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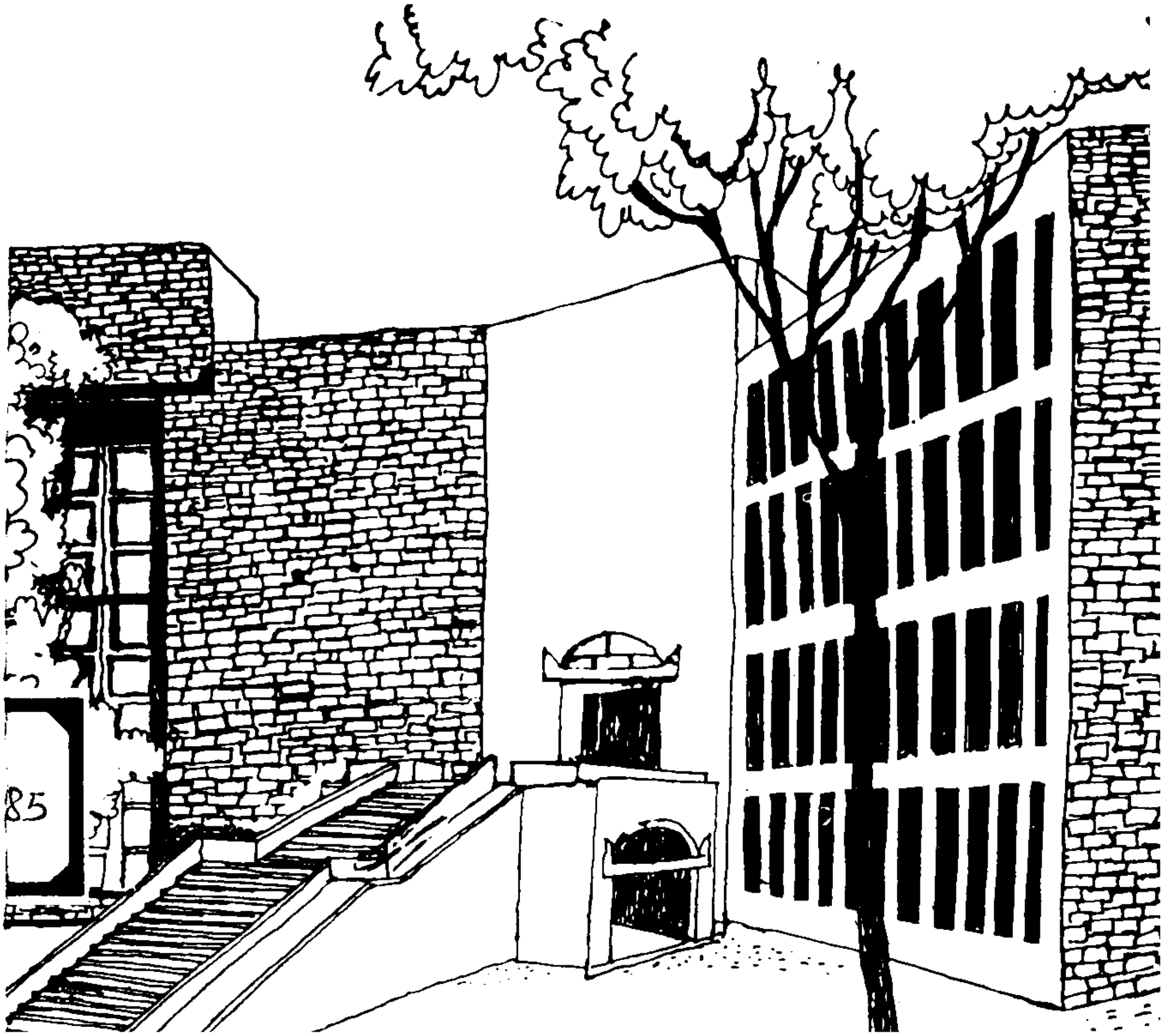

A DIFFERENTIAL GAME IN ADVERTISING

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A B S T R A C T

A general dynamic oligopolistic price-advertising model is formulated and open-loop nash solutions are derived. A detailed discussion of long run equilibrium solutions is given. Conditions for global stability are discussed.

1. Introduction:- In the present paper we analyze optimal advertising and quantity strategies of firms acting in a market with few sellers and differentiated products (heterogenous oligopoly) and state conditions under which rational firm behaviour (in the sense of intertemporal profit maximization) obtains for the dynamic oligopoly case.

