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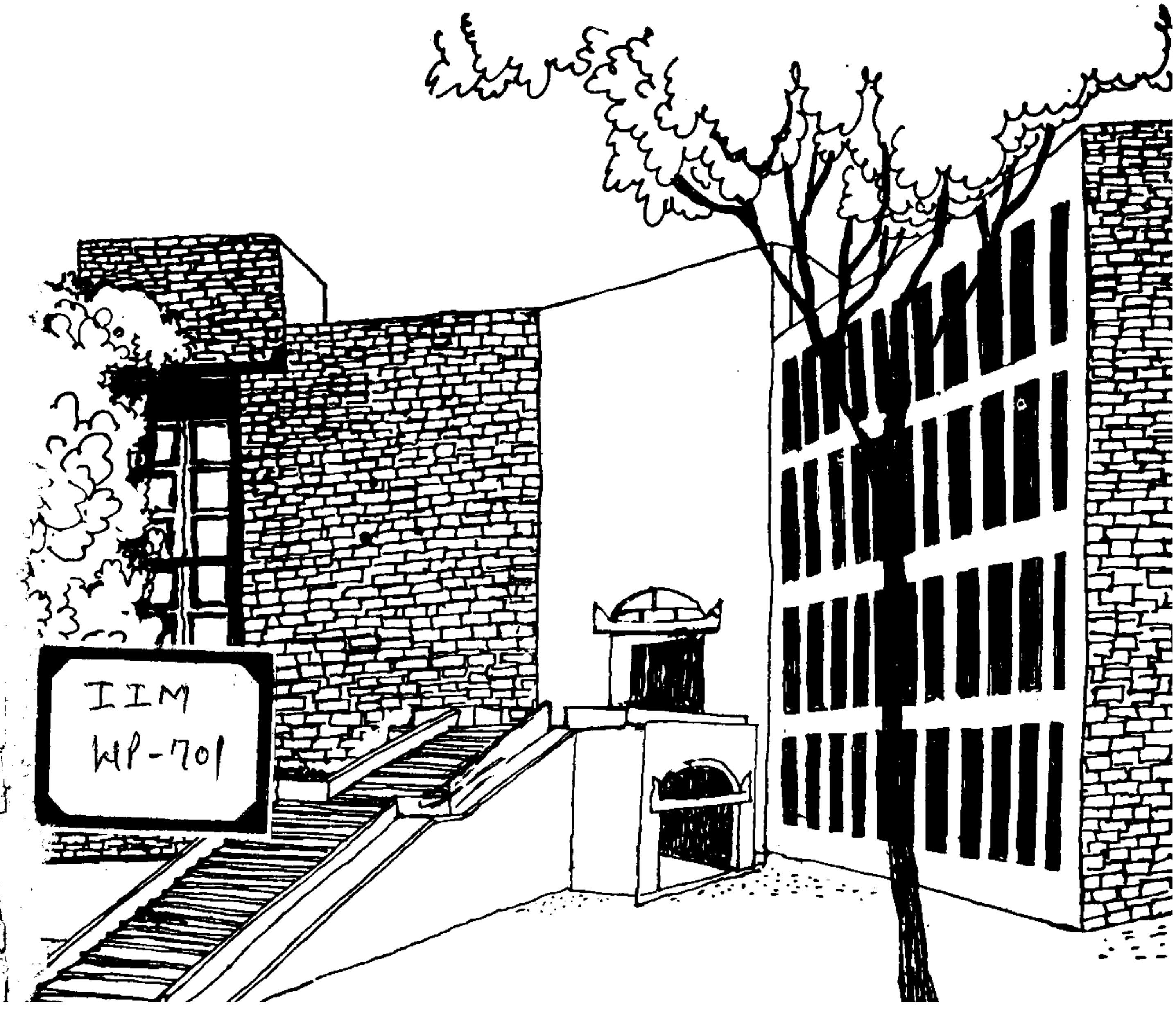

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OPTIMAL MAINTENANCE AND LIFE OF MACHINES, SUBJECT TO TIME VARYING RATE OF DEPRECIATION

By

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ABSTRACT

In this paper we study the optimal maintenance policy and life of machines with time varying decay rate of quality. Many management problems involve the determination of the life or horizon of an activity in combination with the management of it. We study a particular class of such problems, and obtain solutions for optimal maintenance policies and life of machines.

