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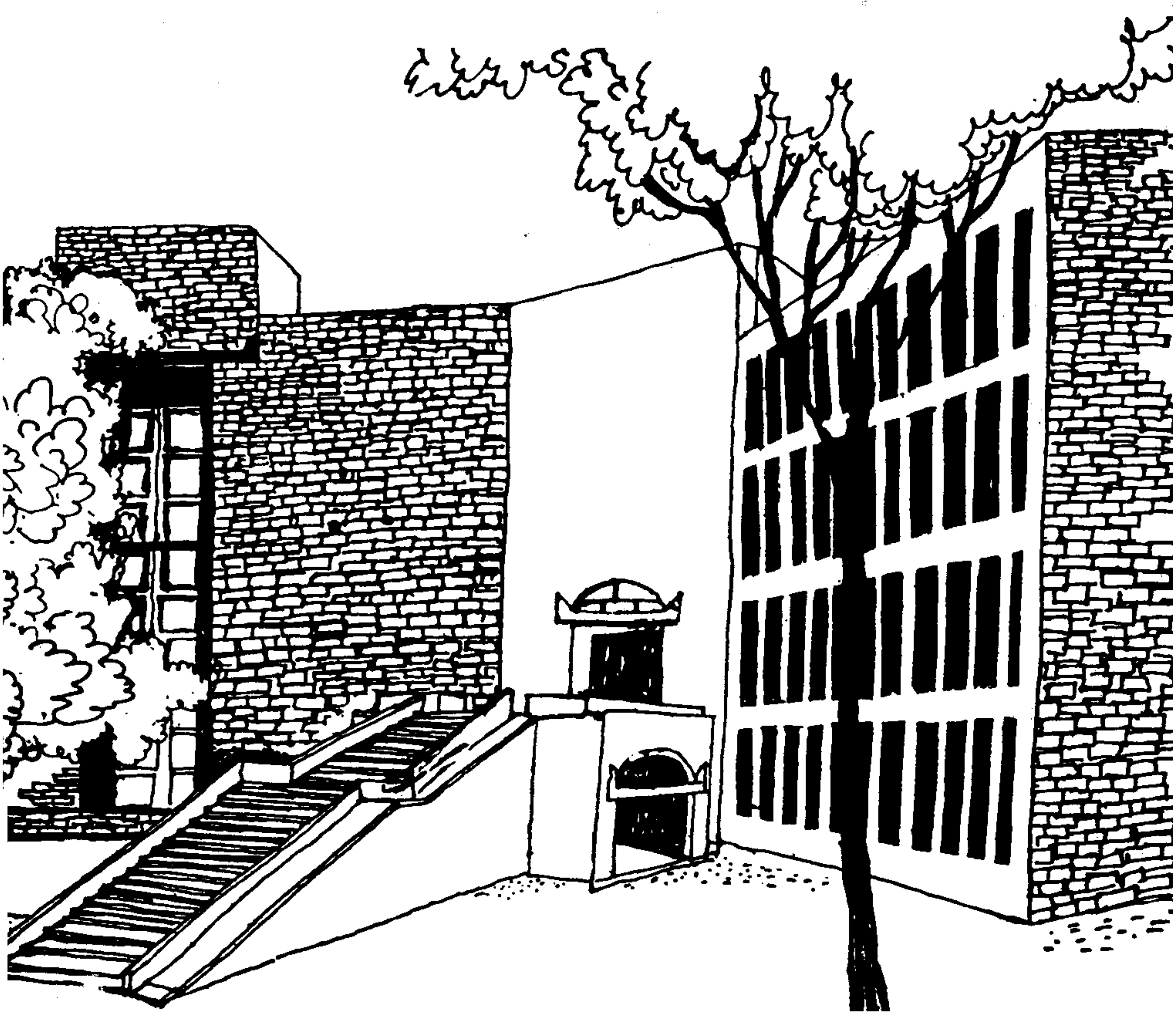

THE AVERAGE WELFARE FAIR AND EGALITARIAN SOLUTIONS FOR BARGAINING PROBLEMS

By

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ABSTRACT

In this paper we present an alternative characterization of the Egalitarian solution for bargaining problems and show that the Egalitarian solution has the property that for any agent the incremental utility from bargaining is atleast as much as the average incremental utility of all other agents. Conversely, we show that any solution which is both weakly Pareto optimal and satisfies this latter property must be egalitarian.