Schmalensee (1972) treats a dynamic oligopoly model, but he only considers the question of optimal advertising policies and assumes a constant price for all firms (monopoly price). The aim of our paper is to study the case of oligopolistic price and advertising competition where we assume that the firms own sales and advertising policies are the instruments under the firms control to influence equilibrium prices of the products. Thus, we are focusing on the question of price and advertising determination in industries selling heterogenous products and analyze it by using a dynamic analytic model.

Fershtman and Kamien (1984) solved a similar duopoly model to the one described by us. Instead of dealing with a heterogenous market they discuss a homogeneous one and focus on the price adjustment speed of the unique market price.

The economic intuition behind our framework comes from the superposition of two distinct areas of research. The first is the dynamic advertising model developed by Nerlove and Arrow(1962) where the goodwill of a firm accumulates in proportion to its advertising outlay minus the depreciation due to forgetting and other reasons. This theme has been developed by Fershtman and Muller(1983) and surveyed extensively by Sethi(1977). The second is the strand of research originating in the work of Arrow and Hurwicz(1958), developed in detail by Arrow and Hahn(1971) on the tantonment process. The Walrasian price adjustment mechanism is synthesized with a dynamic advertising model to provide an adequate framework for dynamic oligopoly.

In terms of differential games, we choose to formulate an open loop solution although it is known to have some limitations; see Spence (1979) or Kydland(1977). The closed loop solutions, however, are known to exist only with severe limitations on the structure and duration of the game, for example, Reinganum(1982). The existence and qualitative properties of zero sum differential games has been extensively investigated. For a review and summary of this line of research, see Friedman(1974).

For open loop, nonzero sum, differential games, Scalzo(1974) first proved existence for any finite duration. Proofs of existence prior to his work were known only for "small" duration. Scalzo's work has been extended by Wilson(1977) and Williams(1980) to games with incomplete information and by Scalzo and Williams(1967) to games with non-linear state equations. All three extensions dealt with the finite horizon case.

In order to construct analytical models of oligopoly markets it is necessary to make assumptions on the behaviour of firms due to actions of their competitors, ie. to make assumptions on their reaction functions. According to the theory of differential games and the solution concepts available we choose the Nash equilibrium approach.

2. The Model

We now formulate the dynamic oligopoly model. Let $N \geq 2$ be the number of firms producing a differentiated product. Assume that at every moment of time there is an equilibrium demand of player i, $f^i(p_1, \ldots, p_N, d_1, \ldots, d_N)$, depending on the prices p_i set by the oligopolists and their advertising rates a_i. We suppose that for $i = 1, \ldots, N$

$$f^i_{p_i} < 0, \quad f^i_{p_j} > 0 \quad \text{for } j \neq i \tag{1a}$$

$$f^i_{a_i} > 0, \; f^i_{a_i a_i} < 0, \; f^i_{a_j} < 0 \text{ for } j \neq i \tag{1b}$$

Further we assume that the determinants of the matrices M_α and M_p defined by

$$M_\alpha = \begin{bmatrix} f^1_{\alpha_1 \alpha_1} & \cdots\cdots & f^1_{\alpha_1 \alpha_N} \\ \vdots & & \\ f^N_{\alpha_N \alpha_1} & \cdots\cdots & f^N_{\alpha_N \alpha_N} \end{bmatrix}, \quad M_p = \begin{bmatrix} f^1_{p_1} & \cdots\cdots & f^1_{p_N} \\ \vdots & & \\ f^N_{p_1} & \cdots\cdots & f^N_{p_N} \end{bmatrix} \tag{1c}$$

have the sign

$$\text{sign}\,(\det M_\alpha) = \text{sign}(\det M_p) = \text{sign}(-1)^N \tag{1d}$$

Thus, in the case of $N = 2$ we have $\det M_\alpha > 0$, $\det M_p > 0$ (cf. Thepot(1983)).

The assumptions(1a,b) are classical ones implying that advertising expenditures are subject to decreasing returns. Assumption (1d) states that a firm's price change(advertising variation, respectively) has a higher impact on its equilibrium demand than on that of the competitors (ie. the "direct" effects of a price (advertising) variation are globally stronger than the "indirect" ones). Notice that for example in a two player model $f^i_{p_j} + f^i_{p_i} < 0$ implies det $M_p > 0$.