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1. Introduction:-

Many management problems require the determination of the life or horizon of an activity in combination with the management of it. An example of this is the maintenance of a machine. If expenses for maintenance are high, then it is likely that the machine will function better and therefore last longer. If we know that it is going to be kept for a long time, it is necessary to spend large amounts on maintenance.

In the following analysis we shall consider a machine maintenance model, with time varying depreciation rate of machinery to solve the problem of determining both the best maintenance policy and the optimal life of machines. We shall first review the history of the theory of optimal replacement decisions, before we present our analysis.

2. Previous work on the replacement decision:-

The analysis of optimal replacement policy was first directed by Taylor(1923) towards determining the service life in such a fashion that unit cost of production was minimized. Hotelling(1925) modified this view by maximizing the present value of the machine's output minus its costs. Thus, T is determined to maximize

$$J(T) = \int_0^T \Psi(t)\exp(-it)dt+S(T)\exp(-iT), \qquad (2.1)$$

where J(T) = the present value of the machine, T = the service life, $\Psi(t)$ = net operating receipts at time t, i = the rate of interest, and S(T) = the salvage value at time T.

Preinrich(1940) extended eq.(2.1) by including the chain of future replacements. That is, he maximized the expression

$$J(T) = \sum_{k=0}^{\infty} \exp(-ikT)\left(\int_0^T \Psi(t)\exp(-it)dt+S(T)\exp(-iT)-I\right), (2.2)$$

where I represents the cost of each new machine in the replacement chain.

The problem of obsolescence has been dealt with by several authors in connection with the problem of optimal service life (see, for example, Bain(1939), Bellman(1955), Liljeblad(1936), and Terborgh (1949)). The best known of these formulations is the one due to

Terborgh(1949) which assumes that an old machine will become inferior compared with modern equipment at a constant rate called the inferiority gradient. The effect of taxes on the replacement decision is discussed by Johansson(1961). For an overall review of replacement see Johansson(1961, p.24) and Smith(1957,1961).

There is often a problem in distinguishing between initial investment and repair cost. By repairing and extending old equipment, nothing may be left of the initial investment, but it is not possible to identify a single time point at which replacement takes place. When a complete replacement does take place, one frequently finds (as was mentioned before) that it is influenced by the repair undertaken. On the other hand, how much one should repair a machine partly depends upon how long it is going to be used. The simultaneous optimal determination of these two factors, repair and service life, has been discussed in Masse'(1962). The importance of this problem for the investment decision after tax has been stressed by Johansson(1961, p.133).

The model we study is a generalization of the model presented by Kamien and Schwartz(1981). Their model has been developed along the lines suggested by Thompson(1968).

Maintenance expenditures are assumed to be subject to the constraints

$$0 \leq v(t) \leq \bar{v}, \tag{3.4}$$

where $\bar{v}$ is a constant upper limit on maintenance expenditures.

3.2 Optimization for a fixed time T

Let us first find the optimal maintenance policy for a fixed value of the resale time. This amounts to maximizing

$$\tilde{J}_T(v(\cdot)) = + \int_0^T \left[\Psi(x(t)) - v(t) \right] \exp(-it)\,dt + S(x(T))\exp(-iT) \tag{3.5}$$

subject to $\dot{x} + \alpha(t)x = v(t)$

$x(o) = x_o.$

The Hamiltonian is

$$H = \left[\Psi(x) - v \right] + \lambda \left[v - \alpha(t)x \right] = \left[\Psi(x) - \lambda\alpha(t)x \right] + v(-1 + \lambda(t))$$

