W. P.: 329

Working Paper

विद्याविनियोगाद्विकासः
IIM
AHMEDABAD

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD

ANALYSIS OF A TWO-UNIT PARELLEL REDENDANT SYSTEM WITH PHASE TYPE FAILURE AND GENERAL REPAIR

By

N. Ravichandran

W P No. 329

August 1980

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD

Analysis of a Two-Unit Parellel Redendant System With Phase Type Failure And General Repair

N. Ravichandran
Indian Institute of Management
Ahmedabad, India

Abstract:

Explicit expressions for the Laplace transform of the reliability and availability of a general two-unit parellel redundant system are obtained. The Mean time to system failure and steady state availability are deduced as special cases. Some particular cases of our result are also obtained.

Introduction:

Several attempts have been made in the past to analyse two-unit parellel redundant systems [1, 2]. Gaver [8] initiated the study of a parellel redundant system which was subsequently continued by Kodame et al [3], Linton [4] and others. For the most recent developments in the analysis of these systems see [5, 6]. Also for the difficulties that arise, when one tries to obtain computationally feasible solution for a general two-unit parellel redundant system see [4, 5]. The object of this contribution is to analyse a more general system than the one discussed in [6].

Essentially we consider a two-unit parellel redundant system where the failure and repair times of one of the units are general random variables and those of the other unit are phase type [7] and general respectively. We give below a brief review of phase type distributions and urge the reader to refer to [7] for further details.

A Note on Phase Type Distributions:

A (continuous) probability distribution of phase type (PH-distribution) on (o,∞) is one which can be obtained as that of time till absorption in a finite state continuous time Markov Chain with one absorbing state and all other states are transient. To be specific, consider a Markov Chain on $\{1, 2 \ldots\ldots m + 1\}$ with initial probability vector $(\underline{\alpha}, o)$ where $\underline{\alpha} \varepsilon R^m$, is such that $\underline{\alpha} \geq \underline{o}$: $\underline{\alpha}e = 1$ and infinitesimal generator

$$Q = \begin{pmatrix} T & T^o \\ \underline{o} & o \end{pmatrix}$$

Where T is a non-singular $m \times m$ matrix with $T_{ii} \leq o$ and $T_{ij} \geq o$ for $i \neq j$ and $T\underline{e} + \underline{T}^o = o$ where $e = (1, 1, 1, \ldots.1)^t \varepsilon R^m$. For such a Markov Chain, the time till absorption in $(m + 1)$ state has cdf.

$F(x) = 1 - \underline{\alpha} \exp (Tx) \underline{e}$, $x \geq o$. The pair $(\underline{\alpha}, T)$ is called a representation of the PH-distribution F(.); we say F(.); we say that the PH-distribution is in phase $j (1 \leq j \leq m)$ if the underlying Markov Chain is in state j.

System Description:

1) The system consists of two-units connected in parellel. Either unit performs the system operation satisfactorily. At time $t = o$ both the units are switched on line.

2) The failure time of unit 2, the failure and repair times of unit 1, are general random variables. The failure time of unit 2 is a phase type distribution with representation $(\underline{\alpha}_i, T)$ and its pdf is denoted by $f_i(.)$, that of unit 1 is denoted by $f(.)$, while the pdf of the repair time of the units is denoted by $g(.)$.

3) There is only one repair facility and on failure a unit is taken to the repair facility, where the repair of the failed unit is entertained immediately, if there is no unit undergoing repair. Otherwise, the newly failed unit waits in the queue to get repaired.

4) The switch is perfect and switchover is instantaneous.

5) Each unit is 'new' after repair.

We require the following additional notations.

T - Transition (rate) probability matrix of a Markov Chain with m states.

$\underline{\alpha}_o$ - A general initial condition vector.

$\underline{\alpha}_j$ - A vector of dimension m in which the j^{th} component is 1 and all other components are zero.

$*$ - Convolution.

$c^{(n)}(t)$ - n fold Convolution of the function $c(t)$ with itself in (o,t).