The intertemporal sales strategy of the i^{th} firm is denoted by $s_i(.)$. The sales strategy may or may not correspond to the percieved or equilibrium demand because of costs of change or costs of breaking habits. However we postulate a simple price adjustment mechanism

$$\dot{p}_i = \alpha_i \left[f^i(p_1, \ldots, p_N, a_1, \ldots, a_N) - s_i\right] \qquad (2)$$

of the usual tatonment variety. Here, α_i is a positive constant adjustment coefficient of player i. Moreover the production costs $c_i(s_i)$ are assumed to be convex with

$$c_i(0) = 0,\ c_i'(0) = 0,\ c_i'(s_i) > 0 \text{ for } s_i > 0,\ c_i''(s_i) > 0 \qquad (3)$$

Finally, k_i are the constant unit costs of advertising, and r_i denotes the discount rate of the i^{th} firm in the oligopoly.

Player i faces the problem of choosing the time paths of s_i and a_i so as to maximize the present value of the stream of profits

$$J_i = \int_0^\infty e^{-r_i t} \left[p_i s_i - c_i(s_i) - k_i a_i \right] dt \tag{4}$$

under the system dynamics(2), where the initial price $p_i(o)=p_{io} \geq o$ is given. [redacted].

3. Optimality Conditions:-

To calculate open - loop nash solutions for the differential game define in Section 2, we define the current-value Hamiltonian for player i(ef. eg. Bazar and Olsder(1982)):

$$H^i = p_i s_i - c_i(s_i) - k_i a_i + \sum_{j=1}^{N} \lambda_j^i \alpha_j (\rho^j - s_j)$$

The adjoint variable λ_j^i does not here have the usual interpretation of a shadow price of current sales. The model with quantity adjustments as studied by Dockner and Feichtinger (1985) allows such an interpretation. This is one of the few sacrifices we have to make for increased realism in the advertising policy that we propose to obtain, by departing from the tradition followed by our predecessors. From now on unless otherwise stated we restrict ourselves to the case of N = 2 firms.

Necessary conditions for an open-loop nash equilibrium are given by the adjoint equations (See(5)) and the Hamiltonian maximizing conditions (See(6)).

$$\dot{\lambda}_i^i = r_i \lambda_i^i - H_{p_i}^i = r_i \lambda_i^i - s_i - \sum_{j=1}^{N} \lambda_j^i \alpha_j \rho_{p_i}^j \tag{5}$$

$$\dot{\lambda}_j^i = r_i \lambda_j^i - H_{p_j}^i = r_i \lambda_j^i - \sum_{k=1}^{N} \lambda_k^i \alpha_k \rho_{p_j}^k \quad \text{for } j \neq i$$

It turns out as in Kamien and Schwarz(1981) (pp.122-123) that the costate variables are redundant, ie. $\lambda^i_j = 0$ for $j \neq i$. Thus, we get by definition $\lambda_i = \lambda^i_i$

$$H^i = p_i s_i - c_i(s_i) - k_i a_i + \lambda_i \alpha_i (f^i - s^i_i).$$

The maximizing conditions for interior solutions are

$$H^i_{s_i} = \quad p_i - c_i'(s_i) - \alpha_i \lambda_i = 0 \qquad (6a)$$

$$H^i_{a_i} = -k_i + \lambda_i \alpha_i f^i_{a_i} = 0 \qquad (6b)$$

To get a maximum of the Hamiltonian with respect to p_i and a_i we normally need to assume that the second order conditions are met. However this is guaranteed by our assumptions already made above:

$$H^i_{s_i s_i} = -c_i''(s_i) < 0, \; H^i_{a_i a_i} = {}_i a_i f^i_{a_i a_i} < 0$$

$$\text{and } H^i_{s_i s_i} H^i_{a_i a_i} - (H^i_{s_i a_i})^2 = -c_i''(s_i) . \lambda_i \alpha_i f^i_{a_i a_i} - 0 <$$

We merely require the concavity of the notional demand function in a_i. We do not require the concavity of notional demand in prices. Hence a larger class of notional demand functions get accommodated in our framework. Wirl(1985) studies the implications of concave and convex price-dependent demand functions respectively but for the monopoly case. It turns out that convex demand functions with quantity adjustments lead to volatile prices (ie. a "zig-zag" policy). As mentioned earlier, it is an easy check to verify that the sufficient conditions for optimality are satisfied.

