

ARBITRATION GAMES AND APPROXIMATE EQUILIBRIA

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ABSTRACT

In this paper we define an arbitration game in the context of a Bayesian Collective Choice Problem and derive an approximate equilibrium for such games under conditions of 'bounded rationality'.

1. Introduction :- The basic object of study in this paper is a Bayesian Collective-choice problem which has found useful applications in the study of bargaining and auction processes. A Bayesian Collective-choice problem is an incomplete information game in which outcomes are jointly feasible for all players together. A mechanism is used to provide the Collective choice, given information provided by the agents. The model has been discussed neatly in Myerson (1983) and related references therein.

The formal model we study here, although defined for the two person situation is valid for a larger number of agents. Consider a situation with two agents $i = 1, 2$ and let T^i denote the set of possible types of agent i . Let $T = T^1 \times T^2$ and let A be the set of possible outcomes or group choices. Each agent has a von Neumann - Morgenstern utility function, $u^i(a, t)$ which denotes the payoff to i if a is the group choice and if $t = (t_1, t_2)$ is the vector of agents' types. Each agent also has a probability function $p^i(t_{-i} | t_i)$ where $t_{-i} \in T_{-i}$, where

$$\begin{aligned} T_{-1} &= T^2 \text{ if } i=1 \\ &= T^1 \text{ if } i=2 \end{aligned}$$

$p^i(t_{-i} | t_i)$ denotes the subjective probability that player i would assign to the event t_{-i} if i 's actual type were t_i . The tuple $= [T, A, u^1, p^1, u^2, p^2]$ is called a Bayesian Collective choice problem.

Conventionally, the method by which outcomes are selected as a function of players' types, is by using the concept of a game-form. A game-form is a pair, $\langle M, g \rangle$ where $M = M^1 \times M^2$. The set M^i is the possible messages agent i can use and M is called the language. For reasons which become apparent as we proceed, we define the outcome rule $g : M \rightarrow A$ as a function which assigns to each message 'm' a group choice $g(m) \in A$.

Given a game form $\langle M, g \rangle$, agents choose messages as a function of their type and their common knowledge. We call a mapping $x^i: T^i \rightarrow M^i$ a strategy for i .

A modest literature on optimal Bayesian mechanisms has developed around the work of Arrow (1979), d'Aspremont and Gerard-Varet (1979, 1982), Laffont and Maskin (1979), Riordan (1984) and Maskin (1986), which models the game form as a solution to an optimization problem solved by a social planner. They consider revelation games, i.e. ones in which $M^i = T^i$ and the mechanism which is used to make the group choice is a solution to a utilitarian social welfare function.

Motivated by similar considerations we define the concept of an arbitration game which generalizes in several respects the construct of an optimal Bayesian mechanism. Our point of departure is the assumption that the game form (or mechanism) $\langle M, g \rangle$ is controlled by a Bayesian Statistician. This Bayesian Statistician is defined by

- (i) a social value function $W: A \times T \rightarrow \mathbb{R}$
- (ii) a posterior probability function $f(\cdot | m): T \rightarrow \mathbb{R} \forall m \in M$ which summarizes his belief that the agents' type is given by 't' conditional on his having received a message $m \in M$.

The ordered pair $(\alpha, \langle w, f \rangle)$ is common knowledge and for the purposes of this paper is called an arbitration game.

Example :- Optimal Bayesian Mechanisms : $w(a, t) = u^1(a, t) + u^2(a, t)$

$$\forall (a, t) \in A \times T; M^i = T^i, i = 1, 2 \text{ and } f(t|t') = 1 \text{ if } t = t' \\ = 0 \text{ if } t \neq t'$$

The further proviso of it being a private values model is often appended to an optimal Bayesian mechanism.

The purpose of this paper is to obtain solutions to arbitration games under conditions of bounded rationality. The approximate solutions we propose satisfies two countervailing conditions : it is reasonably precise, and it is easy to compute. The rationale for imposing these two conditions is that it lends credibility to a theory of rational behaviour described by complex maximization problems. Such is the merit of the certainty equivalence method which we apply, and which has been discussed at great length in Laffont (1989), chapter 3. A byproduct of our analysis is an alternative theory of rational behaviour where a study of multi-agent decision making is based on the developments in statistical inference.

