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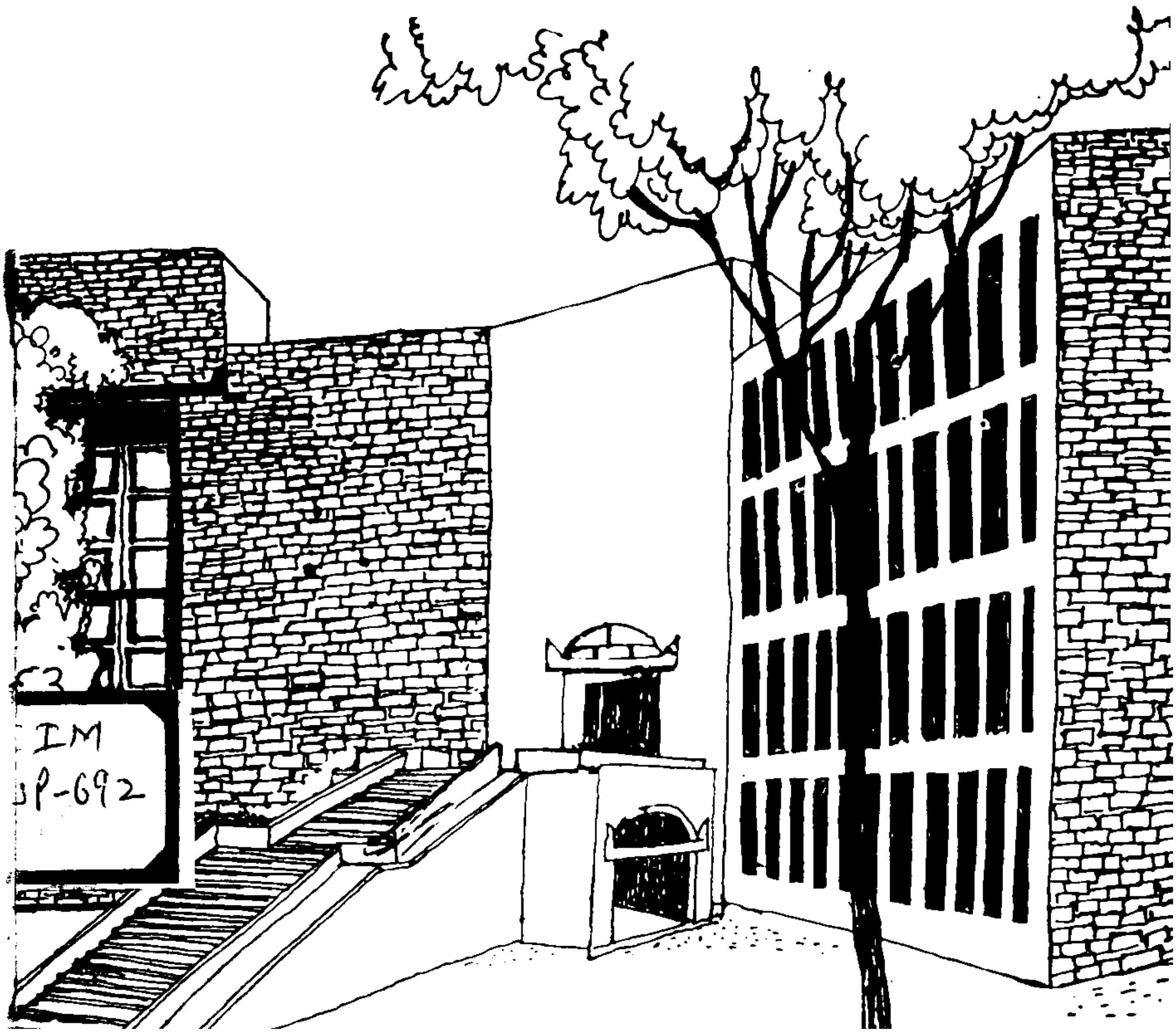

OPTIMUM ORDERING INTERVAL OF INVENTORY WITH RANDOM PRICE FUNCTIONS: A SAMPLE PATH ANALYSIS

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ABSTRACT

In this paper we extend the analysis of optimum ordering interval for inventory, carried out by Mukherjee(2), to incorporate random price schedules observed by firms. We obtain the expression for optimum cycle length by minimizing the expected total cost per unit time. In effect we carry out a sample path analysis. We also study the relationship between optimal interval and probability distributions in the polar case of constant decay rate and a Bernoulli probability measure.

