INCENTIVE EFFICIENCY OF CORRELATED EQUILIBRIA WITH STATE DEPENDENT PAYOFFS

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ABSTRACT

In this paper we extend the framework of a finite game to incorporate state dependent payoffs, prove the existence of a correlated equilibrium in such a set up and obtain a characterization of all incentive efficient correlated equilibria. Finally we prove the existence of correlated equilibria for games with state dependent preferences and single experimentation by the players and indicate a characterization of all incentive efficient equilibria for such games.

1. Introduction: A finite game is given as follows : Let $N = \{1,2,...,n\}$ be a finite set of players. For each $i \in N$, let S^i be a finite set of (pure) strategies of i. Let S be the set of n-tuples of strategies : $S = S^1 \times S^2 \times ... \times S^n$; an element of S is $s = (s^i)_{i \in N}$. For each $i \in N$ and $s \in S$ let

$$s^{-i} = (s^1,...,s^{i-1}, s^{i+1},...,s^n)$$

denote the strategies played by players other than i; hence $s^{-i} \in S^{-i} = \prod_{j \neq i} S^j$ and $s = (s^{-i}, s^i)$. For each $i \in N$, let $h^i = S \to \mathbb{R}$ be the payoff to i when the n-tuple of strategies s is played.

A correlated equilibrium (Aumann 1974, 1987; Hart and Schmeidler 1989) consists of a probability vector $p = (p(s))_{s \in S}$ on S such that the following is satisfied for all $i \in N$ and all $r^i, t^i \in S^i$:

$$(1) \qquad \sum_{s^{-i} \in S^{-i}} p(s^{-i}, r^i)[h^i(s^{-i}, r^i) - h^i(s^{-i}, t^i)] \geq 0.$$

Interpretation of the above solution concept in terms of joint randomization of strategies by a referee is available in the above mentioned references and will not be repeated here. In the above framework the following theorem is significant :

Theorem 1: Every finite game has a correlated equilibrium.

Proof: See Hart and Schmeidler (1989).

In this paper we propose to extend the above framework and definitions to incorporate state dependent payoffs. In such a context we shall establish the existence of a correlated equilibrium and obtain necessary and sufficient conditions for a correlated equilibirum to be incentive efficient.

2. Existence of Correlated Equilibrium with State Dependent Preferences: Let $\Omega = \{\theta_1, \ldots, \theta_k\}$ be a finite set of states of nature and let $\pi : \Omega \to [0,1]$ be such that $\sum_{k=1}^{K} \pi(\theta_k) = 1$. Here π is a probability distribution determining the choice of the state of nature. If there is an objective basis for such a probability distribution, we assume that it is common knowledge. If the probability distribution is based on personal beliefs of the players, then as in Aumann 1987, we invoke the assumption that such prior beliefs are the same for all the players. We now define a finite game with state dependent payoffs.

Let $h^i : S \times \Omega \to \mathbb{R}$ be a state dependent payoff function for player i. A trivial extension of the above definition of a correlated equilibrium is to define a correlated equilibrium with state dependent preferences as follows:

Let $\phi^i = \{d^i / d^i : \Omega \to S^i\}$ and $\phi = \phi^1 \times \phi^2 \times \ldots \times \phi^n$. ϕ^i is a finite set containing $|S^i|^{|\Omega|}$ elements. Let $d^{-i} = (d^1, \ldots, d^{i-1}, d^{i+1}, \ldots, d^n)$ denote the decision rules (strategies of this game with uncertain states) played by everyone but i; thus $d^{-i} \in \phi^{-i} = \prod_{j \neq i} \phi^j$ and $d = (d^{-i}, d^i)$. A correlated equilibrium with state dependent payoffs can then be defined as a probability vector $p = (p(d))_{d \in \phi}$ on ϕ such that the following is satisfied for all $i \in N$ and all $\bar{d}^i$, $\underline{d}^i \in \phi^i$:

$$\sum_{d^{-i} \in \phi^{-i}} p(d^{-i}, \bar{d}^i) \sum_{k=1}^{K} \pi(\theta_k)[h^i(d^{-i}(\theta_k), \bar{d}^i(\theta_k)) - h^i(d^{-i}(\theta_k), \underline{d}^i(\theta_k))] \geq 0.$$

Viewing $\sum_{k=1}^{K} \pi(\theta_k) h^i(d(\theta_k))$ as the expected payoff to player i, when decision rules $d = (d^1, \ldots, d^n)$ are used, this solution concept can be interpreted along traditional lines, except that now randomization takes place over the space of decision rules and not on strategy spaces. The following theorem is then immediate:

<u>Theorem 2:</u> Every finite game with state dependent payoffs has a correlated equilibrium with state dependent payoffs.