2. Equilibrium for An Arbitration game :-

Let us assume that given an arbitration game $(\alpha, \langle w, f \rangle)$,

- (i) A is a compact, convex subset of a p-dimensional Euclidean space \mathbb{R}^p with nonempty interior
- (ii) T^i is a closed, convex subset of an n_i -dimensional Euclidean space \mathbb{R}^{n_i} for $i = 1, 2$.
- (iii) M^i is a compact subset of a q_i -dimensional Euclidean space \mathbb{R}^{q_i} for $i = 1, 2$.
- (iv) $w : A \times T \rightarrow \mathbb{R}$ is a continuous function
- (v) $u^i : A \times T \rightarrow \mathbb{R}$ is a continuous function for $i = 1, 2$.
- (vi) p^1, p^2 and f are conditional probability density functions.

In the above framework the arbitrator solves the following problem :

$$(1) \quad \max_{a \in A} \int_T w(a, t) f(t|m) dt$$

$$\forall m \in M.$$

Let $g^* : M \rightarrow A$ be a solution to (1).

An equilibrium for the arbitration game $(\alpha, \langle w, f \rangle)$ is an ordered pair of strategies $\langle x_1^*(.), x_2^*(.) \rangle$ such that

$$(a) \int_{T_2} u_1(g^*(x_1^*(t_1), x_2^*(t_2)), t) p^1(t_2 | t_1) dt_2 \geq \int_{T_2} u_1(g^*(m_1, x_2^*(t_2)), t) p^1(t_2 | t_1) dt_2$$

$$\forall m_1 \in M^1$$

$$(b) \int_{T_1} u_2(g^*(x_1^*(t_1), x_2^*(t_2)), t) p^2(t_1 | t_2) dt_1 \geq \int_{T_1} u_2(g^*(x_1^*(t_2), m_2), t) p^2(t_1 | t_2) dt_1$$

$$\forall m_2 \in M^2$$

(c) $g^* : M \rightarrow A$ solves (1).

An interesting question that can be posed in this framework is whether the revelation principle would continue to be valid for the class of arbitration games. The answer is in the affirmative as proved in Lahiri (1990 a). In fact the entire problem of adverse selection in principal - agents problems can be now dealt with in the above framework if we instead solve (1) subject to (a) and (b). This has been discussed in Lahiri (1990 b), where an analogous revelation principle has been established.

3. Solution :- The approximate solution to arbitration games that we propose relies on the following assumptions some of which are invoked for simplicity of exposition.

Assumption 1 :- $W : A \times T \rightarrow \mathbb{R}$ is thrice continuously differentiable in all arguments and $W(\cdot, t) : A \rightarrow \mathbb{R}$ is strictly concave $\forall t \in T$.

Assumption 2 :- (a) $u^i : A \times T \rightarrow \mathbb{R}$, does not depend on t_j for $j \neq i$, $i = 1, 2$

(b) $u^i : A \times T \rightarrow \mathbb{R}$, is thrice continuously differentiable as a function of $(a, t_i) \in A \times T^i$, $i = 1, 2$.

Assumption 3 :- (a) $\forall m \in M$, the distribution specified by the density function $f(\cdot/m) : T \rightarrow \mathbb{R}$ has a finite mean denoted by $\bar{t}(m)$. Thus $\bar{t} : M \rightarrow T$ is a well defined function (owing to the convexity of T).

(b) $\bar{t} : M \rightarrow T$ is a twice continuously differentiable function.

Assumption 4 :- (a) $\forall t_1 \in T_1$, the distribution specified by the density function $p^1(\cdot | t_1) : T_2 \rightarrow \mathbb{R}$ has a finite mean denoted by $\bar{t}_1(t_1)$. Thus, $\bar{t}_1 : T_1 \rightarrow T_2$ is a well defined function (owing to the convexity of T_1).

(b) $\forall t_2 \in T_2$, the distribution specified by the density function $p^2(\cdot | t_2) : T_1 \rightarrow \mathbb{R}$ has a finite mean denoted $\bar{t}_2(t_2)$. Thus, $\bar{t}_2 : T_2 \rightarrow T_1$ is a well defined function (owing to the convexity of T_2).

The following theorem is immediate :

Theorem 1 :- Suppose assumption 1 holds. Given any $\delta > 0$, there exists functions $\bar{g} : M \rightarrow A$, $\epsilon : M \rightarrow \mathbb{R}_{++}$ and $\gamma : M \rightarrow \mathbb{R}_{++}$ such that if the Hessian matrix of the function $a \mapsto W(a, \bar{t}(m))$ is non-singular for all $m \in M$ and if

$$\left\{ \begin{array}{l} \int_T f(t|m) dt > 1 - \gamma(m) \quad \forall m \in M, \\ \{t \in T \mid \|\bar{t}(m) - t\| < \epsilon(m)\} \end{array} \right.$$

then

$$\left| W(\bar{g}(m), \bar{t}(m)) - \max_{a \in A} \int_T W(a, t) f(t|m) dt \right| < \delta$$

$$\text{where, } W(\bar{g}(m), \bar{t}(m)) = \max_{a \in A} W(a, \bar{t}(m)) \quad \forall m \in M$$

Proof :- The proof of this theorem follows from the certainty equivalence principle, enunciated in theorem 3, chapter 3 of Laffont (1989).

