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Abstract

Meeting the profit target is often preferred over maximizing expected profit in uncertain business environments. Research into the newsboy problem with satiation objective started quite early. However, the progress has been slow, particularly in the multi-product setting. We study the general multi-product newsboy problem with satiation objective. A discrete formulation is adopted. Computational methods for evaluating and maximizing the satiation probability (i.e., probability of meeting the profit target) are developed. Difficulties associated with the conventional continuous formulation are also discussed.

1 Introduction

Traditionally, operations research/management models consider cost minimization (or profit maximization) as the objective of the decision maker. This objective does not fit into many real life situations, particularly when outcome uncertainty is high. Simon (1959) discussed problems associated with the economic models that maximize expected profit in uncertain environments. He argued that firms and managers would try to satiate (the profit target) rather than to maximize (expected profit). Kahneman & Tversky (1979), in their prospect theory, argued that human decisions are based on gains and losses w.r.t. a reference point (i.e., the profit target), rather than the final value. If gains are given little weightage and losses are given large weightage, the objective is satiation of the profit target[1].

There are empirical evidences in support of the satiation objective. Lanzillotti (1958) explored pricing objectives of twenty business organization; achieving targeted return on investment (which is equivalent to profit target if cost is known) was the most cited choice. Shipley (1981) found similar result in a survey of pricing objectives of 728 British manufacturing firms; about two-third of the participants identified meeting the profit target as their main objective.

Satiation objective in the newsboy problem was first considered by Irwin & Allen (1978); they adopted a continuous formulation and identified the necessary conditions for maximization of the satiation probability (i.e., probability of meeting the profit target) in the single-product

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[1]Prospect theory says that the value function (which the decision maker tries to maximize) is concave for gains, convex for losses, and steeper for losses than for gains. If the concave function for gains is linear with zero slope (little weightage) and the convex function for losses is linear with infinite slope (large weightage), the objective is maximization of probability of meeting the profit target (i.e., satiation of the profit target).

case. These conditions depend on the demand distribution function. Ismail & Louderback (1979) solved the single-product satiating newsboy problem (SPSNP) for normally distributed demand by iterative search; later, H.-S. Lau (1980) derived a closed-form solution for the same problem. Sankarasubramanian & Kumaraswamy (1983) derived closed-form solutions for the SPSNP with uniform and exponential demand distributions. Norland (1980); H.-S. Lau (1980) showed that the optimum order quantity in the SPSNP does not depend on the demand distribution function if stock-out cost is zero; they derived closed-form solution for this special case.

Solution of the SPSNP paved the way for the development of different extensions. For example, A. H.-L. Lau & Lau (1988b) considered price-dependent demand, Khouja (1995) considered price mark-down by the newsboy to sell excess inventory, and Khouja & Robbins (2003) considered advertisement-dependent demand in the SPSNP. There are more such papers, but our focus, in this paper, is on the multi-product extension of the SPSNP.

A. H.-L. Lau & Lau (1988a) were the first to explore the multi-product newsboy problem with satiation objective. They considered the two-product problem with zero stock-out costs and uniform demand; closed-form solution was obtained for the case of identical products (i.e., identical cost and demand parameters for both products). Li et al. (1990) studied a more general version of the two-product problem by allowing the products to be non-identical; everything else were similar to the problem solved by A. H.-L. Lau & Lau (1988a). They proposed an efficient algorithm for finding the optimum order quantities. Later, they solved a similar problem with independent exponential product demands (Li et al., 1991).

To the best of our knowledge, no progress has happened in the domain of the multi-product satiating newsboy problem (MPSNP) after the works of A. H.-L. Lau & Lau (1988a); Li et al. (1990, 1991). However, the study of Shao & Ji (2006) requires a mention here. They solved the MPSNP with zero stock-out cost using fuzzy simulation. Product demands were modelled by fuzzy variables and probability of an event was replaced by credibility of an event. However, we can not use their method for the MPSNP with stochastic demand due to the differences between fuzzy and stochastic modelling. Besides, they did not solve the problem optimally.

Our understanding of the MPSNP is limited to restricted two-product problems. In this paper, we study the general MPSNP; non-zero stock-out costs are considered and no restriction is imposed on product demands. We adapt a discrete formulation of the problem. It allows us to develop computational method for finding and maximizing the satiation probability in the general case. We also discuss limitations of the conventional continuous formulation of the MPSNP (adapted by A. H.-L. Lau & Lau, 1988a; Li et al., 1990, 1991).

2 The discrete formulation

Following notations are used in this paper.

n	Number of products (positive integer).
m_i	Unit profit for the i^{th} product (positive).

c_i	Unit purchase cost less salvage value for the i^{th} product (positive).
s_i	Unit stock-out goodwill loss for the i^{th} product (positive).
X_i	Stochastic demand for the i^{th} product (integer-valued).
a_i	Lower limit of X_i (non-negative integer).
b_i	Upper limit of X_i (positive integer).
$p_i()$	Marginal probability mass function of X_i.
$P_i()$	Marginal cumulative distribution function of X_i.
Q_i	Order quantity of the i^{th} product (non-negative integer).
$\Pi_i(Q_i, x_i)$	Profit from the i^{th} product for order quantity, Q_i and realized demand, x_i.
T	Profit target. $\underline{T}$ represents the maximum assured target and $\overline{T}$ represents the maximum achievable target.
X	Demand vector. $X = (X_i) = (X_1, X_2, \ldots, X_n)$.
Ω	Sample space of X. $\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$, where Ω_i is the sample space of X_i, i.e., $\Omega_i = \{a_i, a_i + 1, \ldots, b_i\}$ for $i = 1, 2, \ldots, n$.
$p()$	Probability mass function of X. For independent demand, $p(x) = \prod_{i=1}^{n} p_i(x_i)$.
Q	Order quantity vector. $Q = (Q_i) = (Q_1, Q_2, \ldots, Q_n)$.
$\Pi(Q, x)$	Total profit for ordering decision, Q and demand scenario, x.
$I_{\Pi(Q,x)\geq T}$	Indicator of $\Pi(Q, x)$ reaching or exceeding T.
$P_T(Q)$	Satiation probability for ordering decision, Q.

In MPSNP, our objective is to

$$\begin{aligned} \underset{Q\in\mathbb{N}_0^n}{\text{maximize}}\ P_T(Q) &= \sum_{x\in\Omega} I_{\Pi(Q,x)\geq T}\ p(x) \\ &= \sum_{x_1=a_1}^{b_1} \sum_{x_2=a_2}^{b_2} \cdots \sum_{x_n=a_n}^{b_n} I_{\Pi(Q,x)\geq T}\ p(x), \end{aligned} \tag{1}$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{N}_0^n$ is the corresponding n-dimensional space.

The indicator function is defined as

$$I_{\Pi(Q,x)\geq T} = \begin{cases} 1 & \text{if } \ \Pi(Q, x) = \sum_{i=1}^{n} \Pi_i(Q_i, x_i) \geq T, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Individual product profits (for $i = 1, 2, \ldots, n$) are given by

$$\begin{aligned} \Pi_i(Q_i, x_i) = &\ \text{sales revenue } - \text{ over-stocking cost (if any) } - \text{ under-stocking cost (if any)} \\ = &\ m_i \min\{Q_i, x_i\} - c_i \max\{0, Q_i - x_i\} - s_i \max\{0, x_i - Q_i\}. \end{aligned} \tag{3}$$

Note that (2) and (3) are valid for the continuous formulation (i.e., real-valued Q_i and x_i for $i = 1, 2, \ldots, n$). Let us note down some properties of the MPSNP.

Proposition 1. *$P_T(Q)$ maximizing ordering decision, $Q^* \in \Omega$.*

See Appendix A for a proof. The above result reduces the solution space from infinite $\mathbb{N}_0^n$ to finite Ω. This ensures computation in finite time.

Like Q, some restrictions can be placed on T too. By (3), the maximum value that $\Pi_i(Q_i, x_i)$ can take is $m_i b_i$ (when $x_i = Q_i = b_i$). Thus, $\Pi(Q, x) \le \sum_{i=1}^n m_i b_i \Rightarrow \overline{T} = \sum_{i=1}^n m_i b_i$. This target is met only when $Q = b$. Any $T > \overline{T}$ is unachievable.

Lemma 1. *Any $T \le \sum_{i=1}^n \max\{\Pi_i(\lfloor Q_{0i} \rfloor, b_i), \Pi_i(\lceil Q_{0i} \rceil, a_i)\}$ can be achieved with certainty, where $Q_{0i} = \{(m_i + c_i)a_i + s_i b_i\}/(m_i + c_i + s_i)$ for $i = 1, 2, \ldots, n$. The i^{th} element of an ordering decision that ensures the above is $\lfloor Q_{0i} \rfloor$ if $\Pi_i(\lfloor Q_{0i} \rfloor, b_i) \ge \Pi_i(\lceil Q_{0i} \rceil, a_i)$ and $\lceil Q_{0i} \rceil$ if $\Pi_i(\lfloor Q_{0i} \rfloor, b_i) \le \Pi_i(\lceil Q_{0i} \rceil, a_i)$.*

It can be easily verified that $Q_{0i} \in (a_i, b_i)$, i.e., $\lfloor Q_{0i} \rfloor, \lceil Q_{0i} \rceil \in \Omega_i$ for $i = 1, 2, \ldots, n$.