<u>Proof:</u> Follows immediately from Theorem 1 and the finiteness of $|S^i|^{|\Omega|}$ for all i.

The above framework is an easy generalization of the framework of a finite game. However, if the state of nature is revealed to the referee before the play of the game, then this framework is redundant. In such a situation we may invoke the following alternative definition: A <u>correlated equilibrium with state dependent preferences</u> consists of a function $p : S \times \Omega \to [0,1]$ such that

(a) $$\sum_{s \in S} p(s|\theta_k) = 1 \quad \forall\ \theta_k \in \Omega$$

(b) for all $i \in N$ and all $r^i, t^i \in S^i$,

$$\sum_{k=1}^{K} \sum_{s^{-i} \in S^i} p(s^{-i}, r^i|\theta_k)\pi(\theta_k)[h^i(s^{-i}, r^i, \theta_k) - h^i(s^{-i}, t^i, \theta_k)] \geq 0.$$

The interpretation is now as follows : An n-tuple of strategies $r \in S$ is chosen at random by a referee, according the conditional distribution $p(.|.)$, who also observes the realization of the state of nature. Each player i is then told (only) his own coordinate r^i of r, and the game is played. A correlated equilibrium with state dependent payoffs results in the n-tuple of strategies in which each player i always plays the "recommended" r^i and this is a Nash equilibrium in this extended game (: all players are assumed to know the conditional distribution $p(.|.)$). In this framework, the following theorem can be established.

Theorem 3: Every finite game with state dependent payoffs has a correlated equilibrium with state dependent payoffs.

Proof: Fix $\theta_k \in \Omega$ and consider the finite game with strategy spaces S^i for player i, $i \in N$ and payoff function $h^i(.,\theta_k) : S \to \mathbb{R}$ for player i, $i \in N$. By Theorem 1, this finite game has a correlated equilibrium i.e. there exists a function $p(.|\theta_k) : S \to [0,1]$ such that

(a) $$\sum_{s \in S} p(s|\theta_k) = 1$$

(b) $\forall\ i \in N$ and all $r^i, t^i \in S^i$,

$$\sum_{s^{-i} \in S^{-i}} p(s^{-i}, r^i|\theta_k)[h^i(s^{-i}, r^i, \theta_k) - h^i(s^{-i}, t^i, \theta_k)] \geq 0.$$

Let θ_k vary over Ω. Therefore

$$\sum_{k=1}^{K} \pi(\theta_k)[\sum_{s^{-i} \in S^{-i}} p(s^{-i}, r^i \theta_k) - h^i(s^{-i}, t^i, \theta_k)] \geq 0$$

$\forall\ i \in N,\ r^i, t^i \in S^i$.

Hence $p(.|.)$ is a correlated equilibrium with state dependent preferences.

In the remainder of the paper we shall work with the second definition of a correlated equilibrium with state dependent payoffs.

3. Incentive Efficient Correlated Equilibrium: We say that a correlated equilibrium with state dependent payoffs $p(.|.)$ is incentive efficient if there does not exist any other correlated equilibrium with state dependent payoffs $\bar{p}(.|.)$ such that for all $i \in N$

$$\sum_{k=1}^{K} \pi(\theta_k) \sum_{s\varepsilon S} \bar{p}(s|\theta_k) \quad h^i(s,\theta_k) \geq \sum_{k=1}^{K} \pi(\theta_k) \sum_{s\varepsilon S} p(s|\theta_k) h^i(s,\theta_k),$$

with strict inequality holding for atleast one $i\varepsilon N$. As in Myerson (1983) we can say that a correlated equilibrium with state dependent payoffs $p(.|.)$ is incentive efficient if and only if there exists a vector $\lambda = (\lambda_1,\ldots,\lambda_n)$ such that every $\lambda_i > 0$ and $p(.|.)$ is an optimal solution to the following problem:

$$\underset{\bar{p}(.|.)}{\text{maximise}} \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{s\varepsilon S} \lambda_i \; \bar{p}(s|\theta_k)\pi(\theta_k) h_i(s,\theta_k)$$

subject to

(i) $$\sum_{s\varepsilon S} \bar{p}(s|\theta_k) = 1 \text{ and } \bar{p}(s|\theta_k) \geq 0 \quad \forall \; s\varepsilon S \text{ and } \theta_k \; \varepsilon \; \Omega$$

(ii) $$\sum_{k=1}^{K} \pi(\theta_k) \sum_{s^{-i}\varepsilon S^{-i}} \bar{p}(s^{-i},r^i|\theta_k)[h^i(s^{-i},r^i,\theta_k)-h^i(s^{-i},t^i,\theta_k)] \geq 0$$

$$\forall \; i, \; \forall \; r^i \; \varepsilon \; S^i \; \forall \; t^i \; \varepsilon \; S^i.$$

The following theorem characterizes all incentive efficient equilibria, whenever $\pi(\theta_k) > 0$ for all $\theta_k \; \varepsilon \; \Omega$.