An even more interesting result is the following :

Theorem 2 :- Under assumptions 1 and 3 and the conditions of theorem 1, the function $\bar{g} : M \rightarrow A$ is twice continuously differentiable.

$$\text{Proof :- By theorem 1, } \mathcal{D}_a W(a, \bar{t}(m)) \Big|_{a=\bar{g}(m)} = 0 \quad \forall m \in M.$$

By assumptions 1 and 3 the function $m \mapsto \mathcal{D}_a W(a, \bar{t}(m))$ is twice continuously differentiable for all $a \in A$.

By the condition on the Hessian matrix of the function $a \mapsto w(a, \bar{z}(m))$ implied in theorem 1 and carried over to theorem 2,

$$\mathcal{D}_a^2 w(a, \bar{z}(m)) \text{ is globally nonsingular.}$$

Hence, by the implicit function theorem, given $m \in M$, there exists a neighbourhood $N(m)$ of m contained in M such that $\forall m' \in N(m)$,

$$\mathcal{D}_a w(a, \bar{z}(m')) \Big|_{a=\bar{g}(m')} = 0.$$

By concavity of w in a ,

$$w(\bar{g}(m'), \bar{z}(m')) = \max_{a \in A} w(a, \bar{z}(m')).$$

Further, $\bar{g} : N(m) \rightarrow A$ is twice continuously differentiable. This being true for all $m \in M$, we obtain the desired result.

Q.E.D.

Significant to our analysis is the equilibrium for the arbitration game which results in a noncooperative setting.

Theorem 3 :- Suppose that the arbitration game $(\alpha, \langle w, f \rangle)$ has an equilibrium. Suppose that, the Hessian matrices of the functions

$$u^i(\cdot, t_i) : A \rightarrow \mathbb{R}$$

are globally non-singular for all $t_i \in T^i$, $i = 1, 2$.

Further suppose assumptions 1 - 4 as also the conditions of theorem 1 hold, and the problems

$$(a) \max_{m_1 \in M^1} u^1(\bar{g}(m_1, m_2), t_1), \quad m_2 \in M^2, \quad t_1 \in T_1$$

and

$$(b) \max_{m_2 \in M^2} u^2(\bar{g}(m_1, m_2), t_2), \quad m_1 \in M^1, \quad t_2 \in T_2$$

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have unique solutions.

Then, given $\delta > 0$, there exists functions $\bar{x}_i : T^i \rightarrow M^i$, $\epsilon_i : T^i \rightarrow \mathbb{R}_{++}$,

$\eta_i : T^i \rightarrow \mathbb{R}_{++}$, $i = 1, 2$ such that,

whenever,

$$\left\{ \int p^i(t_i | t_i) dt_i > 1 - \eta_i(t_i) \quad , \quad i = 1, 2, \right. \\ \left. \left\{ t_i \in T_i : \| t_i - \bar{t}_i(t_i) \| < \epsilon_i(t_i) \right\} \right\}$$

hold, we have

$$(c) \left| u^1(\bar{g}(\bar{x}_1(t_1), \bar{x}_2(\bar{t}_1(t_1))), t) - \max_{m_1 \in M^1} \int_T u^1(\bar{g}(m_1, \bar{x}_2(t_2)), t_1) p^1(t_2 | t_1) dt_1 \right| < \delta$$

$$(d) \left| u^2(\bar{g}(\bar{x}_1(\bar{t}_2(t_2)), \bar{x}_2(t_2)), t_2) - \max_{m_2 \in M^2} \int_T u^2(\bar{g}(\bar{x}_1(t_1), m_2), t_2) p^2(t_1 | t_2) dt_2 \right| < \delta$$

where,

$$u^1(\bar{g}(\bar{x}_1(t_1), \bar{x}_2(\bar{t}_1(t_1))), t_1) = \max_{m_1 \in M^1} u^1(\bar{g}(m_1, \bar{x}_2(\bar{t}_1(t_1))), t_1)$$

and

$$u^2(\bar{g}(\bar{x}_1(\bar{t}_2(t_2)), \bar{x}_2(t_2)), t_2) = \max_{m_2 \in M^2} u^2(\bar{g}(\bar{x}_1(\bar{t}_2(t_2)), m_2), t_2)$$

Proof :- The proof of this theorem appeals in a direct way to theorem 3, chapter 3 of Laffont (1989), where a certainty equivalence result has been established.