$\bar{c}(t) = 1 - c(t)$

$C(t) = \int_o^t c(u)\, du$

Reliability:

We note that at any time t the individual units can be found in one of the following states: operable and found in phase j(j), operable (o), undergoing repair (r), queueing for repair (qr). We also note state j is possible only for unit 2. Let (x, y) denote the state of the system at any time t, where x denotes the state of the unit 1 and y denotes the state of unit 2. We further define the following events:

$E_{(x,y)}$ Event that at time t unit 1 enters state x and unit 2 enters y. x = o, r, qr; y = j, r, qr.

At time t=o, it is given that both units are just operable. Unit 2 will start functioning from phase j with probability $\tilde{\alpha}_j$, ($\tilde{\alpha}_j$ denotes the j^{th} component of the vector α_o). Let this event be $E_{o,o}$ and define

$$A_{o,o}(t) = \Pr\{\text{System is up } (o,t] \mid E_{o,o} \text{ at } t=o\}$$

and more generally

$$R_{i,j}(t) = \Pr\{\text{System is up in } (o,t] \mid E_{i,j} \text{ at } t=o\}$$

and $$A_{i,j}(t) = \Pr\{\text{System is available at } t \mid E_{i,j} \text{ at } t=o\}$$

We first proceed to obtain the reliability of the system and hence we have to determine the function, $R_{o,o}(t)$ which is given by,

$$R_{o,o}^{\prime\prime}(t) = \bar{F}(t) + \sum_{j=1}^{m} \int_{o}^{t} f(u)\, P_{oj}(u)\, R_{r,j}(t-u)du \quad \ldots\ldots\ldots\ldots($$

Where

$P_{oj}(t)$ = Pr {Unit 2 is found in phase j at time t/unit 2 is operable and found in phase i with proability $\tilde{\alpha}_i$ at t=o}

The above equation is obtained by considering the mutually exclusive and exhaustive events that the unit 2 fails in (o,t) or does not fail in (o,t). In the former case when the unit fails before t say in (u, u + du), the other unit has to be operable and hence will be found in one of the phases j (j = 1, 2...m) and in (u, u + du) the event $E_{r,j}$ occurs. We note that during this failure time interval of unit 1, unit 2 alternates between up and down states and the relevant probability in this case is denoted by the function $P_{oj}(t)$. which by using renewal theretic arguments [9] is easily seen to be

$$P_{oj}^{\prime}(t) = \sum_{i=1}^{m} \tilde{\alpha}_i\, \beta_{ij}(t) + \alpha(t) * \sum_{i=1}^{m} \tilde{\alpha}_i\, \beta_{ij}(t) \quad \ldots\ldots\ldots\ldots(2)$$

Where $$\alpha(t) = \sum_{n=1}^{\infty} [f_\sigma(t) * g_2(t)]^{(n)} \quad \ldots\ldots\ldots\ldots(3)$$

and

$\beta_{ij}(t)$ = Pr{Unit 2 is found in phase j at t and has not failed in (o,t) | unit 2 is found in stage i}

Equation (2) determines the $P_{oj}(t)$ used in (1) once the $\beta_{ij}(t)$'s are obtained. However, from the elementary theory of Markov Chains we have,

$$\beta'_{ij}(t) = +\lambda_{jj}\ \beta_{ij}(t) + \sum_{k=1}^{m} \lambda_{kj}\ \beta_{ik}(t) \qquad \ldots\ldots\ldots\ldots (4)$$

where λ_{ij} is the (i,j) th entry of the matrix T. Thus by using the equations (2)-(4) the function $R_{o,o}(t)$ is determined in terms of $R_{r,j}(t)$ which we now obtain, as

$$R_{r,j}(t) = \sum_{k=1}^{m} \bar{G}(t)\ \beta_{jk}(t) + \sum_{i=1}^{m} \int_{o}^{t} g(u)\ \beta_{ji}(u)\ R_{o,i}(t-u)du \ldots (5)$$

The above equation is obtained by considering the following mutually exclusive and exhaustive cases;

(i) The repair of unit 1 is not completed in (o,t) and hence for the reliability of the system the other unit should not fail in (o,t).

(ii) The repair of unit 2 is completed in (u, u + du) and hence for the system to be operable in (o,t) it is necessary that unit 2 is operable in (o,u).