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1. Introduction:- The purpose of this paper is to introduce the notion of a random price function of the simplest type in a model of optimum ordering interval for inventory, as studied by Mukherjee(2). In this sense, the purpose of this analysis is more expository than inventive.

We consider the case of perishable goods with exponential decay as considered by our predecessors. We obtain a solution for the optimum ordering interval which is similar in spirit to the result obtained earlier, with the sole difference being that we include in our environment, random price schedules. With this modification, the optimal ordering intervals are specified in terms of expectations and not as deterministic quantities. We carry out below a sample path analysis of ordering interval for inventory.

2. The Model:- Let (Ω, F, P) be a probability space and $p : \Omega \to \mathbb{R}_+$ be a random variable, which gives the selling price of the commodity for every realization of a state of nature. Let $d(p)$ be the known demand rate when the price is p, so that $d(p(w))$ is the demand rate when the state of nature is $w \in \Omega$. Let $I(t,w)$ be the inventory at time t, corresponding to a state of nature w, $\lambda(t)$ the stock decay rate at time t and shortages are not allowed. In this sense, I is a stochastic process.

The differential equation describing the behaviour of the system is

$$\frac{d}{dt} I(t,w) = -\lambda(t) I(t,w) - d(p(w)) \qquad (1)$$

The solution of this differential equation leads to

$$I(t,w)\exp.\left(\int_0^t \lambda(x)dx\right) = I(0,w) - d(p(w)) \int_0^t \exp.\left(\int_0^y \lambda(x)dx\right) dy \qquad (2)$$

In order to find the expression for $I_w(t,w)$, the inventory process without decay at time t, the differential equation of the system will be

$$\frac{d}{dt} I_w(t,w) = -d(p(w))$$

the solution of which gives

$$I_w(t,w) = I(0,w) - t\ d(p(w)).$$

Therefore $Z(t,w)$, the stock loss process due to decay at time 't', is given by

$$Z(t,w) = I_w(t,w) - I(t,w) = I(0,w) - t\ d(p(w)) - I\ (t,w).$$

On substituting the value $I(0,w)$ from (2) in the above equation, we get

$$Z(t,w) = I(t,w)\left[\exp.\left(\int_0^t \lambda(x)dx\right) - 1\right] - t\,d(p,w) + d(p,w)\int_0^t \exp.\left(\int_0^y \lambda(x)dx\right) dy. \quad (3)$$

If T is the cycle length (ordering interval), the order process, $Q_T(w)$, required to satisfy the demand during a cycle of length T is equal to

$$Q_T(w) = Z(T,w) + Td(p(w)) \quad (4)$$

In case of instantaneous replenishment we have

$$I(0,w) = Q_T(w) \text{ and } E_w\left[I_T(w)\right] = 0.$$

where $E_w[\cdot]$ denotes expectation.

$$\therefore\ Q_T(w) = d(p(w))\left[\int_0^T \exp.\left(\int_0^t \lambda(x)dx\right) dt\right].$$

noting that since $I(o,w) = Q_T(w)$, we get

$$I(t,w) = d(p(w))\exp.\left(-\int_0^t \lambda(x)dx\right)\left[\int_t^T \exp.\left(\int_0^y \lambda(x)dx\right) dy. \quad (5)\right.$$

If the purchase cost per unit, the set-up cost and the unit stock holding cost be denoted by C, K and h, respectively, then for a fixed price level p, the cost per unit time C (T, p) is

$$C(T,p) = \frac{k}{T} + \frac{Cd(p)}{T}\left[\int_0^T \exp.\left(\int_0^t (x)dx\right) dt\right] + \frac{h}{T}\int_0^T d(p)\exp.\left(-\int_0^t \lambda(x)dx\right)$$

$$\left[\int_t^T \exp.\left(\int_0^y \lambda(x)dx\right) dy\right] dt \quad (6)$$