Theorem 4: Suppose that p is a correlated equilibrium with state dependent preferences. Then p is incentive efficient if and only if there exists vectors λ and β, $\lambda \; \varepsilon \; \mathbb{R}^n$, $\beta \; \varepsilon \; \mathbb{R}^{\sum_{i\varepsilon N} |S^i|^2}$ such that

$$\lambda_i > 0 \text{ and } \beta_i(r^i|t^i) \geq 0, \quad \forall i, \; \forall \; r^i \; \varepsilon \; S^i, \; \forall \; t^i \; \varepsilon \; S^i,$$

$$\beta_i(t^i|r^i) \; [\sum_{s^{-i}\varepsilon S^{-i}} \sum_{k=1}^{K} \pi(\theta_k)p(s^{-i},r^i|\theta_k)(h^i(s^{-i},r^i,\theta_k)-h^i(s^{-i},t^i,\theta_k))] = 0$$

$$\forall \; i, \; \forall \; r^i \; \varepsilon \; S^i, \; \forall \; t^i \; \varepsilon \; S^i,$$

and

$$\sum_{s\varepsilon S} p(s|\theta_k) \sum_{i=1}^{n} v_i(s,\theta_k, \lambda, \beta) = \max_{s\;\varepsilon S} \sum_{i=1}^{n} v_i(s,\theta_k,\lambda, \beta) \quad \forall \; \theta_k \; \varepsilon \; \Omega;$$

where

$$v_i(s,\theta_k,\lambda,\beta) = \lambda_i h^i(s,\theta_k) + \sum_{t^i \varepsilon S^i} \beta_i(t^i|s^i)(h^i(s,\theta_k) - h^i(s^{-i},t^i\theta_k))\beta_i(t^i|r^i).$$

<u>Proof</u>: Let

$$L = \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{s\varepsilon S} \lambda_i p(s|\theta_k)\pi(\theta_k)h^i(s,\theta_k)$$

$$+ \sum_{i=1}^{n} \sum_{r^i\varepsilon S^i} \sum_{t^i\varepsilon S^i} \beta_i(t^i|r^i) \sum_{k=1}^{K} \pi(\theta_k) \sum_{s^{-i}\varepsilon S^{-i}} p(s^{-i},r^i|\theta_k)$$

$$[h^i(s^i,r^i,\theta_k) - h^i(s^{-i},t^i,\theta_k)]$$

L is the Lagrangean of the maximization problem.

The set of all correlated equilibrium is a compact convex set and the objective function is linear. Hence, the Lagrangean saddle point conditions are necessary and sufficient for a maximum.

Simplifying the Lagrangean we get,

$$L = \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{s\varepsilon S} \lambda_i h_i(s,\theta_k)p(s|\theta_k)\ \pi(\theta_k)$$

$$+ \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{s\varepsilon S} \beta_i(t^i|s^i) \sum_{k=1}^{K} \pi(\theta_k)p(s|\theta_k)[h^i(s,\theta_k) - h^i(s^{-i},t^i,\theta_k)]$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{K} \sum_{s\varepsilon S} [\lambda_i h_i(s,\theta_k) + \sum_{t^i\varepsilon S^i} \beta_i(t^i|s^i)(h^i(s,\theta_k) - h^i(s^{-i},t^i,\theta_k)].p(s|\theta_k)\pi(\theta_k)$$

$$= \sum_{i=1} \sum_{k=1} \sum_{s\varepsilon S} v_i(s,\theta_k,\lambda\ ,\beta)p(s|\theta_k)\pi(\theta_k).$$

The complementary slackness conditions require that,

$$\beta_i(t^i|s^i) \sum_{s^{-i}\varepsilon S^{-i}} \sum_{k=1}^{K} \pi(\theta_k)p(s|\theta_k)[h^i(s,\theta_k) - h^i(s^{-i},t^i,\theta_k)] = 0.$$

Since $p(.|.)$ must maximize,

$\sum_{k=1}^{K} [\sum_{i=1}^{n} \sum_{s \in S} v_i(s,\theta_k,\lambda,\beta)p(s|\theta_k)]\pi(\theta_k)$ we get that

$$\sum_{s \in S} p(s|\theta_k) \sum_{i=1}^{n} v_i(s,\theta_k,\lambda,\beta) = \max_{s \in S} \sum_{i=1}^{n} v_i(s,\theta_k,\lambda,\beta) \quad \forall \quad \theta_k \in \Omega.$$

This proves the theorem.