Proposition 2. *Maximum assured target, $\underline{T} = \sum_{i=1}^n \max\{\Pi_i(\lfloor Q_{0i} \rfloor, b_i), \Pi_i(\lceil Q_{0i} \rceil, a_i)\}$, where $Q_{0i} = \{(m_i + c_i)a_i + s_i b_i\}/(m_i + c_i + s_i)$ for $i = 1, 2, \ldots, n$.*

See Appendix B for proofs of the above results. Note that if $T \le \underline{T}$ or $T \ge \overline{T}$, the problem has trivial solutions. Further computation is needed only if $T \in (\underline{T}, \overline{T})$.

3 Satiation probability

For maximizing satiation probability, we should be able to compute it first. One way to compute satiation probability for given T and Q is to check $I_{\Pi(Q,x) \ge T}$ $\forall x \in \Omega$ and calculate $\sum_{x \in \Omega} I_{\Pi(Q,x) \ge T}\, p(x)$. However, since the number of elements in Ω increases exponentially with the number of products, time requirement for this method grows exponentially with n. Time requirement increases further if demand ranges $(b_i - a_i + 1)$ are high. Results in the following subsection help in reducing the time requirement for computing $P_T(Q)$.

3.1 An important result

First, let us define profit cushion and cost distance.

Definition 1. *Profit cushion at $Q \in \Omega$ for given $T \in \mathbb{R}$, $PC_T(Q)$ and cost distance of $x \in \mathbb{R}^n$ from given $Q \in \Omega$, $CD_Q(x)$ are defined as*

$$PC_T(Q) = \sum_{i=1}^n m_i Q_i - T,$$

$$CD_Q(x) = \sum_{i=1}^n \left[(m_i + c_i)\max\{0, Q_i - x_i\} + s_i \max\{0, x_i - Q_i\}\right].$$

Though $x \in \Omega \subset \mathbb{R}^n$ for the discrete formulation, we consider $x \in \mathbb{R}^n$ in the above definition to facilitate some results to follow. Cost distance is additive in nature. It is the sum of cost distances of individual products, i.e. $CD_Q(x) = \sum_{i=1}^n CD_{Q_i}(x_i)$.

Profit cushion is the gap between the maximum achievable profit for an ordering decision and the given target. Cost distance is the amount of cushion lost due to mismatch between demand and supply. Following result connects $PC_T(Q)$, $CD_Q(x)$, and $I_{\Pi(Q,x)\geq T}$.

Lemma 2. *$I_{\Pi(Q,x)\geq T} = 1$ if and only if $CD_Q(x) \leq PC_T(Q)$.*

The above result can be verified easily. Individual product profit, i.e., (3) can be rewritten as $\Pi_i(Q_i, x_i) = m_iQ_i - (m_i + c_i)\max\{0, Q_i - x_i\} - s_i \max\{0, x_i - Q_i\} = m_iQ_i - CD_{Q_i}(x_i)$ for $i = 1, 2, \ldots, n$. Then, $\Pi(Q, x) = \sum_{i=1}^n \Pi_i(Q_i, x_i) = T + PC(Q) - CD_Q(x)$. By (2), $I_{\Pi(Q,x)\geq T} = 1$ if and only if $\Pi(Q, x) \geq T$. Hence, $I_{\Pi(Q,x)\geq T} = 1$ if and only if $CD_Q(x) \leq PC_T(Q)$.

By definition, $CD_Q(x)$ is non-negative. If $PC_T(Q) < 0$, $P_T(Q) = 0$ as $CD_Q(x) \geq 0 > PC_T(Q)$ $\forall x \in \Omega$. On the other hand, if $PC_T(Q) \geq 0$, $P_T(Q) \geq 0$ as $CD_Q(Q) = 0 \leq PC_T(Q) \Rightarrow I_{\Pi(Q,Q)\geq T} = 1 \Rightarrow P_T(Q) \geq p(Q)$. To calculate the exact value of $P_T(Q)$ in the later case, we still need to check $I_{\Pi(Q,x)\geq T}$ $\forall x \in \Omega$. We can do better. With the help of the following definition, Theorem 1 precisely identifies the subset of Ω where $I_{\Pi(Q,x)\geq T} = 1$, thereby eliminating the necessity to check $I_{\Pi(Q,x)\geq T}$ completely.

Definition 2. *Terminal demand points for the i^{th} product ($i = 1, 2, \ldots, n$) for given Q and T such that $PC_T(Q) \geq 0$ are defined as*

$$\underline{x}^{(i)}(Q,T) = \left(Q_1, \ldots, Q_{i-1}, Q_i - \frac{PC_T(Q)}{m_i + c_i}, Q_{i+1}, \ldots, Q_n\right),$$
$$\overline{x}^{(i)}(Q,T) = \left(Q_1, \ldots, Q_{i-1}, Q_i + \frac{PC_T(Q)}{s_i}, Q_{i+1}, \ldots, Q_n\right).$$

Clearly, $\underline{x}_i^{(i)}(Q,T) = Q_i - PC_T(Q)/(m_i + c_i)$, $\overline{x}_i^{(i)}(Q,T) = Q_i + PC_T(Q)/s_i$, and $\underline{x}_{j\neq i}^{(i)}(Q,T) = \overline{x}_{j\neq i}^{(i)}(Q,T) = Q_j$ for all $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$. Note that $\underline{x}_i^{(i)}$ and $\overline{x}_i^{(i)}$ need not be in Ω_i (i.e., they can take non-integer values and can be outside the demand limits).

We call $\underline{x}^{(i)}$ and $\overline{x}^{(i)}$ as the terminal demand points because $I_{\Pi(Q,x)\geq T} = 0$ if $x_i \notin [\underline{x}_i^{(i)}, \overline{x}_i^{(i)}]$. Without loss of generality, let us assume that $x_1 \notin [\underline{x}_1^{(1)}, \overline{x}_1^{(1)}]$, i.e., either $x_1 < Q_1 - PC_T(Q)/(m_1 + c_1)$ or $x_1 > Q_1 + PC_T(Q)/s_1$. In the first case, $PC_T(Q) < (m_1 + c_1)(Q_1 - x_1) = CD_{Q_1}(x_1) \leq CD_Q(x)$. In the second case, $PC_T(Q) < s_1(x_1 - Q_1) = CD_{Q_1}(x_1) \leq CD_Q(x)$. In both cases, $CD_Q(x) > PC_T(Q)$. Then by Lemma 2, $I_{\Pi(Q,x)\geq T} = 0$.

Theorem 1. *$I_{\Pi(Q,x)\geq T} = 1$ if and only if $x \in \mathrm{co}D$, where $\mathrm{co}D$ is the convex hull of D. If $PC_T(Q) \geq 0$, D is the set of terminal demand points, else $D = \emptyset$.*

See Appendix C for a proof. The above theorem plays a pivotal role in calculating $P_T(Q)$. It also explains why continuous formulation of the MPSNP is difficult to solve.

3.2 Computation of satiation probability

Using Theorem 1, we can rewrite (1) as

$$P_T(Q) = \sum_{x \in \text{co}D \cap \Omega} p(x), \quad \text{where } D \text{ is defined in Theorem 1.} \tag{4}$$

The above expression eliminates the necessity to check $I_{\Pi(Q,x)\geq T}$ for finding satiation probability. However, we need to identify $x \in \text{co}D \cap \Omega$ for the calculation of $P_T(Q)$.

For the single-product case, $\text{co}D \cap \Omega$ is simply an interval; hence, the computation of $P_T(Q)$ is straight forward. The problem with the multi-product case is that $\text{co}D \cap \Omega$ can assume many different shapes and most of these shapes are irregular[2]. Exploiting the additive structure of $CD_Q(x)$, we can express $\text{co}D \cap \Omega$ as following.

$$\mathbb{S} = \left\{ x : x \in \mathbb{N}_0^n,\ \underline{d}_i \leq x_i \leq \overline{d}_i \text{ for } i = 1, 2, \ldots, n \right\}, \tag{5}$$
$$\text{where } \underline{d}_i = \max\left\{ a_i, Q_i - \frac{pcl_i(x_1, \ldots, x_{i-1})}{m_i + c_i} \right\},\ \overline{d}_i = \min\left\{ b_i, Q_i + \frac{pcl_i(x_1, \ldots, x_{i-1})}{s_i} \right\},$$
$$\text{and } pcl_i(x_1, \ldots, x_{i-1}) = PC_T(Q) - \sum_{j=1}^{i-1} CD_{Q_j}(x_j) \quad \text{for } i = 1, 2, \ldots, n.$$

See Appendix D for a proof that $\mathbb{S} = \text{co}D \cap \Omega$. Structure of $\mathbb{S}$ is similar to that of a hyperrectangle and its elements can be identified with some quick calculations. pcl_i is the amount of profit cushion left for the i^{th} product after compensating for the demand-supply mismatch of the first $i-1$ products. Note that $pcl_1 = PC_T(Q)$.

Since $\mathbb{S} = \text{co}D \cap \Omega$, (4) can be rewritten as

$$P_T(Q) = \sum_{x_1=\lceil \underline{d}_1 \rceil}^{\lfloor \overline{d}_1 \rfloor} \sum_{x_2=\lceil \underline{d}_2(x_1) \rceil}^{\lfloor \overline{d}_2(x_1) \rfloor} \sum_{x_3=\lceil \underline{d}_3(x_1,x_2) \rceil}^{\lfloor \overline{d}_3(x_1,x_2) \rfloor} \cdots\cdots \sum_{x_n=\lceil \underline{d}_n(x_1,\ldots,x_{n-1}) \rceil}^{\lfloor \overline{d}_n(x_1,\ldots,x_{n-1}) \rfloor} p(x_1, \ldots, x_n), \tag{6}$$

where $\underline{d}_i(x_1, \ldots, x_{i-1})$ and $\overline{d}_i(x_1, \ldots, x_{i-1})$ for $i = 1, 2, \ldots, n$ are defined in (5).

