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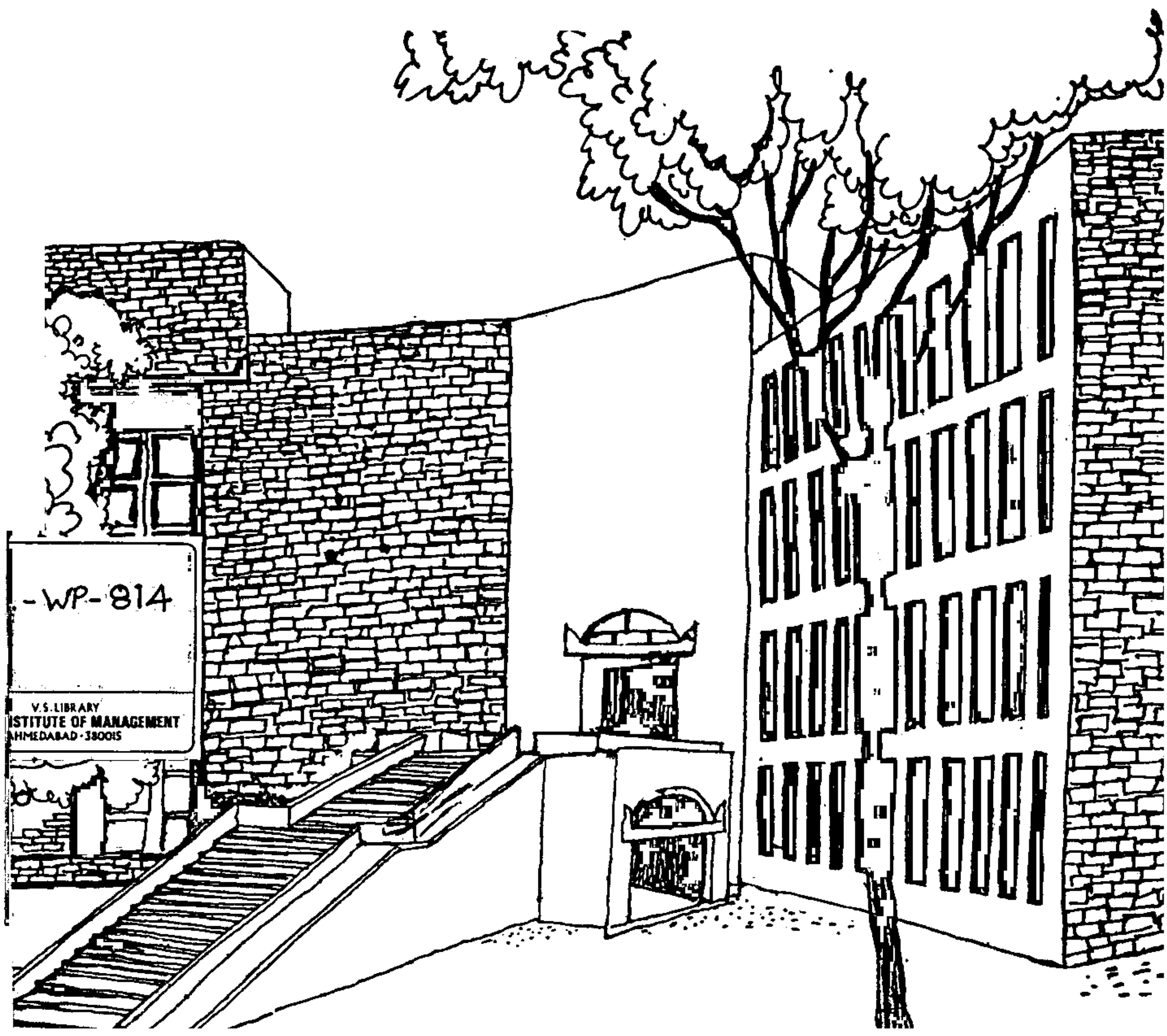

CONTINUITY OF BARGAINING SOLUTIONS DEFINED WITH RESPECT TO A CRITERION

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Abstract

In this paper we study continuity properties of bargaining solutions which are defined with respect to a criterion function. Sufficient conditions are obtained for the solutions to be continuous and the exercise is carried out in a fairly general framework.