From (6b) we get the result that

$$\lambda_i \geqq 0 \text{ for } t \in [0, \infty] \qquad (7)$$

which is a criteria required for the necessary conditions to prove sufficient for optimality.

From (6a) and (6b) we obtain the following theorem.

<u>Theorem 1</u>:- Firm 'i' will adopt an advertising policy which equates its own equilibrium marginal productivity of advertising to the ratio of unit advertisement costs and the difference between the price of its product and the marginal cost ie.

$$f^i_{a_i} = \frac{k_i}{p_i - c_i'(s_i)} \qquad (8)$$

With concave neoclassical percieved demand functions for advertising, satisfying Inada conditions, advertisement outlay reduce to zero if price equals marginal costs, thus corroborating what is to be expected of perfectly competitive price taking behaviour as a special case. A departure from the price equals to marginal cost rule will enhance the necessity for advertising.

It is interesting to note that the parallel Dorfman-Steiner-Theorem (see Dorfman and Steiner(1954)), which states that firm 'i' will advertise until the ratio of advertising expenditures to revenue is equal to the ratio of advertising elasticity of equilibrium demand to price elasticity of equilibrium demand is not quite consistent with the requirements of general competitive analysis, if and when they do hold.

4. A Special Case:- In this section we consider the case where equilibrium demand is only a function of advertising. Thus we suppress the price dependency and study optimal advertising strategies. This can be justified by an argument of Schmalensee(1976) who write that "it is generally accepted 'stylized fact' that price competition is relatively rare in markets with few sellers and differentiated products" (Schmalensee(1976), p.493).

In this case, our necessary conditions reduce to

$$\dot{\lambda}_i = r_i \lambda_i - s_i \tag{9}$$

$$p_i - c_i'(s_i) - \alpha_i \lambda_i = 0 \tag{10}$$

$$-k_i + \lambda_i \alpha_i f^i_{a_i} = 0 \tag{11}$$

Theorem 2:- At a steady state ie. $\dot{p}_i = 0 = \dot{s}_i$,

$$s_i = \frac{r_i k_i}{\alpha_i f^i_{a_i}} \tag{12}$$

Proof:- From (10)

$$\dot{p}_i(t) = c_i''(s_i) + \alpha_i \dot{\lambda}_i = c_i''(s_i)\dot{s}_i + \alpha_i \left[r_i \lambda_i - s_i \right]$$

From (11)

$$\dot{p}_i(t) = c_i''(s_i)\, \dot{s}_i + \alpha_i \left[\frac{r_i k_i}{\alpha_i f^i_{a_i}} - s_i \right]$$

Now set $p_i = 0 = \dot{s}_i$

$$\therefore \frac{r_i k_i}{\alpha_i f^i_{a_i}} = s_i$$ as was required to be proved.

Observing that

$$f^i_{a_i} = \frac{k_i}{p_i - c_i'(s_i)}$$ as in Theorem 1,

we obtain,

$$s_i = \frac{r_i\,(p_i - c_i'(s_i))}{\alpha_i} \tag{13}$$

Assuming constant marginal costs, we get an interesting steady state sales strategy:

$$s_i = \frac{r_i\,(p_i - c_i)}{\alpha_i} \tag{14}$$

where $c_i(s_i) = c_i s_i$, c_i being the unit cost of production.

This result postulates that, the equilibrium steady state sales strategy is directly proportional to the difference between price and unit cost of the product. Thus, if prices equal unit costs as in the competitive case, then there are no sales; this bears out a reasonable hunch that any incentive to sell in an oligipolistic market, comes from a mark-up on unit costs. It may be worthwhile to observe that if f exhibits the neoclassical Inada properties, then advertisement outlays reduce to zero if prices equal unit costs. In sum, under assumptions, competitive behaviour in an oligopolistic market, reduces to general inaction on the part of the sellers.

