix} \qquad (3.1)$$

and these work out to

$$-\alpha_{2,1} = -(\lambda_1+\lambda_2+\mu) - \sqrt{(\lambda_1+\lambda_2+\mu)^2 - 4(\lambda_1\lambda_2 - \mu^2)}$$

$$-\alpha_{2,2} = -(\lambda_1+\lambda_2+\mu) + \sqrt{(\lambda_1+\lambda_2+\mu)^2 - 4(\lambda_1\lambda_2 - \mu^2)}$$

and

$$-\beta_{2,1} = -(\lambda_1+\lambda_2+\mu) - \sqrt{(\lambda_1+\lambda_2+\mu)^2 - 4\lambda_1\lambda_2}$$

$$-\beta_{2,1} = -(\lambda_1+\lambda_2+\mu) + \sqrt{(\lambda_1+\lambda_2+\mu)^2 - 4\lambda_1\lambda_2} \qquad (3.2)$$

and after some simplification we get

$$P_o(t) = \frac{\lambda_1\lambda_2}{\prod_{k=1}^{2}(\alpha_{2,k})} - \frac{\lambda_1\lambda_2}{\alpha_{21}(\alpha_{2,2}-\alpha_{2,1})} e^{-\alpha_{2,1}t} - \frac{\lambda_1\lambda_2}{\alpha_{2,2}(\alpha_{2,1}-\alpha_{2,2})} e^{-\alpha_{2,2}t}$$

$$P_1(t) = \frac{\lambda_2\mu}{\prod_{k=1}^{2}(\alpha_{2,k})} - \frac{\lambda_2(\mu-\alpha_{2,1})}{\alpha_{2,1}(\alpha_{2,2}-\alpha_{2,1})} e^{-\alpha_{2,1}t} - \frac{\lambda_2(\mu-\alpha_{2,2})}{\alpha_{2,2}(\alpha_{2,1}-\alpha_{2,2})} e^{-\alpha_{2,2}t}$$

$$p_2(t) = \frac{\mu^2}{\prod_{k=1}^{2}(\alpha_{2,k})} - \frac{(\mu-\alpha_{21})^2 - \lambda_1\alpha_{2,1}}{\alpha_{2,1}(\alpha_{2,2} - \alpha_{2,1})} e^{-\alpha_{2,1}t}$$

$$- \frac{(\mu-\alpha_{2,2})^2 - \lambda_1 \alpha_{2,2}}{\alpha_{2,2}(\alpha_{2,1} - \alpha_{2,2})} e^{-\alpha_{2,2}t}$$

and

$$q_1(t) = - \frac{\lambda_2}{(\beta_{2,2}-\beta_{2,1})} e^{-\beta_{2,1}t} + \frac{\lambda_2}{(\beta_{2,1}- \beta_{2,2})} e^{-\beta_{2,2}t}$$

$$q_2(t) = \frac{\lambda_1+\mu - \beta_{2,1}}{(\beta_{2,2} - \beta_{2,1})} e^{-\beta_{2,1}t} + \frac{\lambda_1+\mu - \beta_{2,2}}{(\beta_{2,1} - \beta_{2,2})} e^{-\beta_{2,2}t} \qquad (3.3)$$

Conclusion:

Thus in this article, we have provided a procedure by which the transient behaviour of a general reliability system can be studied. The method is illustrated for special case $n = 2$. The procedure is applicable for all $n \geq 2$. However, there is difficulty in obtaining the results analytically as we need to know the roots of the polynomial of degree 'n'. But, when the parameters of the problem are known, the roots of the polynomial which are known to be real and distinct can be obtained by using any of the available computer programming for obtaining the roots of a polynomial. We have illustrated this for the case $n = 3$.

Case I: Cold Standby Situation

Here we consider $\lambda_i = \lambda$. Then from (2.7a) and (2.7b) considering n = 3 we get the following eigen values for different λ's and μ's.

Table 1

Cold Standby	$\alpha_{3,1}$	$\alpha_{3,2}$	$\alpha_{3,3}$	$\beta_{3,1}$	$\beta_{3,2}$	$\beta_{3,3}$
a) $\lambda=1/50$ $\mu=1/25$	0.02000	0.06000	0.1000	0.00194	0.04388	0.09419
b) $\lambda=1/100$ $\mu=1/60$	0.00840	0.02666	0.04492	0.00119	0.01974	0.04240
c) $\lambda=1/120$ $\mu=1/100$	0.00542	0.01833	0.03124	0.00140	0.01398	0.02963

Case II: Warm Standby Situation

Here we consider $\lambda_i = (i-1)\lambda + \lambda_s$. Then from (2.7a) in (2.7b) considering n=3 we get the following eigen values in Table 2 for different values of λ, λ_s and μ .

Table 2

Warm Standby	$\alpha_{3,1}$	$\alpha_{3,2}$	$\alpha_{3,3}$	$\beta_{3,1}$	$\beta_{3,2}$	$\beta_{3,3}$
a) $\lambda=1/50$ $\lambda_s=1/10$ $\mu=1/25$	0.06497	0.15826	0.25677	0.04753	0.13999	0.25247
b) $\lambda=1/100$ $\lambda_s=1/15$ $\mu=1/40$	0.04132	0.10099	0.16269	0.03107	0.09934	0.15959

Case III: Parallel Redudancy Situation

Here we consider $\lambda_i = i\lambda$. Then from (2.7a) and (2.7b) considering n = 3, we get the following eigen values in Table 3 for the different values of λ and μ.

Table 3

Parallel Redudancy	$\alpha_{3,1}$	$\alpha_{3,2}$	$\alpha_{3,3}$	$\beta_{3,1}$	$\beta_{3,2}$	$\beta_{3,3}$
a) $\lambda = 1/50$ $\mu = 1/15$	0.04360	0.10055	0.17584	0.00362	0.07675	0.17297
b) $\lambda = 1/100$ $\mu = 1/30$	0.02180	0.05028	0.08792	0.00181	0.03837	0.08649

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