1. Introduction :- In this paper, we will study the continuity properties of solutions to bargaining problems, as defined by Nash (1950), which are obtained on the basis of some given criteria. We restrict our attention to two-person bargaining problems. Formally, a (two-person) bargaining game S is a proper subset of the non-negative orthant ($\mathbb{R}^2_+$) satisfying :

(1) S is compact and convex,

(2) $0\ (=(0,0)) \in S$ and $x > 0$ for some $x \in S$;

(3) S is comprehensive, i.e. for all $x \in S$ and $y \in \mathbb{R}^2_+$, if $y \leq x$, then $y \in S$.

Let B denote the family of all bargaining games. When interpreting an $S \in B$, one must think of the following game situation. Two players (bargainers) may cooperate and agree on a feasible outcome x in S, giving utility x_i to player $i = 1, 2$, or they may fail to cooperate, in which case the game ends in the disagreement outcome 0. So for any $S \in B$, the disagreement outcome is fixed at 0. To facilitate arriving at a concensus, we assume the existence of an impartial arbitrator. Thus S consists of all the compromises that an arbitrator deciding the case may choose.

Compactness of is required for mathematical convenience and is intuitively plausible in most bargaining situations; convexity stems from allowing lotteries in an underlying bargaining situation. The requirement $x > 0$ for some x serves to give each player an incentive to cooperate. Not all of the restrictions in (1) - (3) are necessary for all of our results, but assuming them simplifies matters and, moreover, none of them goes against intuition.

Following Kaneko (1980) a (two-person) bargaining solution is a set valued map $F : B \twoheadrightarrow \mathbb{R}^2_+$ assigning to each $S \in B$ a set of outcomes $F(S) \subseteq S$ and such that Axiom 0 holds :

Axiom 0 : $F(S)$ depends only on (the shape of) S.

Aniom 0 states explicitly that F does not depend on an underlying bargaining situation (i.e., a set of lotteries and a pair of utility functions mapping these into the plane). By most authors, this is implicitly assumed or taken for granted (however, cf. Shapley (1969), Lahiri (1989)).

The (possibly) fictions arbitrator that we have assumed, is supposed to be equipped with a criterion function. Using this criterion function the arbitrator can order alternative compromises that is available in the game under consideration. A criterion function is a real valued map R : B

$R : B \times \mathbb{R}_+^2 \to \mathbb{R}$), satisfying the condition :

$\forall S \in B$, $\forall$ x, $y \in \mathbb{R}_+^2$, $x > y \Rightarrow R(S, x) > R(S, y)$.

The intuitive interpretation of a criterion function R is the following : given a game $S \in B$ and outcomes x, $y \in \mathbb{R}^2$, $R(S, x) \geq R(S, y)$ implies that the arbitrator considers the outcome x to be atleast as good as the outcome y, under the prevailing game situation. $R(S, x) > R(S, y)$ has the interpretation that x is preferred to y by the arbitrator given the game S.

R itself defines for the particular game situation under consideration the preferences of the arbitrator over alternative compromises. A heuristic and elegant discussion of bargaining solutions defined with respect to a criterion function is available in Wittman (1979).

Let $S \in B$. We say that

1) $x \in S$ is an unanimity (ideal efficient, or ideal maximal) point of S if $x \geq y$ for every $y \in S$.

The set of unanimity points of S is denoted by I Max S.

2) $x \in S$ is an efficient (or Pareto-optimal, or nondominated) point of S if $y \geq x$ for some $y \in S$, then $x \geq y$;

The set of efficient points of S is denoted by Max S ;

3) x is a weakly efficient (or weakly Pareto - optimal, or weakly nondominated) point of S if there does not exist $y \in S$ with $y > x$.

The set of weakly efficient points of S is denoted by W Max S .

Let $F : B \twoheadrightarrow \mathbb{R}_+^2$ be a bargaining solution. We say that F is unanimously generated by a criterion function R if $F(S) = \{x \in S / R(S,x) \geq R(S,y) \forall y \in S\} \subseteq \text{I Max } S$, whenever I Max $S \neq \phi$. We say that F is generated by the criterion function R if $F(S) = \{x \in S / R(S,x) \geq R(S,y) \forall y \in S\} \subseteq \text{Max } S, \forall S \in B$

We say that F is weakly generated by the criterion function R if $F(S) = \{x$

R if $F(S) = \{x \in S / R(S, x) \geq R(S, y) \forall y \in S\} \subseteq \text{W Max } S$

The meaning of the concepts defined above is quite obviours. Thus we are assuming that the solution to each game is obtained by the arbitrator, after he solves an optimization problem.