We are yet to determine the functions $R_{o,i}(t)$ used in equation(5) and they are given by,

$$R_{o,i} = \bar{F}(t) + \sum_{j=1}^{m} \int_{o}^{t} f(u)\ P_{ij}(u)\ R_{r,j}(t-u)du \ \ldots\ldots\ldots (6)$$

Where $P_{ij}(t)$= Pr{Unit 2 is found in phase j at t | it was in phase i at t=o}

and is given by $P_{ij}(t) = f_i(t) * P_{oj}(t)$ .... (7)

Now, the set of equations (1)-(7) can be written in the form:

$$R_i(t) = a_i(t) + \sum_{j=1}^{m} V_{ij}(t) * R_j(t) \qquad \dots\dots\dots\dots\dots (7a)$$

On taking Laplace transform and solving for the $R^*_{o,o}(s)$ we obtain the Laplace transform of the reliability of the system and the mean time to system failure is given by $R^*_{o,o}(o)$.

Availability:

We next proceed to obtain the availability of the system. Since it is given that at t=o event $E_{o,o}$ occurs we need to determine the function $A_{o,o}(t)$. To obtain the Availability of the system in addition to the functions defined earlier we need the following function describing the behaviour of unit 2 during the failure time interval of unit 1.

$$P_{\alpha r}(t,x)dx = Pr\{\text{Unit 2 is under repair and the elapsed repair time lies in } (x, x+dx) \mid \text{unit 2 is found in state } \alpha \text{ at } t=o\}, \quad \alpha = o,1 \dots m, r$$

The functions $P_{\alpha r}(t,x)$ are given by,

$$P_{or}(t,x) = f_o(t-x)\,\bar{G}(x) + \alpha(t-x)\,\bar{G}(x) \qquad \dots\dots\dots\dots\dots (8)$$

$$P_{jr}(t,x) = f_j(t-x)\,\bar{G}(x) + \left[\int_0^{t-x} f_j(t-x-u)\,\alpha(u)du\right]\bar{G}_2(x) \quad \dots (9)$$

$$P_{rj}(t) = \int_0^t g(u)\,P_{oj}(t-u)du; \qquad 1 \le j \le m \qquad \dots\dots\dots\dots (10)$$

and

$$P_{rr}(t,x) = \bar{G}(x)\,\delta(t-x) + \left[\int_0^{t-x} \alpha(u)\,g(t-x-u)du\right]\bar{G}(x) \quad \dots\dots (11)$$

Using the equations (8)-(11) the set of equations governing the availability of the system are given by,

$$A_{o,o}(t) = \overline{F}(t) + \sum_{j=1}^{m} \int_0^t f(u)\, P_{oj}(u)\, A_{r,j}(t-u)\, du$$

$$+ \int_0^t du \int_0^u f(v)\, dv \int_0^v P_{or}(v,x)\, \frac{g(u-v+x)}{\overline{G}(x)}\, A_{r,o}(t-u)du \ldots (12]$$

$$A_{r,o}(t) = \sum_{j=1}^{m} \overline{G}(t)\, \beta_{oj}(t) + \sum_{j=1}^{m} \int_0^t g(u)\, \beta_{oj}\, A_{o,j}(t-u)\, du$$

$$+ \int_0^t g(u)\, \overline{F}_o(u)\, A_{o,r}(t-u)\, du \quad \ldots\ldots\ldots\ldots\ldots (1$$

$$A_{o,r}(t) = \overline{F}(t) + \sum_{j=1}^{m} \int_0^t f(u)\, P_{rj}(u)\, A_{r,j}(t-u)du$$

$$+ \int_0^t du \int_0^u f(v)\, dv \int_0^v P_{rr}(v,x)\, \frac{g(u-v+x)}{\overline{G}(x)}\, dx\, A_{r,o}(t-u) \ldots (14$$

$$A_{o,j}(t) = \overline{F}(t) + \sum_{k=1}^{m} \int_0^t f(u)\, P_{jk}(u)\, A_{r,}\,(t-u)du$$

$$+ \int_0^t du \int_0^u f(v)\, dv \int_0^v P_{jr}(v,x)\, \frac{g(u-v+x)}{\overline{G}(x)}\, dx\, A_{r,o}(t-u) \ldots (1$$

$$A_{r,j}(t) = \sum_{k=1}^{m} \overline{G}(t)\, \beta_{jk}(t) + \int_0^t g(u)\, \beta_{jk}(u)\, A_{o,k}(t-u)\, du$$

$$+ \int_0^t g(u)\, \overline{F}_j(u)\, A_{o,r}(t-u)\, du \quad \ldots\ldots\ldots\ldots\ldots$$