The necessary condition for the optimum cycle T_p is obtained from the solution of $\frac{\partial}{\partial T} E_w \left[C(T, p(w)) \right] = 0$

$$= -\frac{k}{T^2} - \frac{c}{T^2} E_w \left[d(p(w)) \right] \left[\int_0^T \exp.\left(\int_0^t \lambda(x)dx \right) dt \right]$$

$$+ \frac{c}{T} E_w \left[d(p(w)) \right] \left[\exp. \int_0^T \lambda(x)dx \right]$$

$$- \frac{h}{T^2} \int_0^T E_w \left[I(t,w) \right] dt + \frac{h}{T} \cdot \frac{\partial}{\partial T} \int_0^T E_w \left[I(t,w) \right] dt \qquad (7)$$

now, $$\frac{\partial}{\partial T} \int_0^T E_w \left[I(t,w) \right] dt = E_w \left[I(T,w) \right] = 0.$$

$$\therefore 0 = -\frac{k}{\hat{T}^2} - E_w \left[d(p(w)) \right] \frac{c}{\hat{T}^2} \int_0^{\hat{T}} \left(\exp. \int_0^t \lambda(x)dx \right) dt$$

$$+ \frac{c}{\hat{T}} E_w \left[d(p(w)) \right] \left(\exp. \int_0^{\hat{T}} \lambda(x)dx \right)$$

$$- \frac{h}{\hat{T}^2} \int_0^{\hat{T}} E_w \left[I(t,w) \right] dt$$

Therefore, the optimum cycle $\hat{T}$ is given by

$$\hat{T} = \frac{k + C\,E_w\left[d(p(w))\right]\left[\int_0^T \exp.\left(\int_0^t \lambda(x)dx\right) dt\right] + h\int_0^T E_w\left[I(t,w)\right] dt}{C\,E_w\left[d(p(w))\right] \exp. \int_0^T \lambda(x)dx}$$

Since k, C and $d(p) > 0$, $E_w\left[d(p(w))\right] > 0$; and since the integration is taken of an exponential function over the positive range (0, T), we find that $\hat{T} > 0$. Now to find the second order necessary condition that $\hat{T}$ gives the minimum cost function, we find from (7) that

$$T^2 \frac{\partial E_w}{\partial T}\left[C(T,p(w))\right] = -k - C\,E_w\left[d(p(w))\right]\int_0^T \exp\left(\int_0^t \lambda(x)dx\right)dt$$

$$+ C\,E_w\left[d(p(w))\right] T\left[\exp\int_0^T \lambda(x)dx\right]$$

$$- h\int_0^T E_w\left[I(t,w)\right] dt, \qquad (8)$$

$$\therefore 2T\frac{\partial E_w}{\partial T} C(T,p(w)) + T^2\frac{\partial^2 E_w}{\partial T^2} C(T,p(w))$$

$$= -C\,E_w\left[d(p(w))\right] \exp.\left(\int_0^T \lambda(x)dx\right) + C E_w\left[d(p(w))\right]\left[\exp.\int_0^T \lambda(x)dx\right]$$

$$+ TC \exp.\lambda(T)\,E_w\left[d(p(w))\right] - h\,E_w I(T,w)$$

$$= - C E_w \left[d(p(w))\right] \exp.\left(\int_0^T \lambda(x)\,dx\right) + C E_w \left[d(p(w))\right] \left[\exp. \int_0^T \lambda(x)\,dx\right]$$

$$+ TC \exp. \lambda(T)\; E_w \left[d(p(w))\right]$$

Since at $T = \hat{T}$, $\frac{\partial}{\partial T} E_w \left[C(T, p(w))\right] = 0$, therefore

$\frac{\partial^2}{\partial T^2} E_w \left[C(T, p(w))\right] = \frac{C}{T}\, e^{\lambda(T)}\, E_w \left[d(p(w))\right]$ which is >0 at $T = \hat{T}$.

In order to find the optimum price policy $p : \Omega \longrightarrow \mathbb{R}_+$ for a fixed period length we maximize the expected profit rate

$$E_w\left[f(T, p(w))\right] = E_w \left[p(w)\, d(p(w))\right] - E_w \left[C(T, p(w))\right] \quad (9)$$