3. Examples of Solutions generated by a criterion function :-

(A) Nash's Bargaining Solution (cf. Nash [1950]) : $\forall S \in B$,
$R(S, x) = x_1 x_2 \quad \forall \quad x = (x_1, x_2) \in \mathbb{R}_+^2$.

(B) Kalai Smorodinsky's Bargaining Solution (cf. Kalai-Smorodinsky [1975]):
$\forall S \in B$, let $M_i(S) = \sup \{x_i \in \mathbb{R} / \exists\, x = (x_1, x_2) \geq 0, x \in S\}$ for $i = 1, 2$.
Let $R(S, x) = \min\left\{\frac{x_1}{M_1(S)}, \frac{x_2}{M_2(S)}\right\} \forall S \in B$ and $\forall x \in \mathbb{R}_+^2$, $x = (x_1, x_2)$.

In both (A) and (B), R generates F.

(C) The set-valued egalitarian solution

Let $R(S, x) = \min\{x_1, x_2\} \forall S \in B$ and $\forall x \in \mathbb{R}_+^2$, $x = (x_1, x_2)$.

In (C) R generates F weakly.

(D) The Yu p-solution (cf. Yu [1973]) : For $p \in (1, \infty)$, let
$R(S, x) = -\sqrt[p]{(x_1 - M_1(S))^p + (x_2 - M_2(S))^p}$ and for $p = \infty$, let
$R(S, x) = -\operatorname{Max}\{|x_1 - M_1(S)|, |x_2 - M_2(S)|\}$. Here once again, R is defined $\forall S \in B$ and $\forall x = (x_1, x_2) \in \mathbb{R}_+^2$

In (D) R generates F

4. <u>Set - Valued Maps</u> :-

In this section we recapitalate some preliminary concepts on set-valued maps which will be required in the sequel.

Suppose that E_1 and E_2 are two real topological vector spaces. Let G be a set valued map from E_1 to E_2 which means that $G(x)$ is a set in E_2 for each $x \in E_1$. The following notation will be used for set - valued maps :

$\operatorname{dom} G = \{x \in E_1 / G(x) \neq \phi\}$

$\operatorname{Graf} G = \{(x, y) \in E_1 \times E_2 / y \in G(x), x \in \operatorname{dom} G\}$.

Definition 1 : Let X be a subset of dom G. We say that

1) G is upper semi - continuous at $x_0 \in X$ if for each neighbourhood V of $G(x_0)$ in E_2, there is a neighbourhood U of x_0 in E_1 such that
$G(x) \subseteq V$, for all $x \in U \cap \operatorname{dom} G$;

2) G is lower semi - contiuous at $x_0 \in X$ if for any $y \in G(x_0)$, any neighbourhoc V of y in E_2, there is a neighbourhood U of x_0 in E_1 such that
$G(x) \cap V \neq \phi$, for each $x \in U \cap \operatorname{dom} G$.

3) G is continuous at x_0 if it is upper and lower semi - continuous at that point; and G is upper (Resp., lower,....) semi - continuous on X if it is upper (resp., lower,....) semi - continuous at every point of X

4) G is closed if graf G is closed;

5) Whenever "N" denotes some property of sets in E_2, we say that G is "N" - valued on X if G(x) has the property "N", for every $x \in X$.

Good sources of information on set valued maps are Aubin and Ekeland (1984) or Berge (1962). We develop here only what we require in the sequel.

Theorem 1 : Assume that X is a compact set in E_1 and G is an "N" - valued, upper semi - continuous map from E_1 to E_2 with $X \subseteq$ dom G, where "N" may be closed, bounded or compact. Then G (X) has the property "N" in E_2

Proof :- First let "N" be closed and let $\{a_\alpha : \alpha \in I\}$ be a net from G (X) with $\lim a_\alpha = a$. We have to prove that there is some $x \in X$ such that

$$a \in G(x).$$

Let $x_\alpha \in X$, $a_\alpha \in G(x_\alpha)$. We may assume that $\lim x_\alpha = x \in X$.

For any neighbourhood V of G(x) in E_2, there is some $\beta \in I$ such that

$$G(x_\alpha) \subseteq V, \text{ for all } \alpha \geq \beta .$$

In particular,

$$a_\alpha \in V, \text{ for all } \alpha \geq \beta$$

Since V is arbitrary and G is closed valued, we conclude that $a \in G(x)$.