Equations (12)-(15) can be reduced to the set of equation similar to (7a). The limiting availability is obtained by taking $\lim_{t \to \infty} A_{o,o}(t) = \lim_{s \to o} sA^{*}_{o,o}(s)$ as demonstrated in [7].

Remark:

In the above analysis we have freely used the functions $P_{..}(.)$, $P_{..}(.,.)$ which by relation (2) depend very much on $\beta_{ij}(t)$ and we have not provided exact analytical expressions for these functions and all that we have given will help one to obtain only the Laplace transforms of the functions $\beta_{ij}(t)$ and hence that of the $P_{ij}(t)$ functions using this, quantities involving Convolutions of the functions like $f(t)\ P_{kj}(t)$ in the above equations can be discussed provided $f(.)$ is of exponential type (namely negative exponential, Erlang of order k, double exponential). However, if one is interested only in the mean time to system failure and availability (steady state) it can be obtained by means of numerical procedures. Since one is interested in calculating integrals of the form $\int_0^\infty f(u)\ P_{ij}(u)\ du$ or $\int_0^\infty t\ f(u)\ P_{ij}(u) du$. These integrals can be numerically computed since the functions $P_{ij}(t)$ can be obtained numerically by using the one of the several methods available to solve linear differential equations with constant coefficients, satisfied by $\beta_{ij}(t)$.

Special Cases:

We now obtain the reliability of the system for the case in which $f(t)$, $g(t)$ are arbitrary and the life time of unit 2 is a phase type distribution with 2 phases. We also specialise to the case where

$$T = \begin{bmatrix} -\lambda_1 & \lambda_2 \\ 0 & -\lambda_2 \end{bmatrix} \qquad T^o = \begin{pmatrix} 0 \\ \lambda_2 \end{pmatrix}$$

and $\alpha = (1,o)$. Then the state of the system in this case is of the form (x,y) where $x=o,r,qr$ and $y=1,2,r,qr$. Since we are interested in the reliability of the system the states corresponding to x or $y = qr$ is not possible.

The equations governing the reliability in this case are given by

$$R_{o,i}(t) = \bar{F}(t) + \sum_{j=1}^{2} \int_0^t f(u)\, P_{ij}(u)\, R_{r,j}(t-u)du$$

and

$$R_{r,j}(t) = \sum_{k=1}^{2} \bar{G}(t)\, \beta_{jk}(t) + \sum_{k=1}^{2} \int_0^t g(u)\, \beta_{jk}(u)\, R_{o,k}(t-u)du$$

Where

$$P_{kj}(t) = \beta_{kj}(t) + \int_0^t f_k(u)\, P_{rj}(t-u)\, du$$

$$P_{rj}(t) = \int_0^t g(u)\, P_{1j}(t-u)du$$

and

$$P_{1j}(t) = \beta_{1j}(t) + \int_0^t \alpha(u)\, \beta_{1j}(t-u)du$$

with

$$\alpha(t) = \sum_{n=1}^{\infty} \left\{ \int_0^t f_1(u)\, g(t-u)du \right\}^{(n)}$$

$$f_2(t) = \lambda_2 e^{-\lambda_2 t}$$

$$f_1(t) = \int_0^t \lambda_1 e^{-\lambda u} \quad \lambda_2 e^{-\lambda_2 (t-u)du}$$

$$\beta_{11}(t) = e^{\lambda_1 t}; \quad \beta_{22}(t) = e^{-\lambda_2 t}$$

$$\beta_{12}(t) = \int_0^t \lambda_1 e^{-\lambda_1 u} \quad e^{-\lambda_2 (t-u)}\, du$$