Now, we in a position to exploit the power of Theorem 1 in computing the satiation probability. Algorithm 1 computes $P_T(Q)$ by implementing (6). Comments in Algorithm 1 (and the next) are enclosed inside $\langle\rangle$ and the font colour is gray.

Computation time of Algorithm 1 can be represented as: $t_{SP} = \alpha + \alpha_0 n + \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n$, where $\alpha, \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ are input-independent constants and $\beta_1, \beta_2, \ldots, \beta_n$ are input-dependent variables. β_i for $i = 1, 2, \ldots, n$ are given by

$$\beta_i = \sum_{x_1=\lceil \underline{d}_1 \rceil}^{\lfloor \overline{d}_1 \rfloor} \sum_{x_2=\lceil \underline{d}_2(x_1) \rceil}^{\lfloor \overline{d}_2(x_1) \rfloor} \sum_{x_3=\lceil \underline{d}_3(x_1,x_2) \rceil}^{\lfloor \overline{d}_3(x_1,x_2) \rfloor} \cdots\cdots \sum_{x_i=\lceil \underline{d}_i(x_1,\ldots,x_{i-1}) \rceil}^{\lfloor \overline{d}_i(x_1,\ldots,x_{i-1}) \rfloor} 1,$$

[2]Objects, whose elements are difficult to identify, are referred to as irregular objects. A hyperrectangle is a regular object, while a convex hull (in multi-dimension) is irregular.

Algorithm 1 Calculate $P_T(Q)$ for general demand

Input: n, T, Q $\langle Q = (Q_1, Q_2, \ldots, Q_n)\rangle$, *parameters* $\langle a_i, b_i, m_i, c_i, s_i$ for $i = 1, 2, \ldots, n\rangle$, and $p()$ $\langle n$-dimensional pmf array$\rangle$

Output: $P_T(Q)$

1. $P_T(Q) \leftarrow 0$
2. $pcl_1 \leftarrow \sum_{i=1}^{n} m_i Q_i - T$
3. $\underline{d}_1 \leftarrow \max\{a_1, Q_1 - pcl_1/(m_1 + c_1)\}$, $\bar{d}_1 \leftarrow \min\{b_1, Q_1 + pcl_1/s_1\}$
4. **for** $x_1 = \lceil \underline{d}_1 \rceil$ **to** $\lfloor \bar{d}_1 \rfloor$ **do**
5. $pcl_2 \leftarrow pcl_1 - [(m_1 + c_1)\max\{0, Q_1 - x_1\} + s_1 \max\{0, x_1 - Q_1\}]$
6. $\underline{d}_2 \leftarrow \max\{a_2, Q_2 - pcl_2/(m_2 + c_2)\}$, $\bar{d}_2 \leftarrow \min\{b_2, Q_2 + pcl_2/s_2\}$
7. **for** $x_2 = \lceil \underline{d}_2 \rceil$ **to** $\lfloor \bar{d}_2 \rfloor$ **do**
8.   .
9.   .
10. $pcl_n \leftarrow pcl_{n-1} - [(m_{n-1} + c_{n-1})\max\{0, Q_{n-1} - x_{n-1}\} + s_{n-1}\max\{0, x_{n-1} - Q_{n-1}\}]$
11. $\underline{d}_n \leftarrow \max\{a_n, Q_n - pcl_n/(m_n + c_n)\}$, $\bar{d}_n \leftarrow \min\{b_n, Q_n + pcl_n/s_n\}$
12.   **for** $x_n = \lceil \underline{d}_n \rceil$ **to** $\lfloor \bar{d}_n \rfloor$ **do**
13. $P_T(Q) \leftarrow P_T(Q) + p(x_1, \ldots, x_n)$
14.   **end for**
15.   .
16.   .
17. **end for**
18. **end for**
19. **return** $P_T(Q)$

where $\underline{d}_j(x_1, \ldots, x_{j-1})$ and $\bar{d}_j(x_1, \ldots, x_{j-1})$ for $j = 1, 2, \ldots, i$ are defined in (5).

Clearly, $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$. Then $\beta_n < t_{SP}$ and $\beta_1 + \beta_2 + \cdots + \beta_n \leq n\beta_n$. Furthermore, $n\beta_n$ dominates t_{SP} for sufficiently large input. Note that β_n is the cardinality of $\mathbb{S}$ defined in (5). Thus, $t_{SP} = \Omega(|\mathbb{S}|)$ and $t_{SP} = O(n|\mathbb{S}|)$.

$|\mathbb{S}|$ grows exponentially with the number of products. $|\mathbb{S}|$ is increasing in demand ranges ($b_i - a_i + 1$ for $i = 1, 2, \ldots, n$) and decreasing in cost parameters (c_i and s_i for $i = 1, 2, \ldots, n$). $|\mathbb{S}|$ is increasing in profit cushion which implies that $|\mathbb{S}|$ is decreasing in profit target (as $PC_T(Q) = \sum_{i=1}^{n} m_i Q_i - T$ is decreasing in T). Dependence of $|\mathbb{S}|$ on m_i (for $i = 1, 2, \ldots, n$) is not so clear. In the worst case, $|\mathbb{S}|$ can be as high as $|\Omega| = \prod_{i=1}^{n}(b_i - a_i + 1)$.

Space requirement of Algorithm 1 is high too. The storing of probability array (p) consumes space of order $|\Omega| = \prod_{i=1}^{n}(b_i - a_i + 1)$. Space requirement of other inputs and operations of the algorithm is far less and grows linearly with n. However, in many real life situations, demand is modelled as random variable with some standard distribution. In such cases, we can reduce space requirement by removing p from the input and calculating $p(x_1, \ldots, x_n)$ in line 13 of the algorithm before adding it to $P_T(Q)$.

The case of independent demand

For the independent demand case, (6) can be rewritten as

$$\begin{aligned} P_T(Q) &= \sum_{x_1=\lceil \underline{d}_1 \rceil}^{\lfloor \bar{d}_1 \rfloor} \sum_{x_2=\lceil \underline{d}_2 \rceil}^{\lfloor \bar{d}_2 \rfloor} \cdots \sum_{x_{n-1}=\lceil \underline{d}_{n-1} \rceil}^{\lfloor \bar{d}_{n-1} \rfloor} \sum_{x_n=\lceil \underline{d}_n \rceil}^{\lfloor \bar{d}_n \rfloor} \prod_{i=1}^{n} p_i(x_i) \\ &= \sum_{x_1=\lceil \underline{d}_1 \rceil}^{\lfloor \bar{d}_1 \rfloor} \sum_{x_2=\lceil \underline{d}_2 \rceil}^{\lfloor \bar{d}_2 \rfloor} \cdots \sum_{x_{n-1}=\lceil \underline{d}_{n-1} \rceil}^{\lfloor \bar{d}_{n-1} \rfloor} \prod_{i=1}^{n-1} p_i(x_i) \left\{ P_n(\lfloor \bar{d}_n \rfloor) - P_n(\lceil \underline{d}_n \rceil - 1) \right\}, \end{aligned} \tag{7}$$

where $\underline{d}_i(x_1,\ldots,x_{i-1})$ and $\bar{d}_i(x_1,\ldots,x_{i-1})$ for $i = 1,2,\ldots,n$ are defined in (5).

Algorithm 2 computes $P_T(Q)$ for the case of independent demand by implementing (7). It is very similar to Algorithm 1, except for the absence of the innermost for-loop.

Algorithm 2 Calculate $P_T(Q)$ for independent demand

Input: n, T, Q $\langle Q = (Q_1, Q_2, \ldots, Q_n)\rangle$, *parameters* $\langle a_i, b_i, m_i, c_i, s_i$ for $i = 1, 2, \ldots, n\rangle$, $p_i()$ for $i = 1, 2, \ldots, n-1$ ⟨1-dimensional pmf arrays⟩, and $P_n()$ ⟨1-dimensional cdf array⟩

Output: $P_T(Q)$

$P_T(Q) \leftarrow 0$

$f_0 \leftarrow 1$

$pcl_1 \leftarrow \sum_{i=1}^{n} m_i Q_i - T$

$\underline{d}_1 \leftarrow \max\{a_1, Q_1 - pcl_1/(m_1 + c_1)\}$, $\bar{d}_1 \leftarrow \min\{b_1, Q_1 + pcl_1/s_1\}$

for $x_1 = \lceil \underline{d}_1 \rceil$ **to** $\lfloor \bar{d}_1 \rfloor$ **do**

  $f_1 \leftarrow f_0 \times p_1(x_1)$

  $pcl_2 \leftarrow pcl_1 - [(m_1 + c_1)\max\{0, Q_1 - x_1\} + s_1 \max\{0, x_1 - Q_1\}]$

  $\underline{d}_2 \leftarrow \max\{a_2, Q_2 - pcl_2/(m_2 + c_2)\}$, $\bar{d}_2 \leftarrow \min\{b_2, Q_2 + pcl_2/s_2\}$

  .

  .

  for $x_{n-1} = \lceil \underline{d}_{n-1} \rceil$ **to** $\lfloor \bar{d}_{n-1} \rfloor$ **do**

    $f_{n-1} \leftarrow f_{n-2} \times p_{n-1}(x_{n-1})$

    $pcl_n \leftarrow pcl_{n-1} - [(m_{n-1} + c_{n-1})\max\{0, Q_{n-1} - x_{n-1}\} + s_{n-1} \max\{0, x_{n-1} - Q_{n-1}\}]$

    $\underline{d}_n \leftarrow \max\{a_n, Q_n - pcl_n/(m_n + c_n)\}$, $\bar{d}_n \leftarrow \min\{b_n, Q_n + pcl_n/s_n\}$

    $P_T(Q) \leftarrow P_T(Q) + f_{n-1} \times \{P_n(\lfloor \bar{d}_n \rfloor) - P_n(\lceil \underline{d}_n \rceil - 1)\}$

  end for

  .