By holding T fixed, we can find the necessary condition for optimal price policy $P_T : \Omega \longrightarrow \mathbb{R}_+$ from the solution of $\frac{\partial}{\partial p} \left[f(T, p(w))\right] = 0$, which gives the optimal policy $\hat{P}_T : \Omega \longrightarrow \mathbb{R}_+$ as follows:

$$\left[d(p(w))\right] + \left[p(w)\, d'(p(w))\right] - \left[\frac{d'(p(w))C}{T}\left(\int_0^T \exp\left(\int_0^t \lambda(x)\,dx\right) dt\right]\right.$$

$$- \left(\frac{h}{T}\, d'(p(w)) \int_0^T \exp.\left(- \int_0^t \lambda(x)\,dx\right)\left[\int_t^T \exp\left(\int_0^y \lambda(x)\,dx\right) dy\right] dt\right.$$

$$= 0.$$

$$\text{or } \left[d'(\hat{P}_T(w))\right] \left\{ -\frac{c}{T}\left[\int_0^T \exp. \int_0^t \lambda(x)\,dx\right] dt \right.$$

$$\left. - \frac{h}{T}\int_0^T \exp.\left(-\int_0^t \lambda(x)\,dx\right)\left[\int_t^T \exp.\left(\int_0^y \lambda(x)\,dx\right) dy\right] dt \right\}$$

$$+ \left[\hat{P}_T(w)\; d'(\hat{P}_T(w))\right] + \left[d(\hat{P}_T(w))\right] = 0$$

$$\text{Let } A_T = +\frac{c}{T}\left[\int_0^T \exp.\left(\int_0^t \lambda(x)\,dx\right) dt\right] + \frac{h}{T}\int_0^T \exp.\left(-\int_0^t \lambda(x)\,dx\right)\left[\int_t^T \exp.\left(\int_0^y \lambda(x)\,dx\right) dy\right] dt\Bigg]$$

$$\therefore\; - \left[d'(\hat{P}_T(w))\right] A_T + \left[\hat{P}_T(w)\; d'(\hat{P}_T(w))\right] + \left[d(\hat{P}_T(w))\right] = 0$$

In order to find the second order necessary condition that $\hat{P}_T : \Omega \longrightarrow \mathbb{R}_+$ is locally the profit maximizing price policy, we find

$$\frac{\partial^2}{\partial p^2}\left[f(T, p(w))\right] \text{ at } \hat{P}_T(w) \text{ is } \frac{\partial}{\partial p}\left\{\left[d(\hat{P}_T(w))\right] + \left[\hat{P}_T(w)\; d'(\hat{P}_T(w))\right] - \left[d'(\hat{P}_T(w))\right] A_T\right\}$$

$$= \left[d'(\hat{P}_T(w)) - d''(\hat{P}_T(w))\,A_T + \hat{P}_T(w)d''(\hat{P}_T(w)) + d'(\hat{P}_T(w))\right]$$

$$= \left[2d'(\hat{P}_T(w)) + \hat{P}_T(w)d''(\hat{P}_T(w)) - d''(\hat{P}_T(w))\,A_T\right]$$

$$= 2d'(\hat{P}_T(w)) - \frac{d(\hat{P}_T(w))}{d'(\hat{P}_T(w))}\,d''(\hat{P}_T(w)) \leq 0$$

This inequality is surely satisfied if $d'(p) < 0$ and $d''(p) < 0$.

3. An Example:- Let us consider as in Cohen (1), $\lambda(t)$ to be a constant function.

So,

$$C(T,p(w)) = \frac{k}{T} + \frac{Cd(p(w))}{T}\left[\int_0^T e^{\lambda t}dt\right] + \frac{h}{T}\int_0^T d(p(w))e^{-\lambda t}\left(\int_t^T e^{\lambda y}dy\right) dt$$

$$= \frac{k}{T} + \frac{Cd(p(w))}{\lambda T}(e^{\lambda T}-1) + \frac{h}{T}d(p(w))\int_0^T \frac{e^{-\lambda t}}{\lambda}(e^{\lambda T}-e^{\lambda t})dt$$

$$= \frac{k}{T} + \frac{Cd(p(w))}{\lambda T}(e^{\lambda T}-1) + \frac{h}{\lambda T}d(p(w)\left[\frac{e^{\lambda T}}{-\lambda}(e^{-\lambda T}-1) - T\right]$$

$$= \frac{k}{T} + \frac{Cd(p(w))(e^{\lambda T}-1)}{\lambda T} - \frac{he^{\lambda T}}{\lambda^2 T}\,d(p(w))\left[e^{-\lambda T}-1+\lambda Te^{-\lambda T}\right]$$