Relabeling the states as $(o,1) \to 1$: $(o,2) \to 2$ $(r,1) \to 3$ and $(r,2) \to 4$. The set of equations governing the reliability of the system is given by

$$R_i(t) = \alpha_i(t) + \int_0^t V_{ij}(u)\, R_j(t-u)\, du.$$

where

$$\alpha_1(t) = \alpha_2(t) = \bar{F}(t)$$

$$\alpha_3(t) = \bar{G}(t)\ [\beta_{11}(t) + \beta_{12}(t)]$$

$$\alpha_4(t) = \bar{G}(t)\ \beta_{22}(t)$$

$$V_{1,j+2}(t) = f(t)\ P_{1j}(t), \quad j = 1,2$$

$$V_{2,j+2}(t) = f(t)\ P_{2j}(t), \quad j = 1,2$$

$$V_{k,j}(t) = 0 : k = 1,2, \quad j = 1,2$$

$$V_{3i}(t) = g(t)\ \beta_{1i}(t), \ i = 1,2$$

$$V_{3i}(t) = 0, \quad i = 3,4$$

$$V_{42}(t) = g(t)\ \beta_{22}(t)$$

$$V_{41}(t) = 0, \quad i = 1,3,4.$$

Now, the Laplace transform of the reliability of the system conditioned by $E_{(o,1)}$ at t=o is given by

$$R_1^*(s) = N(s) \,/\, D(s)$$

Where

$$D(s) = \begin{vmatrix} 1 & 0 & -V_{13}^*(s) & -V_{14}^*(s) \\ 0 & 1 & -V_{23}^*(s) & -V_{24}^*(s) \\ -V_{31}^*(s) & -V_{32}^*(s) & 1 & 0 \\ 0 & -V_{42}^*(s) & 0 & 1 \end{vmatrix}$$

and $N(s)$ is obtained from $D(s)$ by replacing the first column of $D(s)$ by the transpose of $[\alpha_1^*(s),\ \alpha_2^*(s),\ \alpha_3^*(s),\ \alpha_4^*(s)]$. Further reducing to the special case where $f(t) = \lambda e^{-\lambda t}$ and $T = (-\lambda_1)$ $T^o = (\lambda_1)$ and $\alpha = (1)$ we obtain the reliability of the system conditioned by an $E_{o,1}$ at $t=o$, as

$$R_1^*(o) = \frac{(\lambda+\lambda_1)\,[1-\lambda \overset{*}{g}\,(\lambda+\lambda_1)] + \lambda_2\, \bar{G}\,(\lambda_1)}{\lambda[(\lambda+\lambda_1)\,\{1-\lambda_1 \overset{*}{g}\,(\lambda+\lambda_1)\} - \lambda \overset{*}{g}\,(\lambda_1)\}}$$

which is in agreement with [8].

References

1) S. Osaki and T.Nakagawa (1976); Bibliography for reliability and availability of stochestic systems, IEEE Tran. Rel., Vol.R-25, pp. 284-287.

2) A.Birolini (1975); Some applications of regenerative stochastic process in reliability theory; part 2, Reliability and availability of 2 item redundant systems; Vol. R-24, pp. 336-339.

3) M. Kodamma; H.Deguchi (1974); Reliability Consideration for a 2-unit redundant system with Erlang Failure and general repair distributions; IEEE Trans. Rel. Vol.R-23, pp. 75-81.

4) D.G. Linton (1976); Some advancements in the analysis of two-unit parellel redundant system, Micro Rel. pp.39-46.

5) R.Subramanian and N.Ravichandran (1980); Probablistic analysis of a 2-unit parallel redundant system, Micro Rel. Vol. 19.

6) R.Subramanian and N.Ravichandran(1979); Stochastic behaviour of two-unit parallel redundant system IEEE Tran. Rel. Vol. R-28, pp. 419-420.

7) M.F. Neuts (1975); Probability distribution of phase type; in Liber Amicorum Professor Emeritus H. Florin, Department of Mathematics, University of Lonavin, Belgium, pp. 173-206.

8) D.P. Gaver (1963); Time to failure and availability of parallel systems with repair, IEEE Trans. Rel. Vol. R-12, pp. 30-38.

9) D.R. Cox (1968); Renewal Theory, Methune, London.

...