1. Introduction :- The bargaining problem as proposed by Nash (1950), can be interpreted as a problem of fair division of resources amongst a given set of agents. Expounding this idea and its philosophical underpinnings have been a major intellectual preoccupation amongst welfare economists in the recent past.

From the standpoint of theory, new characterizations of existing solutions to bargaining problems is an important field of study. Such has been a main activity of the literature known as 'axiomatic models of bargaining'.

In this paper we propose an alternative characterization of an important bargaining solution known as the Egalitarian solution. The significance of this characterization lies in its appeal as a criteria of distributive justice (or fairness) as well as an alternative theoretical view of an existing solution .

2. Framework :- In this section we develop a general framework for the analysis of bargaining problems. The framework we propose is a synthesis of the one developed by Nash (1950) and the one developed by Yaari (1981), the latter being a context in which problems of distributive justice can be analyzed.

To capture the essentials of a bargaining problem, we start with a non-empty set X and a list of n real-valued functions $u_1, u_2, \ldots, u_n$, defined on X i.e. $u_i : X \to \mathbb{R}$ for all $i \in \{1, \ldots, n\}$. The Set X is the universe of discourse and its elements are entities among which society could conceivably be called upon to exercise a choice. Elements of X will be called <u>alternatives</u> and the set X of all conceivable alternatives will be referred to as the <u>choice space</u>. The functions $u_1, u_2, \ldots, u_n$ are utility functions, which represent the preferences over the choice space of the individuals who make up the society under consideration. In other words, "Society" consists of n individuals who are identified by the utility functions $u_1, u_2, \ldots, u_n$ all of which are functions assigning real numbers to elements of the choice space.

The choice space X consists of all concievable alternatives that might arise as options in any hypothetical situation. However, every specific choice situations is, characterized by a <u>subset</u> of X, consisting of those alternatives that are feasible in that particular choice situation, taking into account all the extraneous factors. This gives rise to the notion of the <u>set of feasible alternatives</u> (or <u>feasible set</u>, for short) for which the symbol F will be used.

We shall agree to call an n-tuple of utility function $(u_i)_{i=1}^n$, an <u>utility profile</u> for the society, where for each $i \in \{1, \ldots, n\}$, $u_i : X \to \mathbb{R}$ is a real valued function defined on X.

A <u>social choice problem</u> is an ordered pair $S = ((u_i)_{i=1}^n, F)$ where $(u_i)_{i=1}^n$ is an utility profile and $F \subseteq X$ is a feasible set of alternatives.

Let $x_o \in X$ be an alternative which we shall refer to as the <u>status quo point</u>. The underlying interpretation of a status-quo point is that if the agents agree on an alternative in the feasible set then they select this alternative; in the absence of an agreement they remain at x_o.

A <u>bargaining problem</u> is an ordered pair (S, x_o) where $S = ((u_i)_{i=1}^n, F)$ is a social choice problem and $x_o \in X$.

A significant assumption which prevades a major portion of bargaining theory is that given a bargaining problem (S, x_o) where $S=((u_i)_{i=1}^n, F)$, $x_o \in F$ and there exists $x \in F$ such that $u_i(x) > u_i(x_o) \forall\ i=1,\ldots,n$. The meaning of this assumption is that the status quo point is a feasible alternative and there exists another feasible alternative at which every one is better-off than at the status-quo point, so that there is some advantage to be garnered from trying to arrive at a compromise. We shall make this assumption throughout our analysis.

Given a bargaining problem (S, x_o) where $S=((u_i)_{i=1}^n, F)$, one can define a set $\mathcal{U}_F$ in the following manner :

$$\mathcal{U}_F = \left\{ (a_1,\ldots,a_n) / u_1(x)=a_1, u_2(x)=a_2,\ldots,u_n(x)=a_n \text{ for some } x \in F \right\}.$$

$\mathcal{U}_F$ is called the set of <u>feasible utility profiles</u> (or the utility-possibilities set). If one defines a vector-valued function u by writing $u(x)=(u_1(x),\ldots,u_n(x))$ for all $x \in X$, then $\mathcal{U}_F$ is given simply by the statement

$$\mathcal{U}_F = u(F),$$

$\mathcal{U}_F$ being the image of F under u. Each choice that society can feasibly make is translated into an n-tuple of utility numbers and the collection of these n-tuple of utility numbers is the set $\mathcal{U}_F$.

The following assumption justified in detail by Yaari (1981) is standard and for technical reasons will be assumed throughout our analysis :

<u>Principle of Non-Polarization</u> : The set $\mathcal{U}_F$, of feasible utility profiles, is convex, comprehensive and bounded from above.