  .

end for

return $P_T(Q)$

Following the analysis of Algorithm 1, computation time of Algorithm 2, $t_{SP(ID)} = \alpha + \alpha_0 n + \alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_{n-1}\beta_{n-1}$. Clearly, $\beta_{n-1} < t_{SP(ID)}$ and $n\beta_{n-1}$ dominates $t_{SP(ID)}$ for sufficiently large input. Let us define $\mathbb{S}_{-1}$ as

$$\mathbb{S}_{-1} = \left\{ x : x \in \mathbb{N}_0^{n-1},\ \underline{d}_i \le x_i \le \overline{d}_i \text{ for } i = 1, 2, \ldots, n-1 \right\}, \tag{8}$$

where $\underline{d}_i$ and $\overline{d}_i$ for $i = 1, 2, \ldots, n$ follow the definition of (5). $\mathbb{S}_{-1}$ is the projection of n-dimensional $\mathbb{S}$ into $(n-1)$-dimensional space. Clearly, β_{n-1} is the cardinality of $\mathbb{S}_{-1}$. Then $t_{SP(ID)} = \Omega(|\mathbb{S}_{-1}|)$ and $t_{SP(ID)} = O(n|\mathbb{S}_{-1}|)$.

The dependence of $|\mathbb{S}_{-1}|$ on different inputs is very similar to that of $|\mathbb{S}|$. The worst case value of $|\mathbb{S}_{-1}|$ is $\prod_{i=1}^{n-1}(b_i - a_i + 1)$. Clearly, time requirement of Algorithm 2 is less than that of Algorithm 1. Product with small $m_i + c_i, s_i$ and large $b_i - a_i$ should be labelled as the n^{th} product (so that $\lfloor \overline{d}_i \rfloor - \lceil \underline{d}_i \rceil$ is large) to maximize the time saving.

Space requirement of Algorithm 2, too, is less than that of Algorithm 1. Space requirement for storing p_i for $i = 1, 2, \ldots, n-1$ and P_n is of order $\sum_{i=1}^{n}(b_i - a_i)$. The remaining space requirement also grows linearly with n.

4 Enumeration-based optimization

With the help of algorithms for computing satiation probability, we can maximize it by complete enumeration of Ω. For the general case, $P_T(Q)$ is calculated $|\Omega|$ times using Algorithm 1. Since computation time of Algorithm 1, $t_{SP} = O(n|\Omega|)$, computation time for the enumeration-based optimization, $t_{OS} = O(n|\Omega|^2) = O(n\prod_{i=1}^{n} r_i^2)$, where $r_i = b_i - a_i + 1$ for $i = 1, 2, \ldots, n$ are the demand ranges. For the independent demand case, Algorithm 2 is used for the computation of $P_T(Q)$. Then computation time for optimization, $t_{OS(ID)} = O(nr_n \prod_{i=1}^{n-1} r_i^2)$ as computation time of Algorithm 2, $t_{SP(ID)} = O(n\prod_{i=1}^{n-1} r_i)$. Space requirement for the optimization is very similar to that of $P_T(Q)$ computing algorithms.

Numerical results

We test time performance of our optimization method by solving test problems. Factors that influence the computation time for optimization most are i) number of products, ii) demand ranges, and iii) demand type. We solved both independent demand (ID) and dependent demand (DD) problems. Three demand ranges (low, medium, and high) are considered. We started with the two-product problems (2P), but could not go beyond the three-product (3P) problems due to large time requirements. See Appendix E for the details on the test problem generation process. We implemented the algorithms in *GNU Octave 3.6.4 (GCC 4.6.2)* and solved the test problems in *Intel Core i5 (3.30 GHz) processors*.

Table 1 shows the computation time for optimization in three problem classes. A problem class is defined by the number of products (nP, $n = 2, 3, \ldots$) and demand type (ID and DD). It is evident that the 2P ID problems can be solved optimally in quick time. Each of the 100 solved problems took less than two minutes. However, the scenario changed quickly as we moved to the 3P ID problems. The maximum computation time exceeded 3 days while the average computation time was about 9 hours.

2P DD problems, as expected, took more time to solve optimally than their ID counterparts. The maximum computation time reached 14 hours while the average was about 80 minutes. Note that values in Table 1 are inclusive of the computation of probability mass function (p). This kind of implementation increases computation time, but decreases space usage.

Table 1: Computation time for optimization

		CPU Time		
Problem	Number	Min	Max	Avg
2P ID	100	2.3	81.8	25.9
3P ID	50	1186.5	295095.9	32076.4
2P DD	50	25.2	51495.9	4729.9

For practical purposes, where time requirement is a major concern, 2P ID problems can be solved optimally. Some 2P DD and few 3P ID problems can be solved optimally too. However, our optimization method may not be suitable for some 2P DD and most 3P ID problems. This method is not suitable for nP ID, $n \geq 4$ and nP DD, $n \geq 3$ problems.

5 Difficulty with the continuous formulation

Discrete formulation of the MPSNP has allowed us to compute the satiation probability and maximize it. However, computation time for the optimization grows exponentially as the number of products increases. Numerical results showed that the enumeration-based optimization can not be used for problems with three or more product. Let us check if the continuous formulation of the MPSNP is "better" or not.

Some changes are necessary for the continuous formulation. Individual product demands, X_i for $i = 1, 2, \ldots, n$ are real-valued random variables with bounded interval supports, $\Omega_i = [a_i, b_i]$. p_i and P_i are replaced by f_i and F_i respectively for $i = 1, 2, \ldots, n$. Order quantities, Q_i for $i = 1, 2, \ldots, n$ can take non-integer values. The distribution and density functions of demand vector, $X = (X_i)$ are denoted by F and f respectively. Now, our objective is to

$$\begin{aligned}\underset{Q \in \mathbb{R}^n_{\geq 0}}{\text{maximize}}\ P_T(Q) &= \int_{x \in \Omega} I_{\Pi(Q,x) \geq T}\ \mathrm{d}F(x) \\ &= \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_n}^{b_n} I_{\Pi(Q,x) \geq T}\ f(x) \mathrm{d}x_1 \mathrm{d}x_2 \cdots \mathrm{d}x_n, \end{aligned} \tag{9}$$

where $\mathbb{R}_{\geq 0} = \{x : x \in \mathbb{R}, x \geq 0\}$ and $\mathbb{R}^n_{\geq 0}$ is the corresponding n-dimensional space.

Most of our results and definitions for the discrete case are valid for the continuous case without any change. Lemma 1 and Proposition 2 need minor adjustment as order quantities are no more integer-valued. Similarly, members of $\mathbb{S}$ (defined in (5)) are no more in $\mathbb{N}^n_0$. For the continuous case, (6) takes the following form.

$$\begin{aligned} P_T(Q) &= \int_{x \in \mathrm{co}D \cap \Omega} \mathrm{d}F(x) \quad \text{(by Theorem 1)} \\ &= \int_{\underline{d}_1}^{\overline{d}_1} \int_{\underline{d}_2(x_1)}^{\overline{d}_2(x_1)} \cdots \int_{\underline{d}_n(x_1,\ldots,x_{n-1})}^{\overline{d}_n(x_1,\ldots,x_{n-1})} f(x)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n, \end{aligned} \tag{10}$$

where $\underline{d}_i$ and $\overline{d}_i$ for $i = 1, 2, \ldots, n$ follow the definition of (5).

Complexity of the above expression of $P_T(Q)$ is evident. Each integration limit in (10) can assume two values; $\underline{d}_i$ can be a_i or $Q_i - pcl_i/(m_i + c_i)$ and $\overline{d}_i$ can be b_i or $Q_i + pcl_i/s_i$ for $i = 1, 2, \ldots, n$, where $pcl_i(x_1, \ldots, x_{i-1})$ follows the definition of (5). Depending on the location of Q in Ω and model parameters, each integration in (10) can assume upto four different forms. Thus, the multiple integration can assume upto 4^n different forms.

The above observation can be understood geometrically too. $P_T(Q)$ is basically the probability of $X \in \mathrm{co}D \cap \Omega$. Thus, the expression of $P_T(Q)$ changes with the shape of $\mathrm{co}D \cap \Omega$. Ω is a hyperrectangle and co$D$ is a convex hull with $2n$ vertices, each laying on an axis in the n-dimensional space if Q is considered as the origin. Any number of vertices, ranging from 0 to $2n$, can be outside the hyperrectangle giving rise to different shapes for $\mathrm{co}D \cap \Omega$. Total number of different shapes is $\binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{2n} = 2^{2n} = 4^n$.

To maximize satiation probability, Ω is split into 4^n number of regions such that each region corresponds to one form of $P_T(Q)$. Note that some of these regions can be empty for a given problem instance. After the splitting of Ω, optimization for each region is carried out (or dominance of $P_T(Q)$ in one region over another is established) and the best among these "regional optima" is the global optima.

The problem with this method is the exponential growth of the number of regions with the number of products. This number is 16 for the two-product case, 64 for the three-product case, 256 for the four-product case, and so on. Beyond two-products, it becomes unmanageable. Even for the two-product case, the method is tedious. Further complexity in evaluating $P_T(Q)$ by (10) arises if f is complex.