Now, let "N" be bounded and let V be an arbitrary neighbourhood of zero in E_2. We have to show that there is some $t > o$ such that

$$G(X) \subseteq t\, V.$$

To this end, for every $x \in X$, consider the set

$$U(x) = \{ y \in X / G(y) \subseteq G(x) + V$$

Which is open in X due to the upper semi - continuity of G. By the compactness of X, there are a finite number of points from X, say $x_1, \ldots\ldots, x_n$, such that $\{U(x_i) : i = 1, \ldots\ldots, n\}$ covers X. Thus,

$$G(X) \subseteq \{ \quad G(x_i) : i = 1, \ldots, n\} + n\, V.$$

Remember that G is bounded - valued, which means that there are some $t_i \geq o$ so that

$$G(x_i) \subseteq t_i V.$$

Take $t = n + t_1 + \ldots\ldots + t_n$ to get the inclusion $G(X) \subseteq t V$.

Further, let " N " be compact and suppose that $\{ V_\alpha : \alpha \in I \}$ where V_α are open, is a cover of $G(X)$. We have to draw a finite subcover from that cover. For $x \in X$, denote by $I(x)$ a finite index set from I which exists by the compactness of $G(x)$ such that $\{ V_\alpha : \alpha \in I(x) \}$ covers $G(x)$. Again the set

$$U(x) = \left\{ y \in X : G(y) \subseteq \bigcup \{ V_\alpha : \alpha \in I(x) \} \right\},$$

is open and we can obtain a finite cover of X, say $\{ U(x_i) : i = 1, \ldots, n \}$. Then the family $\{ V_\alpha : \alpha \in I(x_1) \cup \ldots \cup I(x_n) \}$ forms a finite subcover of $G(X)$

Definition 2 : (Penot (1984) : - G is said to be compact at $x \in \operatorname{dom} G$ if any net $\{ (x_\alpha, y_\alpha) \}$ from graf G possesses a convergent subnet with the limit belonging to graf G as soon as $\{ x_\alpha \}$ converges to x.

Whenever this is true for each $x \in X \subseteq \operatorname{dom} G$, we say that G is compact on the set X.

Definition 3 :- Set now E_1 and E_2 be metric spaces and let G be compact valued on E_1. We say that G is Lipsehitz at $x \in \operatorname{dom} G$ if there is a neighbourhood U of x in E_1 and a positive number r, called a Lipsehitz constant, such that

$$h(G(x), G(y)) \leq r\, d(x, y), \text{ for each } y \in U,$$

where $d(\cdot, \cdot)$ is the metric in E_1, and $h(\cdot, \cdot)$ is the Hausdorff distance between two compact sets in E_2.

Proposition 1 :- If G is Lipsehitz at $x \in \operatorname{dom} G$, then it is compact and continuous at that point.

Proof :- The continuity of the map at x is obvious. We prove that the map is compact. For let $\{ (x_\alpha, y_\alpha) \}$ be a net from graf G with $\{ x_\alpha \}$ converging to

$x \in$ dom G. Consider the net $d(y_\alpha, G(x))$ of real numbers. Since G is lipschitz at the point x,

$$\lim d(y_\alpha, G(x)) = 0.$$

By the compactness of G(x), there is a net $\{z_\alpha\}$ from G(x) such that

$$d(y_\alpha, G(x)) = d(y_\alpha, z_\alpha)$$

and this net may be assumed to converge to some $z \in G(x)$. We have then,

$$d(y_\alpha, z) \leq d(y_\alpha, z_\alpha) + d(z_\alpha, z)$$

and $\{y_\alpha\}$ converges to z, completing the proof.

Now suppose that E_3 is a real topological vector space, H is a set - valu__ map from E_2 to E_3. The composition $H \circ G$ is the map from E_1 to E_2 defined as follows

$$(H \circ G)(x) = \bigcup \{H(y) : y \in G(x)\}.$$

Proposition 2 :- Suppose that the map G is compact at $x \in E_1$ and H is compact on G(x). The map $H \circ G$ is compact at x.

Proof : Immediate from the definition.

5) Continuity of Solutions :-

Let T be a topological space. Let us be given the following map :

$$S : T \twoheadrightarrow B,$$

Where for each $t \in T$, $S(t)$ is a bargaining game.