Let $\Sigma = \left\{ (S, x_o) / (S, x_o) \text{ is a bargaining problem satisfying the principle of non-polarization} \right\}$.

Given the domain of bargaining problems Σ, a <u>solution defined on that domain</u> is a function that associates with every problem in the domain a nonempty subset of feasible alternatives of that problem; these alternatives are interpreted as the set of possible compromises reached by the agents (or recommended to them, in the event of an impartial arbitrator being incharge of deciding on the outcomes). The set of values taken by the solution, when applied to a particular problem, is the <u>solution outcome</u> of the problem. Thus, formally, a <u>solution on Σ</u> is a correspondence $G: \Sigma \twoheadrightarrow X$ such that for all $(S, x_0) \in \Sigma$, $G[(S, x_0)] \subseteq F$.

The intuitive interpretation of the solution G is that of a model of some first stage of negotiations in which a subset of the feasible set is identified, from amongst which the final outcome will eventually be selected, through a process left unspecified.

The main solutions to bargaining problems that have been studied and axiomatized in the literature are the Nash (1950) solution, the Kalai-Smorodinsky (1975) solution and the Egalitarian solution of Kalai (1977), to mention a few. Our purpose in this paper is to obtain a simple alternative characterization of the Egalitarian solution. For conventional axiomatizations and further extensions of this solution one may refer to Kalai (1977), Roth (1979) or Thomson (in press).

3. <u>The Egalitarian Solution</u> :- In almost all of bargaining theory, the least that a bargaining solution is required to satisfy is <u>Weak Pareto Optimality</u>.

<u>Weak Pareto Optimality</u> (WPO) : A bargaining solution $G: \Sigma \twoheadrightarrow X$ is said to be <u>weakly Pareto optimal</u> if for all $(S, x_0) \in \Sigma$ the following is satisfied :

$x \in G[(S,x_0)] \Rightarrow$ there does not exist any $y \in F$ such that

$$u_i(y) > u_i(x) \text{ for all } i \in \{1,\dots,n\}.$$

i.e. given any element of a solution outcome, it is not possible to find any other feasible outcome where everybody is better-off than before.

This principle enjoys wide acceptance, the main reason for which is that a failure to do so would lead to the untenable position that a solution may have to be enforced, against the will of all individuals.

The Egalitarian solution $E : \sum \twoheadrightarrow X$ is defined to be a weakly Pareto optimal solution which satisfies in addition the following criteria :

$$\forall (S,x_0) \in \sum, \; x \in E[(S,x_0)] \Rightarrow \forall \; i,j \in \{1,\dots,n\}, \; u_i(x) - u_i(x_0) = u_j(x) - u_j(x_0).$$

The Egalitarian solution is cardinalist in spirit and says that for a feasible alternative to qualify as a solution outcome it is necessary that the incremental utility derived from bargaining be the same for all agents.

4. The Average Welfare Fair Solution : A solution to bargaining problems that we propose as an alternative characterization of the Egalitarian solution has been motivated by a solution to problems of distributive justice initially suggested by Thomson (1982).

In that paper, an equity criterion for exchange economies is suggested, where "an allocation is equitable if no agent would rather consume the average of what everyone else is consuming." Such an allocation is referred to as an A-envy-free-allocation (where A stands for 'average'). The solution is appealing on grounds of its informational efficiency and distributive justice.

Closer in spirit to the Egalitarian solution, yet confined to the context of a pure exchange economy (allowing for consumption externalities) is the concept of a welfare-egalitarian allocation, proposed by Villar (1988), where all agents' welfare levels are equalized under the assumption that utility functions are interpersonally comparable (in ordinal terms). In this paper, the existence of a Pareto-optimal welfare-egalitarian allocation (called welfare-fair allocation) is established.

Modifying Thomson's (1982) characterization we define a solution $G : \Sigma \twoheadrightarrow X$ to be A-Welfare Egalitarian (where once again A stands for 'average') if $\forall (S,x_0) \in \Sigma, x \in G[(S,x_0)] \Rightarrow \forall i \in \{1,\ldots,n\}$
$u_i(x)-u_i(x_0) \geq \frac{1}{(n-1)} \sum_{j \neq i} [u_j(x)-u_j(x_0)]$.

Intuitively, we declare a feasible alternative to be A-Welfare Egalitarian if the incremental welfare derived by any agent from the bargaining solution is atleast as much as the average incremental utility of all other agents.