The necessary conditions include (3.6)

$$\lambda' = \hat{u} - \frac{\partial H}{\partial x} = -\Psi'(x) + \lambda\alpha(t) + i\lambda \tag{3.7}$$

$$v = \begin{Bmatrix} 0 \\ ? \\ \bar{v} \end{Bmatrix} \text{ whenever } -1 + \lambda(t) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \tag{3.8}$$

$$\lambda(T) = + S'[x(T)] \tag{3.9}$$

The necessary and sufficient conditions are (sufficiency as always comes from the convexity of the problem)

$$\dot{x} + \alpha(t)x = \dot{v} \tag{3.12}$$

$$x(o) = x_0$$

$$-\lambda' + (\alpha(t) + i)\lambda - \Phi'(x) = 0 \tag{3.8a}$$

$$\lambda(T) = + S'[x(T)]$$

$$v(t) = \begin{Bmatrix} 0 \\ \text{undetermined} \\ \bar{v} \end{Bmatrix} \quad \text{if} \quad \lambda(t) \begin{Bmatrix} < \\ = \\ > \end{Bmatrix} 1 \tag{3.9a}$$

The best maintenance policy is 'bang-bang'. In order to discuss more explicitly how the above system of equations can be solved, we make some further assumptions about the parameters in the next section.

Solution in a particular case:-

Let us assume that

$$\Phi(x) = \Phi x \tag{3.10}$$

and

$$S(x) = Kx \tag{3.11}$$

where K and Φ arecconstants, $K \leq 1$.

The constant Φ measures how a unit of quality gets transformed into operating receipts per unit of time.

The adjoint equation becomes

$$-\lambda' + (\alpha(t) + i)\lambda - \Phi = 0 \tag{3.12}$$

$$\lambda(T) = +K \tag{3.13}$$

The solution of which is given by the following:

$$-e^{-\int_0^t(\alpha(s)+i)\,ds}\,\dot{\lambda}(t)+e^{-\int_0^t(\alpha(s)+i)\,ds}\,(\alpha(t)+i)\,\lambda(t)-e^{-\int_0^t(\alpha(s)+i)\,ds}\,\bar{I}=0 \quad (3.14)$$

or

$$\frac{d}{dt}\left[-e^{-\int_0^t(\alpha(s)+i)\,ds}\,\lambda(t)\right]=\bar{I}\,e^{-\int_0^t(\alpha(s)+i)\,ds} \quad (3.15)$$

or

$$-e^{-\int_0^t(\alpha(s)+i)\,ds}\,\lambda(t)+\lambda(o)=\bar{I}\int_0^t e^{-\int_0^T(\alpha(s)+i)\,ds}\,dT$$

or

$$-\exp.\left[-\int_0^t(\alpha(s)+i)\,ds\right]\lambda(t)+\lambda(o)=\bar{I}\int_0^t \exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right]dT \quad (3.16)$$

Now $\lambda(T)=+K$ implies

$$-K\exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right]+\lambda(o)=\bar{I}\int_0^T \exp.\left[-\int_0^\tau(\alpha(s)+i)\,ds\right]d\tau \quad (3.17)$$

$$\therefore \lambda(o)=\bar{I}\int_0^T \exp.\left[-\int_0^\tau(\alpha(s)+i)\,ds\right]d\tau+K\exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right] \quad (3.18)$$

$$\therefore -\exp.\left[-\int_0^t(\alpha(s)+i)\,ds\right]\lambda(t)+\bar{I}\int_0^T \exp.\left[-\int_0^\tau(\alpha(s)+i)\,ds\right]d\tau$$

$$+K\exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right]=\bar{I}\int_0^t \exp.\left[-\int_0^\tau(\alpha(s)+i)\,ds\right]d\tau$$

$$\therefore\ \lambda(t) = -\exp.\left[+\int_0^t (\alpha(s)+i)ds\right]\left\{-K\exp.\left[-\int_0^T (\alpha(s)+i)ds\right] - I\int_0^T \exp.\left[-\int_0^\tau (\alpha(s)+i)ds\right]d\tau + I\int_0^t \exp.\left[-\int_0^T (\alpha(s)+i)ds\right]d\tau\right. \quad (3.19)$$

$\lambda(o) > 1$ implies and is implied by

$$I\int_0^T \exp.\left[-\int_0^\tau (\alpha(s)+i)ds\right]d\tau + K\exp.\left[-\int_0^T (\alpha(s)+i)ds\right] > 1$$