5. Characterization of Stationary Equilibrium Points:-

To study long run equilibrium solutions we compute the steady state of equations (2) and (5)

$$\overset{o}{p}_i = 0 \iff s_i^{\infty} = f^i \qquad (15a)$$

$$\dot{\lambda}_i = 0 \iff \lambda_i^{\infty} = \frac{s_i^{\infty}}{r_i - \alpha_i f^i_{p_i}} \qquad (15b)$$

Substitution of (15a) and (15b) into conditions (6a) and (6b) yield:

$$p_i^{\infty} = c_i' + \frac{\alpha_i s_i^{\infty}}{r_i - \alpha_i f^i_{p_i}} \qquad (16a)$$

$$f^i_{a_i} = \frac{k_i(r_i - \alpha_i f^i_{p_i})}{\alpha_i s_i^{\infty}} \qquad (16b)$$

Now substituting $(r_i - \alpha_i f^i_{p_i})$ from (16b) into (16a) we get

$$p_i^{\infty} = c_i'(s_i^{\infty}) + \frac{\alpha_i s_i^{\infty} k_i}{f^i_{a_i} \alpha_i s_i^{\infty}} = c_i'(s_i^{\infty}) + \frac{k_i}{f^i_{a_i}} \qquad (17)$$

which shows that $f^i_{a_i}$ and hence a_i is a constant. Denote the latter by a_i^{∞}.

If we interpret $\frac{p_i^{\infty} - c_i'(s_i^{\infty})}{p_i^{\infty}}$ according to Lerner's degree of monolopy power as a measure of oligopoly power of firm i (cf. Lerner(1934)), we obtain the condition that in the long-run the oligopoly power of firm 'i' is directly proportional to ratio of its unit cost of advertising, to its marginal productivity of advertising. Further, at a steady state the familiar property of prices being equal to marginal costs (units costs if the cost function is linear) plus a mark up is once again retrieved.

6. Global Stability Results for Steady State Solutions:-

Now we study the global stability properties of nash equilibrium solutions given by the necessary conditions(2),(5) and (6). We show by using a theorem of Haurie and Leitmann(1984) that all uniformly bounded solutions converge to the unique steady state solution defined by (15), (16) and (17). According to Haurie and Leitmann(1984) we use vector Lyapunov functions to derive this global stability result. To simplify the proofs we treat the case of N = 2 players.

By use of the implicit function theorem we get from the maximizing conditions (6a) and (6b) that

$$a_i = \tilde{a}_i\ (p_i, \lambda_i) = (c_i)^{-1}(p_i - \alpha_i \lambda_i) \quad (18)$$

$$a_i = \tilde{a}_i\ (p_1,\ p_2, \lambda_1, \lambda_2) \quad (19)$$

Infact $f^1_{a_1} = \dfrac{k_1}{\lambda_1 \alpha_1}$ yields $a_1 = \bar{a}_1(p_1,\ p_2,\ \lambda_1\ a_2)$

$f^2_{a_2} = \dfrac{k_2}{\lambda_2 \alpha_2}$ yields $a_2 = \bar{a}_2\ (p_1,\ p_2,\ \lambda_2,\ a_1)$

Substituting one into the other gives the required expression.

Thus, the modified Hamiltonian dynamic system is the differential equation system in $(p_1,\ p_2, \lambda_1, \lambda_2)$ given by

$$\dot{p}_1 = \alpha_1\left[f^1(p_1, p_2, \tilde{a}_1, \tilde{a}_2) - \tilde{a}_1\right]$$

$$\dot{p}_2 = \alpha_2\left[f^2(p_1, p_2, \tilde{a}_1, \tilde{a}_2) - \tilde{a}_2\right]$$

$$\dot{\lambda}_1 = r_1\lambda_1 - \tilde{a}_1 - \lambda_1\alpha_1 f^1_{p_1} \qquad (20)$$

$$\dot{\lambda}_2 = r_2\lambda_2 - \tilde{a}_2 - \lambda_1\alpha_1 f^2_{p_2}$$

It is easy to verify that system (20) possesses a unique steady state solution, which we characterized in the preceeding section. We now transform the solution $Z = (p_1^{\infty}, p_2^{\infty}, a_1^{\infty}, a_2^{\infty})$ into $Z' = (0, 0, 0, 0)$. From the theory of Hamiltonian dynamical systems we know that the canonical system of an optimal control problem can never be stable. All that can be expected is the saddle-point "stability". Here we prove that the dynamical modified Hamiltonian system(20) has the saddle-point property. We state the result in the form that all uniformly bounded ~~xkk~~ solutions of system (20) converge to the steady state Z(or Z'), ie. there exists a global manifold such that all solutions emanating from this manifold converge to the steady state.

Theorem 3:- All bounded state and costate solutions corresponding to open-loop nash equilibria must converge to $(p_1^{\infty}, p_2^{\infty}, a_1^{\infty}, a_2^{\infty})$.

The proof is a straight forward application of Lemma 6.1 of Haurie and Leitmann(1984) using the same vector valued Lyapunov function.

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