Special cases

Some simplifications are possible by putting restrictions. If stock-out costs (s_i for $i = 1, 2, \ldots, n$) are assumed to be zero, the number of forms that $P_T(Q)$ can take reduces drastically. Note that our model does not permit s_i to be zero (as that would imply division by zero); however, we can choose sufficiently small values for s_i such that $\overline{d}_i = \min\{b_i, Q_i + pcl_i/s_i\} = b_i$ for $i = 1, 2, \ldots, n$. Geometrically, this means that n vertices are always outside Ω. Then the number of forms that $P_T(Q)$ can assume is 2^n. In this case,

$$P_T(Q) = \int_{\underline{d}_1}^{b_1} \int_{\underline{d}_2(x_1)}^{b_2} \cdots \int_{\underline{d}_n(x_1,\ldots,x_{n-1})}^{b_n} f(x)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \tag{11}$$

Assuming zero stock-out costs and uniform demand, Li et al. (1990) solved the two-product problem using the continuous formulation. They split Ω into 4 different regions and maximized $P_T(Q)$ in each region; best among these regional optima was selected as the global optimum. Approach of A. H.-L. Lau & Lau (1988a) was similar to that of Li et al. (1990).

Instead of assuming zero stock-out costs, if we assume unbounded demand upper limit, i.e., $b_i = \infty$ for $i = 1, 2, \ldots, n$, $\overline{d}_i = \min\{b_i, Q_i + pcl_i/s_i\} = Q_i + pcl_i/s_i$ for $i = 1, 2, \ldots, n$. Geometrically, this means that n vertices are always inside Ω. Then

$$P_T(Q) = \int_{\underline{d}_1}^{Q_1+pcl_1/s_1} \int_{\underline{d}_2(x_1)}^{Q_2+pcl_2(x_1)/s_2} \cdots \int_{\underline{d}_n(x_1,\ldots,x_{n-1})}^{Q_n+pcl_n(x_1,\ldots,x_{n-1})/s_n} f(x)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \tag{12}$$

The number of forms that $P_T(Q)$ can take remains 2^n. However, (12) is more complex than (11) due to the presence of $x_1, x_2, \ldots, x_{i-1}$ in the upper integration limits of the i^{th} integral for $i = 2, 3, \ldots, n$. In addition to the unbounded demand upper limit assumption, if we assume zero stock-out costs, (12) takes a form similar to (11). Then

$$P_T(Q) = \int_{\underline{d}_1}^{\infty} \int_{\underline{d}_2(x_1)}^{\infty} \cdots \int_{\underline{d}_n(x_1,\ldots,x_{n-1})}^{\infty} f(x)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \tag{13}$$

Li et al. (1991) solved the two-product problem with zero stock-out costs and independent exponential product demands ($b_1 = b_2 = \infty$). Ω was split into 4 different regions and the best among the regional optima was selected as the global optimum.

The simplest case arises if we assume that $a_i = -\infty$ for $i = 1, 2, \ldots, n$ in addition to the zero stock-out costs and unbounded demand upper limit assumptions. Then $\underline{d}_i = \max\{a_i, Q_i - pcl_i/(m_i + c_i)\} = Q_i - pcl_i/(m_i + c_i)$ for $i = 1, 2, \ldots, n$ and

$$P_T(Q) = \int_{Q_1-\frac{pcl_1}{m_1+c_1}}^{\infty} \int_{Q_2-\frac{pcl_2(x_1)}{m_2+c_2}}^{\infty} \cdots \int_{Q_n-\frac{pcl_n(x_1,\ldots,x_{n-1})}{m_n+c_n}}^{\infty} f(x)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n. \tag{14}$$

Now, $P_T(Q)$ takes only one form (irrespective of n). Even in this case, the continuous formulation need not be simple. Interested readers can try the two-product problem with normal demand (no truncation) and zero stock-out costs.

The above discussion explains why the continuous modelling of the MPSNP is difficult to solve. This difficulty may be one of the reasons why no progress has happened in this direction after the works of A. H.-L. Lau & Lau (1988a); Li et al. (1990, 1991).

6 Conclusion

Even though satiation objective in the newsboy problem started receiving attention quite early, research progress has been slow, particularly in the multi-product setting. Our understanding is limited to some restricted two-product problems (zero stock-out costs and independent product demands). We study the general MPSNP.

Unlike existing literature, we adapt a discrete formulation of the problem and identify its key properties. We show that satiation probability for an ordering decision, Q is the probability that demand lies in a convex hull around Q. "Size" of the convex hull depends on Q, profit target, and model parameters. This result enables us to develop an algorithm for computing the satiation probability in the general MPSNP. Prior to our work, there were no closed-form expression or computational method for calculating the satiation probability.

We maximized satiation probability by complete enumeration and tested time performance of this method. Computation time for solving the test problems suggests that our method can be adopted for two and three-product independent demand problems and two-product dependent demand problems. For larger problems (more products), our method is not suitable. Even though our optimization method is inefficient, the discrete formulation may still be the preferred way. We have explained why the conventional continuous formulation of the MPSNP is difficult to solve, even for the smaller problems.

Our optimization method is generic in nature, i.e., it works for any demand distribution and any parameter values. It is possible to develop faster algorithms by imposing restrictions on these factors. We already have developed algorithm specific to the independent demand case. Still, a lot more can be done in this direction. Another way to tackle the issue of large computation time requirement is to solve the MPSNP using heuristics. Our optimization method can be used to test heuristic accuracy. Metaheuristics can also be tried.

An alternate approach to solve the MPSNP is to find the distribution function of the total profit (as a function of the ordering decision). Let $G_Q()$ be the distribution. Then Q with the lowest $G_Q(T)$ is the optimum solution as $P_T(Q) = 1 - G_Q(T)$. However, G_Q is difficult to evaluate; to the best of our knowledge, there is no study that finds G_Q for the general MPSNP. Using the central limit theorem, Özler et al. (2009) approximated G_Q by normal distribution for the independent demand case. They found the average approximation error to be less than 1% for 25 or more products. Using this approximation, the MPSNP with independent product demands can be attempted.

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Appendix A

By (3), if $Q_i < a_i$, $\Pi_i(a_i, x_i) - \Pi_i(Q_i, x_i) = (m_i + s_i)(a_i - Q_i) > 0 \; \forall x_i \in \Omega_i$. So $Q_i = a_i$ assures greater profit than any $Q_i < a_i$ in every demand scenario. Similarly, if $Q_i > b_i$, $\Pi_i(b_i, x_i) - \Pi_i(Q_i, x_i) = c_i(Q_i - b_i) > 0 \; \forall x_i \in \Omega_i$. Thus $Q_i = b_i$ is always better than any $Q_i > b_i$

in making profit. These are true for every $i = 1, 2, \ldots, n$.

Let $Q \notin \Omega$, i.e., $Q_i \notin \Omega_i$ for at least one $i = 1, 2, \ldots, n$. Let us construct $Q' \in \Omega$ from Q by replacing the "outliers" with their nearest demand limits. Then $\Pi(Q', x) > \Pi(Q, x)\ \forall x \in \Omega$. Therefore $I_{\Pi(Q,x)\geq T} = 1 \Rightarrow I_{\Pi(Q',x)\geq T} = 1\ \forall x \in \Omega$. However, $I_{\Pi(Q,x)\geq T} = 0$ does not necessarily imply $I_{\Pi(Q',x)\geq T} = 0$. Hence, $P_T(Q) \leq P_T(Q')$. Thus, for every $Q \notin \Omega$, $\exists\ Q' \in \Omega$ such that $P_T(Q) \leq P_T(Q')$. Therefore $Q^* \in \Omega$.

Appendix B

Lemma 1: For any Q_i, the worst profit is realized either when $X_i = a_i$ or when $X_i = b_i$. Thus, $\Pi_i(Q_i)_{min} = \min\{\Pi_i(Q_i, x_i) : x_i \in \Omega_i\} = \min\{\Pi_i(Q_i, a_i), \Pi_i(Q_i, b_i)\}$. By (3), $\Pi_i(Q_i, a_i) = m_i a_i - c_i(Q_i - a_i)$ and $\Pi_i(Q_i, b_i) = m_i Q_i - s_i(b_i - Q_i)$. $\Pi_i(Q_i, a_i)$ is decreasing in Q_i, while $\Pi_i(Q_i, b_i)$ is increasing in Q_i. They are equal at $Q_{0i} = \{(m_i + c_i)a_i + s_i b_i\}/(m_i + c_i + s_i)$. Note that Q_{0i} need not be integer-valued. Thus, $\Pi_i(Q_i)_{min} = \Pi_i(Q_i, b_i)$ if $Q_i \leq \lfloor Q_{0i} \rfloor$, and $\Pi_i(Q_i)_{min} = \Pi_i(Q_i, a_i)$ if $Q_i \geq \lceil Q_{0i} \rceil$.

We can further say that if $Q_i \leq \lfloor Q_{0i} \rfloor$, $\Pi_i(Q_i)_{min} = \Pi_i(Q_i, b_i)$ is increasing in Q_i and if $Q_i \geq \lceil Q_{0i} \rceil$, $\Pi_i(Q_i)_{min} = \Pi_i(Q_i, a_i)$ is decreasing in Q_i. So maximum assured profit for the i^{th} product, $\max\{\Pi_i(Q_i)_{min} : Q_i \in \Omega_i\} = \max\{\Pi_i(\lfloor Q_{0i} \rfloor, b_i), \Pi_i(\lceil Q_{0i} \rceil, a_i)\}$. This is achieved by $\lfloor Q_{0i} \rfloor$ if $\Pi_i(\lfloor Q_{0i} \rfloor, b_i) \geq \Pi_i(\lceil Q_{0i} \rceil, a_i)$ and by $\lceil Q_{0i} \rceil$ if $\Pi_i(\lfloor Q_{0i} \rfloor, b_i) \leq \Pi_i(\lceil Q_{0i} \rceil, a_i)$. These arguments are true for every $i = 1, 2, \ldots, n$.