Given a criterion function $R : B \times \mathbb{R}_+^2 \to \mathbb{R}$, this map determines the following solution :

$$F(S(t)) \equiv \{x \in S(t) \mid R(S(t), x) \geq R(S(t), y) \ \forall y \in S(t)\}$$

With the above notation we obtain the following theorem :

Theorem 2 :- The set valued map $R(S(.), S( )) : T \twoheadrightarrow \mathbb{R}$ is

1) Closed if

i) $R(S(\cdot), \cdot)$ is continuous

ii) S is compact, closed

2) Upper semi continuous if

iii) $R(S(\cdot), \cdot)$ is continuous,

iv) S is upper semi - continuous;

3) Lower semi-continuous if

v) $R(S(\cdot), \cdot)$ is continuous,

iv) S is lower semi - continuous;

4) Compact valued if

vii) $R(S(t), \cdot) : \mathbb{R}_+^2 \to \mathbb{R}$ is upper semi - continuous, for every fixed t T,

viii) S is compact valued.

Here $R(S(\cdot), \cdot) : T \times \mathbb{R}_+^2 \to \mathbb{R}$ is a parametrized criterion function.

Proof :- For the first statement, let $\{(t_\alpha, y_\alpha)\}$ be a net from the graph of $R(S(\cdot), S(\cdot))$ converging to (t_0, y_0), some $t_0 \in T$. We have to show that $y_0 \in R(S(t_0), S(t_0))$; i.e. there exists $x_0 \in S(t_0)$ such that $y_0 = R(S(t_0), x_0)$.

Let $y_\alpha = R(S(t_\alpha), x_\alpha)$ where $x_\alpha \in S(t_\alpha)$. By condition (ii), it can be assumed that $\{x_\alpha\}$ converges to some $x_0 \in S(t_0)$. Since R is continuous, $y_0 = R(S(t_0), x_0)$.

For the second statement, let V be a neighbourhood of $R(S(t_0); S(t_0))$ in $\mathbb{R}$.

We have to find a neighbourhood $\mathcal{U}$ of t_0 in T such that

$R(S(t), S(t)) \subseteq V$, for all $t \in \mathcal{U}$.

In view of (iii), for each $x \in S(t_0)$, there neighbourhoods $A(x)$ of t_0 in

T, $C(x)$ of x in $\mathbb{R}^2_+$ such that

$$\bigcup_{t \in A(x)} R(S(t), C(x)) \subseteq V.$$

Since $S(t_o)$ is compact, one can find a number of points, say $x_1, \ldots, x_n$ in $S(t_o)$ such that $\{C(x_1), \ldots, C(x_n)\}$ is an open cover of it. Denote by C the union of $C(x_1), \ldots, C(x_n)$. It is an open neighbourhood of $S(t_o)$ in $\mathbb{R}^2_+$. By the upper semi continuity of $S(\cdot)$, there is a neighbourhood A_o of t_o in T such that

$$S(A_o) \subseteq C$$

Take now, $\mathcal{U} = A_o \cap A(x_1) \cap \ldots \cap A(x_n)$,

$$R(S(t), S(t)) \subseteq V, \text{ for all } t \in \mathcal{U}.$$

Further, for 3), let V be a neighbourhood in $\mathbb{R}$ with

$V \cap R(S(t_o), S(t_o)) \neq \phi$, i. e.

$R(S(t_o), x_o) \in V$, for some $x_o \in S(t_o)$

By condition (V), there are neighbourhoods A_o of t_o in T, and C_o of x_o in $\mathbb{R}^2_+$ such that

$$R(S(t), x) \in V, \text{ for all } t \in A_o, x \in C_o.$$

Since $S(\cdot)$ is lower semi continuous, for the given C_o there is a neighbourhood A of t_o in T such that

$$C_o \cap S(t) \neq \phi, \text{ for all } t \in A.$$

Take $\mathcal{U} = A \cap A_o$ and obtain

$$V \cap R(S(t), S(t)) \neq \phi \text{ for all } t \in \mathcal{U},$$

i. e. $R(S(\cdot), S(\cdot))$ is lower semi - continuous,

The last statement is trivial.

We are now in a position to state and prove an important theorem of this pa

Theorem 3 :- Let $F : B \twoheadrightarrow \mathbb{R}^2_+$ be a bargaining solution defined with respect t criterion function $R : B \times \mathbb{R}^2_+ \to \mathbb{R}$, i. e. $\forall$ $S \in B$,

$$F(S) = \{x \in S / R(S, x) \geq R(S, y) \; \forall \; y \in S\}$$

Then, the map $F(S(\cdot)) : T \twoheadrightarrow \mathbb{R}^2_+$ is

1) Closed if $R(S(\cdot), S(\cdot))$ is closed and lower semi continuous

2) Upper semi continuous if $R(S(\cdot), S(\cdot))$ is continuous and (hence) compact valued.