$\lambda(o) < 1$ implies and is implied by

$$I\int_0^T \exp.\left[-\int_0^\tau (\alpha(s)+i)ds\right]d\tau + K\exp.\left[-\int_0^T (\alpha(s)+i)ds\right] < 1.$$

It then follows that the optimal policy is given by:

if $I\int_0^T \exp\left[-\int_0^\tau (\alpha(s)+i)ds\right]d\tau + K\exp.\left[-\int_0^T (\alpha(s)+i)ds\right] < 1$

then $v(t) = 0,\ 0 \leq t \leq T$

If $I\int_0^T \exp.\left[-\int_0^\tau (\alpha(s)+i)ds\right]d\tau + K\exp.\left[-\int_0^T (\alpha(s)+i)ds\right] > 1$

then $v(t) = \begin{cases} \bar{v}, & 0 \leq t \leq \theta \\ 0, & \theta < t \leq T \end{cases}$

where $\lambda(\theta) = 1$.

The value of θ is defined by

$$1 = \exp.\left[\int_0^{\theta}(\alpha(s)+i)ds\right]\left\{K \exp.\left[-\int_0^{T}(\alpha(s)+i)ds\right] + \Phi\int_0^{T}\exp.\left[-\int_0^{\tau}(\alpha(s)+i)ds\right] d\tau\right\}$$

$$1 = \exp\left[\int_0^{\theta}(\alpha(s)+i)ds\right]\left\{\Phi\int_{\theta}^{T}\exp.\left[-\int_0^{\tau}(\alpha(s)+i)ds\right] d\tau + K\exp.\left[-\int_0^{T}(\alpha(s)+i)ds\right]\right\}$$

$$0 = \int_0^{\theta}(\alpha(s)+i)ds + \log\left\{\Phi\int_0^{T}\exp.\left[-\int_0^{\tau}(\alpha(s)+i)ds\right] d\tau + K\exp.\left[-\int_0^{T}(\alpha(s)+i)ds\right]\right\} \quad (3.20)$$

3.4 Optimal lifetime in the particular case:-

In addition to the previous necessary conditions, we now need to impose the following constraint to choose an optimum value of T:

$$\Phi(x(T))-V(T)+\lambda(T)\left[V(T)-\alpha(T)x(T)\right] = 0. \quad (3.21)$$

(See Kamien and Schwarz(1981)).

In our particular case, this reduces to

$$\Phi\, x(T) - V(T) + K\left[V(T) - \alpha(T)\, x(T)\right] = 0. \quad (3.22)$$

Let us suppose first that T satisfies

$$\Phi\int_0^{T}\exp\left[-\int_0^{\tau}(\alpha(s)+i)ds\right] d\tau + K\exp.\left[-\int_0^{T}(\alpha(s)+i)ds\right] > 1$$

i.e. a switch does occur at an intermediate point.

$\therefore \quad V(T) = 0.$

$\therefore\ \Psi x(T) - K\alpha(T)\,x(T) = 0.$

$\therefore\ x(T) = 0$ or $\Psi - K\alpha(T) = 0.$

Assume, $\Psi - K\alpha(t) \neq 0\ \forall\ t.$

Now,

$$\dot{x} + \alpha(t)x = v(t)$$

implies

$$e^{\int_0^t \alpha(s)ds} \quad \dot{x} + \alpha(t)e^{\int_0^t \alpha(s)ds} \quad x = e^{\int_0^t \alpha(s)ds} v(t)$$

$$\text{or} \quad \frac{d}{dt}\left[e^{\int_0^t \alpha(s)ds} x(t) \right] = e^{\int_0^t \alpha(s)ds} v(t)$$

$$\text{or} \quad e^{\int_0^t \alpha(s)ds} x(t) - x_0 = \int_0^t e^{\int_0^\tau \alpha(s)ds} v(\tau)d\tau$$

$$\text{or} \quad x(t) = e^{-\int_0^t \alpha(s)ds} \left[x_0 + \int_0^t e^{\int_0^\tau \alpha(s)ds} v(\tau)d\tau \right]$$

So, the life of the machine is infinite.