Any $T \leq \sum_{i=1}^{n} \max\{\Pi_i(\lfloor Q_{0i} \rfloor, b_i), \Pi_i(\lceil Q_{0i} \rceil, a_i)\}$ can be split into n product specific targets such that $T_i \leq \max\{\Pi_i(\lfloor Q_{0i} \rfloor, b_i), \Pi_i(\lceil Q_{0i} \rceil, a_i)\}$ for $i = 1, 2, \ldots, n$. Each T_i can be achieved with certainty by $\lfloor Q_{0i} \rfloor$ or $\lceil Q_{0i} \rceil$ or both. Thus, such T can be achieved with certainty.

Proposition 2: Let $\sum_{i=1}^{n} \max\{\Pi_i(\lfloor Q_{0i} \rfloor, b_i), \Pi_i(\lceil Q_{0i} \rceil, a_i)\} = T_0$. By Lemma 1, any $T \leq T_0$ can be achieved with certainty. Let $T > T_0$ and Q^* is the optimum ordering decision. To establish T_0 as the maximum assured target, we need to show that $P_T(Q^*) < 1\ \forall T > T_0$.

Assuming a contradiction, let $P_T(Q^*) = 1$ for some $T > T_0$. Then $\Pi(Q^*, x) \geq T > T_0\ \forall x \in \Omega$. Following the arguments in the proof of Lemma 1, $T_0 = \sum_{i=1}^{n} \max\{\Pi_i(Q_i)_{min} : Q_i \in \Omega_i\} \geq \sum_{i=1}^{n} \Pi_i(Q_i^*)_{min} = \min\{\Pi(Q^*, x) : x \in \Omega\}$. Hence, $T_0 \geq \Pi(Q^*, x)$ for at least one $x \in \Omega$, which is in contradiction with our assumption. Thus, $\underline{T} = T_0$.

Appendix C

If $PC_T(Q) < 0$, by Lemma 2, $I_{\Pi(Q,x)\geq T} = 0\ \forall x \in \mathbb{R}^n$ as $CD_Q(x)$ is a non-negative quantity. Since $D = \emptyset$, co$D = \emptyset$ too. Then our claim is vacuously true.

If $PC_T(Q) \geq 0$, by Definition 2, $D = \bigcup_{i=1}^{n}\{\underline{x}^{(i)}, \overline{x}^{(i)}\}$. Then D and co$D$ are non-empty. If $PC_T(Q) = 0$, all terminal demand points coincide with Q, i.e., $D = \{Q\}$ and co$D = \{Q\}$. In this case, our claim is that $I_{\Pi(Q,x)\geq T} = 1$ if and only if $x = Q$, which is true. If $x = Q$, $CD_x(Q) = 0 = PC_T(Q)$ and by Lemma 2, $I_{\Pi(Q,x)\geq T} = 1$. If $x \neq Q$, $CD_x(Q) > 0 = PC_T(Q)$

and by Lemma 2, $I_{\Pi(Q,x)\geq T} = 0$. Next, we take the case of positive $PC_T(Q)$.

Let x be an arbitrary member of co$D$. Then $x = \sum_{i=1}^{n} \underline{\alpha}_i \underline{x}^{(i)} + \sum_{i=1}^{n} \overline{\alpha}_i \overline{x}^{(i)}$ for some $\underline{\alpha}_i, \overline{\alpha}_i \geq 0$ for $i = 1, 2, \ldots, n$ such that $\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = 1$. Using Definition 2,

$$x_i = Q_i - \underline{\alpha}_i \frac{PC_T(Q)}{m_i + c_i} + \overline{\alpha}_i \frac{PC_T(Q)}{s_i} = Q_i + \delta_i PC_T(Q) \tag{C.1}$$

for $i = 1, 2, \ldots, n$, where $\delta_i = \overline{\alpha}_i / s_i - \underline{\alpha}_i/(m_i + c_i)$. Now, $CD_Q(x)$ can be expressed as

$$\begin{aligned} CD_Q(x) &= \sum_{i=1}^{n} \left[(m_i + c_i) \max\{0, -\delta_i PC_T(Q)\} + s_i \max\{0, \delta_i PC_T(Q)\}\right] \\ &= PC_T(Q) \sum_{i=1}^{n} \left[s_i \max\{0, \delta_i\} - (m_i + c_i) \min\{0, \delta_i\}\right]. \end{aligned} \tag{C.2}$$

$$\text{If } \delta_i \geq 0,\ s_i \max\{0, \delta_i\} - (m_i + c_i) \min\{0, \delta_i\} = s_i \left(\frac{\overline{\alpha}_i}{s_i} - \frac{\underline{\alpha}_i}{m_i + c_i}\right) \leq \overline{\alpha}_i.$$

$$\text{If } \delta_i \leq 0,\ s_i \max\{0, \delta_i\} - (m_i + c_i) \min\{0, \delta_i\} = -(m_i + c_i) \left(\frac{\overline{\alpha}_i}{s_i} - \frac{\underline{\alpha}_i}{m_i + c_i}\right) \leq \underline{\alpha}_i.$$

$$\Rightarrow \sum_{i=1}^{n} \left[s_i \max\{0, \delta_i\} - (m_i + c_i) \min\{0, \delta_i\}\right] \leq \sum_{i=1}^{n} (\underline{\alpha}_i + \overline{\alpha}_i) = 1.$$

Hence, $CD_Q(x) \leq PC_T(Q)$. Then by Lemma 2, $I_{\Pi(Q,x)\geq T} = 1$. Since this holds for an arbitrary $x \in$ co$D$, $x \in$ co$D \Rightarrow I_{\Pi(Q,x)\geq T} = 1$.

Now, we need to prove the converse, i.e., $I_{\Pi(Q,x)\geq T} = 1 \Rightarrow x \in$ co$D$. Instead, we show that $I_{\Pi(Q,x)\geq T} = 0\ \forall x \notin$ co$D$. This result along with the already established $I_{\Pi(Q,x)\geq T} = 1\ \forall x \in$ co$D$ gives us the desire result. This is accomplished in the following steps.

First, we find a way to characterize the boundary points (relative boundary) of co$D$. Note that co$D$ is compact as D, being a finite set, is compact (Hiriart-Urruty & Lemaréchal, 2004, pg. 31). Hence, the closure of co$D$, cl co$D$ = co$D$. Then the boundary of co$D$ is wholly contained in it. Let us split the members of co$D$, i.e., all possible convex combinations of elements in D into following two disjoint sets.

$$A = \left\{ \sum_{i=1}^{n} (\underline{\alpha}_i \underline{x}^{(i)} + \overline{\alpha}_i \overline{x}^{(i)}) : \text{both } \underline{\alpha}_i \text{ and } \overline{\alpha}_i \text{ are non-zero for at least one } i = 1, 2, \ldots, n \right\}.$$

$$B = \left\{ \sum_{i=1}^{n} (\underline{\alpha}_i \underline{x}^{(i)} + \overline{\alpha}_i \overline{x}^{(i)}) : \text{at least one of } \underline{\alpha}_i \text{ and } \overline{\alpha}_i \text{ is zero for every } i = 1, 2, \ldots, n \right\}.$$

The usual restrictions on $\underline{\alpha}_i, \overline{\alpha}_i$ hold, i.e., $\underline{\alpha}_i, \overline{\alpha}_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = 1$. We show that $A \subseteq$ ri co$D$ (the relative interior of co$D$).

Let x be an arbitrary member of A. Then $x = \sum_{i=1}^{n}(\underline{\alpha}_i \underline{x}^{(i)} + \overline{\alpha}_i \overline{x}^{(i)})$, where $\underline{\alpha}_i, \overline{\alpha}_i \geq 0$ for $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = 1$, and both of $\underline{\alpha}_i, \overline{\alpha}_i$ are non-zero for at least one $i = 1, 2, \ldots, n$.

Without loss of generality, let us assume that $\underline{\alpha}_1, \overline{\alpha}_1 > 0$. Depending on the value of $\underline{\alpha}_1 + \overline{\alpha}_1$, two cases arise: $\underline{\alpha}_1 + \overline{\alpha}_1 < 1$ and $\underline{\alpha}_1 + \overline{\alpha}_1 = 1$. If $\underline{\alpha}_1 + \overline{\alpha}_1 < 1$,

$$\begin{aligned} x &= (\underline{\alpha}_1 + \overline{\alpha}_1)\left[\frac{\underline{\alpha}_1 \underline{x}^{(1)}}{\underline{\alpha}_1 + \overline{\alpha}_1} + \frac{\overline{\alpha}_1 \overline{x}^{(1)}}{\underline{\alpha}_1 + \overline{\alpha}_1}\right] + (1 - \underline{\alpha}_1 - \overline{\alpha}_1)\sum_{i=2}^{n}\left[\frac{\underline{\alpha}_i \underline{x}^{(i)}}{1 - \underline{\alpha}_1 - \overline{\alpha}_1} + \frac{\overline{\alpha}_i \overline{x}^{(i)}}{1 - \underline{\alpha}_1 - \overline{\alpha}_1}\right] \\ &= (\underline{\alpha}_1 + \overline{\alpha}_1) y + (1 - \underline{\alpha}_1 - \overline{\alpha}_1) z \quad \text{(say)}. \end{aligned}$$

Clearly, y and z, being convex combinations of elements in D, are in co$D$. Furthermore, y lies in the interior of the line segment connecting $\underline{x}^{(1)}$ and $\overline{x}^{(1)}$ as $\underline{\alpha}_1/(\underline{\alpha}_1 + \overline{\alpha}_1) \in (0,1)$ and $\overline{\alpha}_1/(\underline{\alpha}_1 + \overline{\alpha}_1) \in (0,1)$. If $\underline{\alpha}_1 + \overline{\alpha}_1 = 1$, y remains in the interior of the line segment connecting $\underline{x}^{(1)}$ and $\overline{x}^{(1)}$, while z vanishes, i.e., $x = y$ (as $1 - \underline{\alpha}_1 - \overline{\alpha}_1 = 0$ then).