Proof :- For the first statement, let $\{(t_\alpha, y_\alpha)\}$ be a net from the graph of $F(S(\cdot))$, converging to (t_o', y_o), $t_o \in T$. We have to prove

$$y_o \in F(S(t_o))$$

Indeed by the closedness of $R(S(\cdot), S(\cdot))$, $\exists x_o \in R(S(t_o), S(t_o))$, such that $x_o = R(S(t_o), F(S(t_o))$ If $y_o \notin F(S(t_o))$ there is some $z \in R(S(t_o), S(t_o))$ such that $z > x_o$. Since $R(S(\cdot), S(\cdot))$ is lower semi continuous, for $z \in R(S(t_o), S(t_o))$, there is a net $\{z_\alpha\}$, $z_\alpha \in R(S(t_\alpha), S(t_\alpha))$ such that

$$\lim z_\alpha = z.$$

It follows fr om this limit, that for α large $z_\alpha > R(S(t_\alpha), y_\alpha)$, contradicting the fact that $y_\alpha \in F(S(t_\alpha))$

For the second statement, suppose to the contrary that there is a neighbourhood V of $F(S(t_o))$ in $\mathbb{R}^2_+$ and a net $\{(t_\alpha, y_\alpha)\}$ from the graph of $F(S(\cdot))$ such that

$$\lim t_\alpha = t_o \in T, \; y_\alpha \notin V.$$

Owing to the fact that $R(S(\cdot), S(\cdot))$ is compact valued we may assume that

$R(S(t_\alpha), y_\alpha)$ converges to some $R(S(t_o), y_o) = z_o$, where $\{y_\alpha\}$ converges to y_o. It is easy to verify that $R(S(\cdot), S(\cdot))$ is closed, hence so is $F(S(\cdot))$ by the first statement. Consequently we arrive at the contradiction

$$y_o \in F(S(t_o)) \subseteq V.$$

Combining theorems 2 and 3, we obtain the main theorem of our paper.

Theorem 4 :- Let $F : B \to\to \mathbb{R}^2_+$ be a bargaining solution defined with respect to a criterion function $R : B \times \mathbb{R}^2_+ \to \mathbb{R}$ i. e. $\forall S \in B$,

$$F(S) = \{x \in / R(S, x) \geq R(S, y) \; \forall y \in S\}.$$

Then, the map $F(S(\cdot)) : T \to\to \mathbb{R}^2_+$ is

1) closed if

(i) $R(S(\cdot),\cdot) : T \times \mathbb{R}^2_+ \to \mathbb{R}$ is continuous

(ii) $S : T \to B$ is lower semicontinuous, compact, closed

2) upper semi-continuous if

(iiii) $R(S(\cdot),\cdot) : T \times \mathbb{R}^2 \to \mathbb{R}$ is continuous

(iv) $S : T \to B$ is continuous

Proof :- Follows immediately by combining theorems 2 and 3.

The conventional theory of bargaining treats $F : B \twoheadrightarrow \mathbb{R}^2_+$ as a point valued function (see for instance Kalai (1986)). In such cases we write $F : B \to \mathbb{R}^2_+$ and have the following significant restatement of the above theorem.

Theorem 5 :- Let $F : B \to \mathbb{R}^2_+$ be a bargaining solution defined with respect to a criterion function $R : B \times \mathbb{R}^2_+ \to \mathbb{R}$ i. e. $\forall S \in B$,

$$F(S) = \underset{x \in S}{\operatorname{argmax}} \; R(S, x).$$

Then, the map $F(S(\cdot)) : T \to \mathbb{R}^2_+$ is continuous if $R(S(\cdot),\cdot) : T \times \mathbb{R}^2_+$ is continuous and $S : T \to B$ is continuous.

Proof : Follows immediately from Theorem 4.

6) Conclusion :- The continuity properties analysed above are fairly general. some connection with related literature deserve mentioning.

In Forgo [1981] game theoretical treatment of multicriteria decision making problems have been analysed using a framework which our paper generalizes. Our analysis would be valid in that framework too.

Continuity properties similar to the ones we have established are a necessary first step to more substantive analysis. Now we can rightly pose the question: By <u>how much</u> does the solution set to a bargaining problem change if the game is perturbed slightly ? We have merely shown that small perturbations yield small changes. But a measure of this change would be desirable and hence a possible ground for further research.

A possible application of the results discussed above arises in problems of fair division of a fixed supply of resources amongst two players. By varying the resources continuously, we could trace out a path of allocations between two agents. The consequence of such a procedure is now obvious in light of the above results.

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