We say that a bargaining solution $G : \Sigma \twoheadrightarrow X$ is A-Welfare Fair if it is both Weakly Pareto Optimal and A-Welfare Egalitarian.

The intuitive meaning of an A-welfare fair outcome is patently clear from the above discussion.

5. The Equivalence Theorem :- In this section we establish that a solution is Egalitarian if and only if it is A-Welfare Fair.

Theorem :- A solution $G : \Sigma \rightarrow X$ is A-Welfare Fair if and only if G=E.

<u>Proof</u> :- Let us first show that E is A-Welfare Fair.

$$\forall (S,x_0)\in\Sigma, x\in E\left[(S,x_0)\right], \forall\, i,j\in\{1,\ldots,n\},$$
$$u_i(x) - u_i(x^o) = u_j(x) - u_j(x^o)$$

Hence, trivially, $u_i(x)-u_i(x^o) = \sum_{j\neq i}\left[u_j(x)-u_j(x^o)\right]/(n-1)$ $\forall i\in\{1,\ldots,n\}$. Thus E is A-Welfare Egalitarian. Since E satisfies <u>Weak Pareto Optimality</u> by definition, E is thus <u>A-Welfare Fair</u>.

Conversely suppose $G:\Sigma\twoheadrightarrow X$ is A-Welfare Fair. Hence it is A-Welfare egalitarian. Thus $\forall\, i\in\{1,\ldots,n\}, \forall\, x\in G\left[(S,x_0)\right]$,

$$u_i(x)-u_i(x_0)\geq \frac{1}{(n-1)}\sum_{j\neq i}\left[u_j(x)-u_j(x_0)\right] \quad (*)$$

Suppose towards a contradiction that for some $i'\in\{1,\ldots,n\}$, and some $x\in G\left[(S,x_0)\right]$,

$$u_{i'}(x) - u_{i'}(x_0) > \frac{1}{(n-1)}\sum_{j\neq i'}\left[u_j(x)-u_j(x_0)\right] \quad (**)$$

On the one hand we have the following identity :

$$\sum_{i=1}^{n}\left[u_i(x)-u_i(x_0)\right] = \sum_{i=1}^{n}\frac{1}{(n-1)}\sum_{j\neq i}\left[u_j(x)-u_j(x_0)\right].$$

On the other hand (*) and (**) imply,

$$\sum_{i=1}^{n}\left[u_i(x)-u_i(x_0)\right] > \sum_{i=1}^{n}\frac{1}{(n-1)}\sum_{j\neq i}\left[u_j(x)-u_j(x_0)\right].$$

This contradiction establishes that

$$u_i(x)-u_i(x_0) = \frac{1}{(n-1)}\sum_{j\neq i}\left[u_j(x)-u_j(x_0)\right]$$

$\forall i\in\{1,\ldots,n\}$ and $\forall x\in G\left[(S,x_0)\right]$.

However, G is A-Welfare Fair implies by definition that A is Weakly Pareto Optimal. Hence G = E.

Q.E.D.

This establishes the required equivalence.

As an immediate corollary to the above theorem we have the following :

<u>Corollary</u> :- A solution $G: \sum \twoheadrightarrow X$ is <u>A-welfare egalitarian</u> if and only if

$$\forall (S, x_0) \in \sum, \forall x \in G[(S, x_0)], \forall i, j \in \{1, \dots, n\},$$

$$u_i(x) - u_i(x_0) = u_j(x) - u_j(x_0).$$

<u>Proof</u> :- Contained in the proof of the above theorem.

5. <u>Conclusion</u> :- In this paper we have obtained an interesting characterization of the Egalitarian solution, motivated by a fairness criteria postulated by Thomson (1982). The first point we would like to resolve is whether this characterization could be extended to <u>social choice problems</u>. The answer is in the affirmative and the relevant characterization proceeds by suppressing x_0 and $u_i(x_0)$ throughout our analysis. Since the modification is straightforward, we shall not dwell on it any further.

The second point we would like to address ourselves to is the <u>principle of non-polarization</u>. The primary reason why it has been invoked in our analysis is to permit the existence of the Egalitarian solution. Without this assumption the Egalitarian solution fails to exists.

The third and final point that we would like to make is that the characterization presented in this paper reinforces the links between bargaining problems and problems of fairness or fair division of resources. Notable connections between these two significant areas of research has been outlined in Thomson (in press).

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