D consists of $2n$ number of distinct points in $\mathbb{R}^n$. If we consider Q as the origin, $\underline{x}^{(i)}$ and $\overline{x}^{(i)}$ lie on the i^{th} axis; $\underline{x}^{(i)}$ lies on the negative-side and $\overline{x}^{(i)}$ lies on the positive-side. Then Q is in the interior of co$D$[3]. Furthermore, $Q = \{(m_1+c_1)/(m_1+c_1+s_1)\}\underline{x}^{(1)} + \{s_1/(m_1+c_1+s_1)\}\overline{x}^{(1)}$ lies in the interior of the line segment connecting $\underline{x}^{(1)}$ and $\overline{x}^{(1)}$.

Since y and Q lie in the interior of the line segment connecting $\underline{x}^{(1)}$ and $\overline{x}^{(1)}$, either $y = \beta Q + (1-\beta)\underline{x}^{(1)}$ or $y = \beta Q + (1-\beta)\overline{x}^{(1)}$ for some $\beta \in (0,1]$. So y lies in the half-open line segment connecting $Q \in$ ri co$D$ (closed in this end) and $\underline{x}^{(1)} \in$ cl co$D$ or $\overline{x}^{(1)} \in$ cl co$D$. Hence, $y \in$ ri co$D$ (Hiriart-Urruty & Lemaréchal, 2004, pg. 35). Extending the same logic, if $\underline{\alpha}_1 + \overline{\alpha}_1 < 1$, since $x = \gamma y + (1-\gamma)z$, where $\gamma = \underline{\alpha}_1 + \overline{\alpha}_1 \in (0,1)$, lies in the open line segment connecting $y \in$ ri co$D$ and $z \in$ cl co$D$, $x \in$ ri co$D$. If $\underline{\alpha}_1 + \overline{\alpha}_1 = 1$, $x = y \in$ ri co$D$.

Since an arbitrary $x \in A$ is in ri co$D$, $A \subseteq$ ri co$D$. Then cl co$D -$ ri co$D \subseteq$ cl co$D - A = B$, i.e., B contains all the boundary points of co$D$. Hence, a boundary point of co$D$ can be represented as convex combination of points in D such that at least one of the coefficients corresponding to $\underline{x}^{(i)}$ and $\overline{x}^{(i)}$ is zero for every $i = 1, 2, \ldots, n$.

Now, we show that $CD_Q(x) = PC_T(Q)$ if x is a boundary point of co$D$. It is sufficient to show that $CD_Q(x_B) = PC_T(Q)$ for an arbitrary $x_B \in B$. Let $x_B = \sum_{i=1}^{n}(\underline{\alpha}_i \underline{x}^{(i)} + \overline{\alpha}_i \overline{x}^{(i)})$, where $\underline{\alpha}_i, \overline{\alpha}_i \geq 0$ for $i = 1, 2, \ldots, n$, $\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = 1$, and at least one of $\underline{\alpha}_i, \overline{\alpha}_i$ is zero for every $i = 1, 2, \ldots, n$. Using (C.2), $CD_Q(x_B)$ can be expressed as

$$CD_Q(x_B) = PC_T(Q)\sum_{i=1}^{n}\left[s_i \max\{0, \delta_i\} - (m_i + c_i)\min\{0, \delta_i\}\right],$$

where $\delta_i = \overline{\alpha}_i/s_i - \underline{\alpha}_i/(m_i + c_i)$ for $i = 1, 2, \ldots, n$.

$$\text{If } \underline{\alpha}_i = 0,\ \delta_i = \frac{\overline{\alpha}_i}{s_i} \geq 0 \ \Rightarrow s_i \max\{0, \delta_i\} - (m_i + c_i)\min\{0, \delta_i\} = \overline{\alpha}_i = \overline{\alpha}_i + \underline{\alpha}_i.$$

$$\text{If } \overline{\alpha}_i = 0,\ \delta_i = \frac{-\underline{\alpha}_i}{m_i + c_i} \leq 0 \ \Rightarrow s_i \max\{0, \delta_i\} - (m_i + c_i)\min\{0, \delta_i\} = \underline{\alpha}_i = \underline{\alpha}_i + \overline{\alpha}_i.$$

[3]A formal proof is possible by showing that no hyperplane passing through Q supports co$D$. Then Q can not be a boundary point of co$D$ (Hiriart-Urruty & Lemaréchal, 2004, pg. 54). Since $Q \in$ co$D$, $Q \in$ ri co$D$.

$$\Rightarrow CD_Q(x_B) = PC(Q)\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = PC_T(Q).$$

Now, we show that $CD_Q(x) > PC_T(Q)$ for an arbitrary $x \notin \text{co}D$. Since $Q \in \text{ri co}D$, there exist a boundary point of $\text{co}D$ (say, x_B) that lies in the interior of the line segment connecting x and Q. So $x_B = \lambda x + (1-\lambda)Q$ for some $\lambda \in (0,1)$. Let $x_B = \sum_{i=1}^{n}(\underline{\alpha}_i \underline{x}^{(i)} + \overline{\alpha}_i \overline{x}^{(i)})$, where $\underline{\alpha}_i, \overline{\alpha}_i \geq 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = 1$. By (C.1), for $i = 1, 2, \ldots, n$,

$$x_{Bi} = Q_i + \delta_i PC_T(Q), \quad \text{where } \delta_i = \frac{\overline{\alpha}_i}{s_i} - \frac{\underline{\alpha}_i}{m_i + c_i}.$$
$$\Rightarrow x_i = \frac{1}{\lambda}x_{Bi} - \left(\frac{1}{\lambda} - 1\right)Q_i = Q_i + \delta_i \frac{PC_T(Q)}{\lambda}.$$

Now, $CD_Q(x)$ can be expressed as

$$\begin{aligned} CD_Q(x) &= \sum_{i=1}^{n}\left[(m_i + c_i)\max\left\{0, -\delta_i \frac{PC_T(Q)}{\lambda}\right\} + s_i \max\left\{0, \delta_i \frac{PC_T(Q)}{\lambda}\right\}\right] \\ &= \frac{PC_T(Q)}{\lambda}\sum_{i=1}^{n}\left[s_i \max\{0, \delta_i\} - (m_i + c_i)\min\{0, \delta_i\}\right]. \end{aligned}$$

Since x_B is a boundary point of $\text{co}D$, at least one of $\underline{\alpha}_i, \overline{\alpha}_i$ is zero for each $i = 1, 2, \ldots, n$. Then $\max\{0, \delta_i\} - (m_i + c_i)\min\{0, \delta_i\} = (\underline{\alpha}_i + \overline{\alpha}_i)$ for $i = 1, 2, \ldots, n$.

$$\Rightarrow CD_Q(x) = \frac{PC_T(Q)}{\lambda}\sum_{i=1}^{n}(\underline{\alpha}_i + \overline{\alpha}_i) = \frac{PC_T(Q)}{\lambda} > PC_T(Q).$$

By Lemma 2, $I_{\Pi(Q,x) \geq T} = 0$ as $CD_Q(x) > PC_T(Q)$. Since x is an arbitrary point not in $\text{co}D$, $I_{\Pi(Q,x) \geq T} = 0 \ \forall x \notin \text{co}D$. This completes the proof.

Appendix D

If $PC_T(Q) < 0$, both $\text{co}D$ and $\mathbb{S}$ are empty. $\mathbb{S}$ is empty because $\underline{d}_i > \overline{d}_i$ (i.e., $[\underline{d}_i, \overline{d}_i] = \emptyset$) as $pcl_i < 0$ for $i = 1, 2, \ldots, n$. Then our claim is vacuously true. Next, we consider the case of $PC_T(Q) \geq 0$. We can rewrite $\mathbb{S}$ (Equation 5) as

$$\begin{aligned} \mathbb{S} &= \left\{x : x \in \mathbb{N}_0^n,\ a_i \leq x_i \leq b_i,\ \underline{d}_{0i} \leq x_i \leq \overline{d}_{0i} \text{ for } i = 1, 2, \ldots, n\right\}, \\ &\quad \text{where } \underline{d}_{0i} = Q_i - \frac{pcl_i(x_1, \ldots, x_{i-1})}{m_i + c_i} \text{ and } \overline{d}_{0i} = Q_i + \frac{pcl_i(x_1, \ldots, x_{i-1})}{s_i} \quad \text{for } i = 1, 2, \ldots, n. \\ &= \{x : x \in \mathbb{N}_0^n,\ a_i \leq x_i \leq b_i\ \forall i = 1, 2, \ldots, n\} \cap \{x : x \in \mathbb{R}^n,\ \underline{d}_{0i} \leq x_i \leq \overline{d}_{0i}\ \forall i = 1, 2, \ldots, n\} \\ &= \Omega \cap \mathbb{S}_0, \quad \text{where } \mathbb{S}_0 = \{x : x \in \mathbb{R}^n,\ \underline{d}_{0i} \leq x_i \leq \overline{d}_{0i} \text{ for } i = 1, 2, \ldots, n\}. \end{aligned}$$

Now, it is sufficient to show that $\mathbb{S}_0 = \text{co}D$ to establish $\mathbb{S} = \text{co}D \cap \Omega$.