If however, there exists T such that $\Psi - K\alpha(T) = 0$, then this T gives the optimal life of the machine.

$$\text{If } \Psi\int_0^T \exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right]d\tau + K\exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right] < 1,$$

then $v(t) = \bar{v},\ 0 \leq t \leq T.$

$$\therefore\ \Psi x(T) - \bar{V} + K\left[\bar{V} - \alpha(T)\,x(T)\right] = 0.$$

$$\text{or}\quad X(T)\left[\Psi - \alpha(T)\right] = \bar{V}\left[1-K\right].$$

$$\text{or}\quad X(T) = \frac{\bar{V}\left[1-K\right]}{\Psi - \alpha(T)}.$$

The optimal life of the machine is given by the solution of the equation

$$\frac{\bar{V}\left[1-K\right]}{\Psi - \alpha(T)} = \exp.\left[-\int_0^T\alpha(s)\,ds\right]\left[x_0 + \bar{V}\int_0^T\exp.\left[\int_0^\tau\alpha(s)\,ds\right]d\tau\right]$$

We can now state the following theorem:

<u>Theorem</u>:- Given the assumptions in Section (3.3) if T is the optimal life of a machine then T satisfies:

$$\text{either (a)}\ \Psi\int_0^T\exp.\left[-\int_0^\tau(\alpha(s)+i)\,ds\right]d\tau + K\exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right] > 1$$

$$\text{and}\quad \Psi - K\alpha(T) = 0$$

$$\text{or (b)}\ \Psi\int_0^T\exp.\left[-\int_0^\tau(\alpha(s)+i)\,ds\right]d\tau + K\exp.\left[-\int_0^T(\alpha(s)+i)\,ds\right] < 1$$

and

$$\frac{\bar{v}\,[1-K]}{\Psi-\alpha(T)} = \exp.\left[-\int_0^T \alpha(s)\,ds\right]\left[x_0+\bar{v}\int_0^T \exp.\left[\int_0^\tau \alpha(s)\,ds\right]d\tau\right]$$

$$\text{If } \Psi\int_0^\infty \exp.\left[-\int_0^\tau (\alpha(s)+i)\,ds\right]d\tau + K\exp.\left[-\int_0^\infty (\alpha(s)+i)\,ds\right] > 1$$

and $\Psi \neq K\alpha(t)$ for all $t \geq 0$, then the machine has infinite life.

4. Conclusion: In this paper we have given an introduction to the determination of optimal management and life (horizon) of a type of activity which recurs in practice and involves time varying depreciation. Closed form solutions could not be obtained. However, approximations to optimal solutions can be retrieved by methods of numerical integration.

References:-

1. Bain, J.(1939), "The Relation of the Economic Life of Equipment to Reinvestment Cycles', The Review of Economics and Statistics, 21, No:2, 79-88.

2. Bellman, R.(1955), "Equipment Replacement Policy", Journal of the Society for Industrial and Applied Mathematics, 3, No:3, 133-6.

3. Hotelling, H.(1925), "A General Mathematical Theory of Depreciation", The Journal of The American Statistical Association, 20, No:150, 340-53.

4. Johansson, S.E.(1961), "Skatt - Investering - Värdening", Norstedt.

5. Kamien, M.I. and N. Schwartz.(1981), "The Calculus of Variations and Optimal Control in Economics and Management," North Holland.

6. Liljeblad, R.(1936): 'Om Avskrivning av Fasta Productionsmedel', Nordisk Tidskrift for Teknisk Ökonomi, 2, Nos.1-2, 129-37.

7. Massé, P.(1962): "Optimal Investment Decisions", Prentice Hall.

8. Smith, V.L.(1957): "Economic Equipment Policies : An Evaluation", Management Science, 4, No:1, 20-37.

9. Smith, V.L.(1961): "Investment and Production", Harvard University Press.

10. Terborgh, G.W.(1949): "Dynamic Equipment Policy : A MAPI Study, Mc Graw Hill.

11. Thompson, G.L.(1968): "Optimal Maintenance Policy and Sale Date of a Machine," Management Science, 14, No:9, 543-50.

2. Taylor, J.S.(1923): "A Statistical Theory of Depreciation, Based on Unit Cost," The Journal of the American Statistical Association, 18, No.144, 1010-23.