As an immediate corollary of the above theorems we obtain the following significant necessary condition for an approximate equilibrium $\langle \bar{x}_1(\cdot), \bar{x}_2(\cdot) \rangle$ for an arbitration game $(\alpha, \langle W, f \rangle)$ as enunciated in theorem 3 above.

Corollary :- Given an arbitration game $(\alpha, \langle W, f \rangle)$ let $\bar{g}: M \rightarrow A$ be an approximate solution to the optimization problem in the sense of theorem 1. Let $\langle \bar{x}_1(\cdot), \bar{x}_2(\cdot) \rangle$ be an approximate equilibrium to $(\alpha, \langle W, f \rangle)$ in the sense of theorem 3. Then, provided that assumption 1 - 4, the conditions of theorem 1, the conditions of theorem 3 hold and that $\bar{g}(m) \in \text{int}(A) \forall m \in M, \bar{x}_i(t_i) \in \text{int}(M_i) \forall t_i \in T_i, i = 1, 2$ it is necessary that

$$(i) \mathbb{D}_a u(\bar{g}(m), \bar{t}(m)) = 0 \quad \forall m \in M$$

(ii) $\forall t_1 \in T_1, \bar{x}_1(t_1)$ satisfies

$$\mathbb{D}_a u^1(\bar{g}(m_1, \bar{x}_2(\bar{t}_1(t_1)), t_1), \mathbb{D}_{m_1} \bar{g}(m_1, \bar{x}_2(\bar{t}_1(t_1)))) = 0$$

where $\bar{x}_2(\bar{t}_1(t_1))$ solves

$$\mathbb{D}_a u^2(\bar{g}(\bar{x}_1(\bar{t}_2(t_2)), m_2, \bar{t}_1(t_1)), \mathbb{D}_{m_2} \bar{g}(\bar{x}_1(\bar{t}_2(t_2)), m_2)) = 0$$

and (iii) $\forall t_2 \in T_2, \bar{x}_2(t_2)$ satisfies

$$\mathbb{D}_a u^2(\bar{g}(\bar{x}_1(\bar{t}_2(t_2)), m_2), t_2), \mathbb{D}_{m_1} \bar{g}(\bar{x}_1(\bar{t}_2(t_2)), m_2) = 0$$

where $\bar{x}_1(\bar{t}_2(t_2))$ solves

$$\mathbb{D}_a u^1(\bar{g}(m_1, \bar{x}_2(\bar{t}_1(t_1)), \bar{t}_2(t_2)), \mathbb{D}_{m_1} \bar{g}(m_1, \bar{x}_2(\bar{t}_1(t_1)))) = 0.$$

Proof :- Follows immediately from applying the first order conditions for an interior maximum, once we are equipped with theorems 1 - 3.

3. Conclusion :- Here we will view the problem from a somewhat different angle and evaluate the results, we have obtained from a different perspective in the context of bounded rationality.

Given an arbitration game $(\alpha, \langle W, f \rangle)$ let us view the triplet $\langle \bar{t}, \bar{t}_1, \bar{t}_2 \rangle$ as a statistic which the arbitrator and the agents use in estimating the unknown parameters of the model that each would be interested in. Viewed from this perspective, we have merely prescribed one statistic which they initially use to infer the unknown parameters of the model and then subsequently the problem reduces to a non-cooperative game under certainty. The statistic is the mean of the posterior distribution and we show that under certain conditions it is a 'fairly reliable' method to use in solving a Bayesian Collective choice problem.

The question that now arises is whether we could use some other statistic and once again reduce the problem to a non-cooperative game under certainty? For instance we could use the generalized maximum likelihood approach (see Berger (1985)) in estimating the unknown parameters and then reduce the game as before to a non-cooperative game under certainty. The maximum likelihood estimator has many desirable properties apart from being asymptotically normal, producing an unbiased efficient estimator if any such exists and always being a function of a sufficient statistic. Further, the revelation principle would continue to hold once the game was reduced to a non-cooperative game under certainty as above. Which statistic we should use in computing approximate equilibria depends largely on the arbitration game at hand.

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