Let us consider an arbitrary $x \in \text{co}D$. If $x \notin \mathbb{S}_0$, $x_i \notin [\underline{d}_{0i}, \overline{d}_{0i}]$ for one or more $i = 1, 2, \ldots, n$. If $x_i < \underline{d}_{0i} = Q_i - pcl_i/(m_i + c_i)$, $pcl_i < (m_i + c_i)(Q_i - x_i) = CD_{Q_i}(x_i)$. Similarly, if $x_i > \overline{d}_{0i} = Q_i + pcl_i/s_i$, $pcl_i < s_i(x_i - Q_i) = CD_{Q_i}(x_i)$. In both cases, $pcl_i = PC_T(Q) - \sum_{j=1}^{i-1} CD_{Q_j}(x_j) < CD_{Q_i}(x_i) \Rightarrow PC_T(Q) < \sum_{j=1}^{i} CD_{Q_j}(x_j) \leq CD_Q(x)$. Then by Lemma 2, $I_{\Pi(Q,x) \geq T} = 0$, which is impossible as $x \in \text{co}D$. Hence, $x \in \mathbb{S}_0$. Since the choice of x is arbitrary, $\text{co}D \subseteq \mathbb{S}_0$.

Now, let us consider an arbitrary $x \in \mathbb{S}_0$. Then $\underline{d}_{0n} \leq x_n \leq \overline{d}_{0n}$, i.e., $Q_n - pcl_n/(m_n + c_n) \leq x_n \leq Q_n + pcl_n/s_n$. Then $Q_n - x_n \leq pcl_n/(m_n + c_n)$ and $x_n - Q_n \leq pcl_n/s_n$. Hence, $CD_{Q_n}(x_n) \leq pcl_n = PC_T(Q) - \sum_{j=1}^{n-1} CD_{Q_j}(x_j) \Rightarrow CD_Q(x) \leq PC_T(Q)$. By Lemma 2, $I_{\Pi(Q,x) \geq T} = 1$. Then by Theorem 1, $x \in \text{co}D$. Since the choice of x is arbitrary, $\mathbb{S}_0 \subseteq \text{co}D$. We have already established that $\text{co}D \subseteq \mathbb{S}_0$. Hence, $\mathbb{S}_0 = \text{co}D$.

In the last part of the proof, we do not explicitly use the fact that $\underline{d}_{0i} \leq x_i \leq \overline{d}_{0i}$ for $i = 1, 2, \ldots, n-1$. If one or more of these is/are violated, i.e., $x_j \notin [\underline{d}_{0j}, \overline{d}_{0j}]$ for some $j = 1, 2, \ldots, n-1$, $PC_T(Q) < \sum_{i=1}^{j} CD_{Q_i}(x_i)$. Then $pcl_{j+1} < 0 \Rightarrow \underline{d}_{0j+1} > \overline{d}_{0j+1} \Rightarrow \mathbb{S}_0 = \emptyset$, a contradiction to the non-emptiness assumption of $\mathbb{S}_0$.

Appendix E

An MPSNP for given n is defined by cost parameters, demand parameters, and profit target. Here, we describe which values of these factors we chose to create a pool of versatile problems and how did we select test problems from this pool.

Problem construction

Cost parameters: Critical fractile, $cf = (m+s)/(m+c+s) = 1 - c/(m+c+s)$ has an important role in the profit maximizing (or cost minimizing) newsboy problem. Optimum order quantity in such problems is given by $F(Q^*) = cf$ (Silver et al., 1998, chap. 10), where F is the demand distribution function. Note that $cf \in (0, 1)$. We choose cost parameters such that cf takes low (0.3), medium (0.5), and high (0.7) values. Two sets of parameters are chosen for each cf value (one with $m < s$ and the other with $m > s$). Table 2 shows these cost profiles.

Table 2: Cost profiles

	Profile number					
Parameters	*1*	*2*	*3*	*4*	*5*	*6*
m	2	1	3	2	4	3
c	7	7	5	5	3	3
s	1	2	2	3	3	4
cf	0.3	0.3	0.5	0.5	0.7	0.7

Demand parameters: We consider three demand ranges: i) low ($a = 0, b = 100$), ii) medium ($a = 300, b = 500$), and iii) high ($a = 1000, b = 1500$). Demand distributions are different for independent and dependent demand cases.

Independent demand: We assume uniform distribution (UD) in $[a, b]$ for low demand range. For medium demand range, we assume triangular distribution (TD) in $[a, b]$ with $(1-\theta)a + \theta b$ as the mode ($0 \leq \theta \leq 1$). Finally, for high demand range, we assume truncated normal distribution (TND) in $[a, b]$ with $(1-\theta)a + \theta b$ as the mode ($0 \leq \theta \leq 1$) and $(b-a)/10$ as the standard deviation, σ. We vary location of the mode (θ) for TD and TND so that we get left-skewed ($\theta = 0.3$), symmetric ($\theta = 0.5$), and right-skewed ($\theta = 0.7$) distributions. Note that UD is always symmetric. This way, we get seven demand profiles (see Table 3(a)). Since our model is discrete, we take $p(x) = F(x+0.5) - F(x-0.5)\ \forall x \in \{a, a+1, \ldots, b\}$.

Dependent demand: We assume truncated multivariate normal distribution (TMVND). Here, we can vary location of the mode for all demand ranges. This gives us nine demand profiles (see Table 3(b)). We construct the covariance matrix as: $\Sigma = sd \cdot R \cdot sd$, where $sd_{n\times n}$ is the diagonal matrix of standard deviations and $R_{n\times n}$ is the correlation matrix. We take $\sigma = (b-a)/10$ and generate R randomly. Ideally, we should and take $p(x) = F_{tmvn}(x+0.5) - F_{tmvn}(x-0.5)\ \forall x \in \Omega$, where F_{tmvn} is the distribution function of TMVND. Since F_{tmvn} does not have a closed form expression, calculation of $p(x)\ \forall x \in \Omega$ is time consuming, particularly for large n. Thus, we take $p(x) = f_{mvn}(x)/tf\ \forall x \in \Omega$, where f_{mvn} is the density function of the underlying MVND and $tf = \sum_{x\in\Omega} f_{mvn}(x)$ is the truncation factor. f_{mvn} has closed form expression. With this approximation, we may deviate significantly from TMVND. However, our objective (the study of the dependent demand case) is unaffected by this approximation error.

Table 3: Demand profiles

(a) Independent demand

	Profile number						
Parameters	*1*	*2*	*3*	*4*	*5*	*6*	*7*
a	0	300	300	300	1000	1000	1000
b	100	500	500	500	1500	1500	1500
Dist.	UD	TD	TD	TD	TND	TND	TND
Mode	–	360	400	440	1150	1250	1350

(b) Dependent demand

	Profile number								
Parameters	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
a	0	0	0	300	300	300	1000	1000	1000
b	100	100	100	500	500	500	1500	1500	1500
Mode	30	50	70	360	400	440	1150	1250	1350

Profit target: If cost and demand parameters of a problem are known, we can calculate $\underline{T}$ and $\overline{T}$. If $T \notin (\underline{T}, \overline{T})$, optimum solution to such problem is trivial. A non-trivial profit target can be expressed as: $T = (1-t)\underline{T} + t\overline{T}$, $t \in (0, 1)$. Since $\underline{T}$ can be negative, we take $T = (1-t)\max\{0, \underline{T}\} + t\overline{T}$, $t \in (0, 1)$ to ensure positivity of T. We consider three levels for profit target: i) low ($t = 0.3$), ii) medium ($t = 0.5$), and iii) high ($t = 0.7$). We use profile number *1, 2, 3* to represent these low, medium, and high target levels.

Problem selection

Product profiles: Since cost and demand profiles are independent of each other, we have $n_{pp} = 6 \times 7 = 42$ product profiles for the independent demand case and $n_{pp} = 6 \times 9 = 54$ product profiles for the dependent demand case. Table 4 shows the product profiles and corresponding cost and demand profiles.

Table 4: Product profiles

Cost profiles	Demand profiles								
	1	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*
1	**1**	**7**	**13**	**19**	**25**	**31**	**37**	**43**	**49**
2	**2**	**8**	**14**	**20**	**26**	**32**	**38**	**44**	**50**
3	**3**	**9**	**15**	**21**	**27**	**33**	**39**	**45**	**51**
4	**4**	**10**	**16**	**22**	**28**	**34**	**40**	**46**	**52**
5	**5**	**11**	**17**	**23**	**29**	**35**	**41**	**47**	**53**
6	**6**	**12**	**18**	**24**	**30**	**36**	**42**	**48**	**54**

Using the target and product profiles, an n-product problem can be uniquely specified by a vector of $(n + 1)$ elements. The first element is an integer between 1 and 3; it represents the profit target level. Remaining n elements are integers between 1 and n_{pp}. These n integers represent the product profiles.

Random selection: With 3 target profiles and n_{pp} product profiles, we can construct $n_{tot} = 3 \times \binom{n_{pp}}{n}$ number of n-product problems such that every product in a problem is distinct. We do not allow two products to be identical to ensure versatility. We assign unique serial numbers $(1, 2, \ldots, n_{tot})$ to these problems. *This assignment is done randomly.* If we wish to solve n_{tp} number of test problems, we select problems corresponding to $1, 2, \ldots, n_{tp}$.

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