Let $\Omega = \{w_1, w_2\}$, $P(w_1) = \theta$, $P(w_2) = 1 - \theta$

$$\therefore \; E_w\left[C(T,p(w))\right] = \frac{k}{T} + \left[\frac{C(e^{\lambda T}-1)}{\lambda T} - \frac{he^{\lambda T}}{\lambda^2 T}\left\{e^{-\lambda T}-1+\lambda T e^{-\lambda T}\right\}\right]\left[\theta d(p(w_1)) + (1-\theta)\, d(p(w_2))\right]$$

$\therefore$ The first order necessary condition for optimal ordering interval requires

$$0 = -\frac{k}{T^2} + \left[-\frac{C}{\lambda T^2}(e^{\lambda T}-1) + \frac{Ce^{\lambda T}}{T} + \frac{he^{\lambda T}}{\lambda^2 T^2}\left\{e^{-\lambda T}-1+\lambda T e^{-\lambda T}\right\}\right]\left[\theta d(p(w_1)) + (1-\theta) d\,(p(w_2))\right]$$

because $E_w\left[I_T(w)\right] = 0$

$$\text{or } 0 = -k + \left[-\frac{C}{\lambda}(e^{\lambda T}-1) + TCe^{\lambda T} + \frac{he^{\lambda T}}{\lambda^2}\left\{e^{-\lambda T}-1+\lambda T e^{-\lambda T}\right\}\right]\left[\theta d(p(w_1)) + (1-\theta) d(p(w_2))\right]$$

$$\text{or } 0 = -k + \left[-\frac{C}{\lambda}(e^{\lambda T}-1) + TCe^{\lambda T} + \frac{h}{\lambda^2}\left\{1+\lambda T-e^{\lambda T}\right\}\right]\left[\theta d(p(w_1)) + (1-\theta) d(p(w_2))\right]$$

$$= -k - \left[\frac{C}{\lambda}(e^{\lambda T}-1) - TCe^{\lambda T} - \frac{h}{\lambda^2}\left\{1+\lambda T-e^{\lambda T}\right\}\right]\left[\theta d(p(w_1)) + (1-\theta)\, d\,(p(w_2))\right]$$

Let us consider the truncated Taylor series approximation of $e^{\lambda T}$ i.e. $e^{\lambda T} \simeq 1 + \lambda T$. Then we get,

$$0 = -k - \left[\frac{C}{\lambda}(\lambda T) - TC(1+\lambda T)\right]\left[\theta d(p(w_1)) + (1-\theta)d\,(p(w_2))\right]$$

$$= -k - \left[-\lambda T^2 C\right]\left[\theta d(p(w_1)) + (1-\theta)d(p(w_2))\right]$$

$$= -k + \lambda T^2 C\left[\theta d(p(w_1)) + (1-\theta)d(p(w_2))\right]$$

$$\therefore \hat{T}^2 = \frac{k}{\lambda C\left[\theta d(p(w_1)) + (1-\theta)d(p(w_2))\right]}$$

A comparative static analysis of the above optimal ordering time proceeds as follows:

$$2\hat{T}\,\frac{d\hat{T}}{d\theta} = \frac{-k\left[\lambda C\left\{d(p(w_1) - d(p(w_2))\right\}\right]}{\lambda^2 C^2\left[\theta d(p(w_1)) + (1-\theta)d(p(w_2))\right]^2}$$

$$\therefore \text{sign}\left(\frac{d\hat{T}}{d\theta}\right) = \text{sign}\left(d(p(w_2)) - d\,(p(w_1))\right)$$

If $p(w_2) > p(w_1)$ then $d\,(p(w_2)) - d(p(w_1)) < 0$ under our assumptions.

$\therefore \frac{d\hat{T}}{d\theta} < 0$ i.e. an increase in the probability of occurrence of w_1 and hence in the lower price possibility, decreases the optimum ordering interval. In such a situation more frequent ordering of inventory becomes necessary. Conversely, an increase in the probability of occurrence of the higher price possibility increases the optimum ordering interval. In such a situation less frequent ordering of inventory is necessary.

This observation is plansible in light of the fact that a lower price possibility is associated with a higher demand and hence a requirement for more frequent ordering of the inventory. This adduces to a partial validation of our model.

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