4. Correlated Equilibrium for Games With State Dependent Payoffs and Single Experimentation:

In this final section we extend the framework of games with state dependent payoffs to include the possibility of experimentation by the players with a view to obtaining additional information about the state of nature θ. Our approach is similar to that in Blackwell and Girshick [1954] in modelling what may be referred to as a statistical game.

We assume that each player 'i' is permitted to observe the outcome of an experiment which he is allowed to perform in order to obtain further information about the state of nature θ The observations of player i, are restricted to lie in a (finite) set of outcomes Z^i. Let $Z = Z^1 \times \ldots \times Z^n$. We assume that there exists a function $f : Z \times \Omega \to [0,1]$ denoted $f(.|.)$ such that $\sum_{z \in Z} f(z|\theta_k) = 1 \;\; \forall \; \theta_k \in \Omega$. $f(z|\theta_k)$ denotes the probability of occurrence of $z \in Z$, when the true state of nature is $\theta_k \in \Omega$. Each player i, has now a strategy set $D^i = \{d^i/d^i : Z^i \to S^i\}$, where a typical strategy is a decision rule or a function mapping his observation into a pure strategy S^i for the finite game considered earlier. Let $D=D^1 \times \ldots \times D^n$ and $D^{-i} = \prod_{j \neq i} D^j$, have the usual interpretations. The fact that the players are allowed to observe the outcome of a single experiment means that it is a fixed sample size experiment. In the above framework we have the following definition:

A correlated equilibrium with state dependent payoffs and single experimentation is a function $p : D \times \Omega \to [0,1]$ denoted $p(.|.)$ such that

(i) $\sum_{d \in D} p(d|\theta_k) = 1 \quad \forall\, \theta_k \in \Omega$

(ii) For all $\bar{d}^i$, $\underline{d}^i \in D^i$,

$$\sum_{k=1} \sum_{d^{-i} \in D^{-i}} \sum_{z \in Z} p(d^{-i},\bar{d}^i|\theta_k)\pi(\theta_k)\ f(z|\theta_k)[h^i(d^{-i}(z^{-i}),\bar{d}^i(z^i)) - h^i(d^{-i}(z^{-i}),\underline{d}^i(z^i))] \geq 0.$$

The following theorem can now be proved.

Theorem 5: Every finite game with state dependent payoffs and single experimentation has a correlated equilibrium with state dependent payoffs and single experimentation.

Proof: Consider the following auxiliary finite game with state dependent payoffs and n players. The strategy space of player $i = 1,\ldots,n$ is given by D^i where $D^i = |S^i|^{|Z^i|} < +\infty$; the state dependent payoff function of player i is given by $g^i : D \times \Omega \to \mathbb{R}$ where $g^i(d,\theta_k) = \sum_{z \in Z} h^i(d(z),\theta_k)f(z|\theta_k)$. The game $g = (g^1,\ldots,g^n)$ is a finite game with state dependent payoffs. Hence by Theorem 2, it admits a correlated equilibrium with state dependent payoffs; i.e. there exists a function $p : D \times \Omega \to [0,1]$ denoted $p(.|.)$ such that

(i) $\sum_{d \in D} p(d|\theta_k) = 1 \quad \forall\, \theta_k \in \Omega$

(ii) $\sum_{d^{-i} \in D^{-i}} p(d^{-i},\bar{d}^i|\theta_k)\pi(\theta_k)\ [g^i(d^{-i},\bar{d}^i,\theta_k) - g^i(d^{-i},\underline{d}^i,\theta_k)] \geq 0$

for all $\bar{d}^i$, $\underline{d}^i \in D^i$ and for all $i = 1,\ldots,n$.

Hence p is a correlated equilibrium with state dependent payoffs and single experimentation.

Remark 1: No where in the proofs of Theorems 2, 3 and 5 was the fact that the players have common subjective probability about the occurence of the state of nature used. The results continue to hold with prior beliefs being different for the players. The assumption on common prior beliefs makes the statement and proof of Theorem 4 manageable.

Remark 2: A theorem analogous to Theorem 4 can now be asserted for correlated equilibria with state dependent payoffs and single experimentation.

Remark 3: The interpretation of correlated equilibria with state dependent payoffs and single experimentation is similar to the interpretation prevailing under no experimentation except now the joint randomization takes place over the extended strategy space i.e. the space of decision rules of the players.

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