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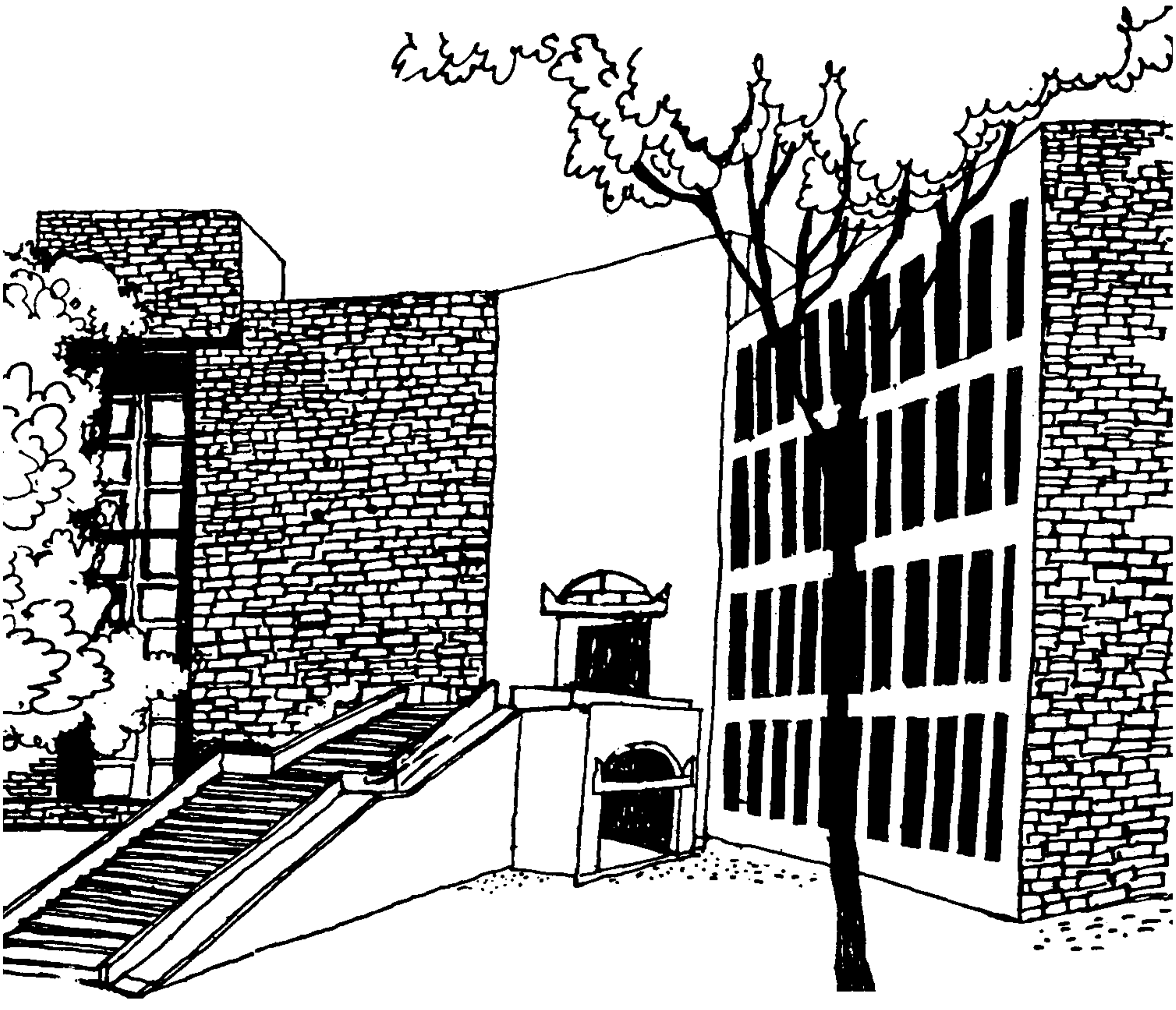

# DISTRIBUTIVE JUSTICE WITH EXTERNALITIES AND PUBLIC GOODS

By

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## Abstract

We analyze in this paper the distribution of a fixed amount of perfectly divisible private goods among a fixed number of agents and with a certain portion of the private goods allocated for the creation of public goods. Each agent's preferences exhibit a type of consumption externality made precise in the paper, and we focus our attention on the existence of efficient and egalitarian allocations of the goods.

**1. Introduction** : Following Villar (1988) and Nieto (1991), we analyze in this paper the distribution of a fixed amount of perfectly divisible private goods among a fixed number of agents and with a certain portion of the private goods allocated for the creation of public goods. Each agent's preferences exhibit a type of consumption externality made precise in the paper, and we focus our attention on the existence of efficient and egalitarian allocations of the goods. Our paper and the results contained therein are a generalization of the private good model formulated by Villar (1988) and an appropriate modification of the public good economy discussed in the same paper, to accommodate additional economic insights and realism.

## 2. Notion and definitions :

Let there be l perfectly divisible (private) goods to be distributed among n individuals (denoted generically by i, j, ...) and suppose there are t perfectly divisible (public) goods which are produced using the private good. The consumption of this last set of goods is assumed to be the same for all the individuals. Each individual is completely represented by his/her preferences over bundles of goods and their distribution. Let G be a compact subset of $\mathbb{R}^t_+$. We assume that the consumption levels of the public goods are restricted to belong to the set G. This set is invoked both for mathematical convenience and for additional realism.

We say that $(x, g) \in \mathbb{R}^{ln}_+ \times G$ is an allocation, i.e., a distribution of l goods among the n agents and a level of consumption of the public goods; $x_i$ will denote the quantities of private goods attributed to agent i under allocation (x, g).

Preferences of individual i are represented by his/her utility function

$u_i : \mathbb{R}^{ln}_+ \times G \to \mathbb{R}$

A vector of utility functions, $\{u_i\}_{i=1,\ldots,n}$, one for each individual, is denoted by u, and called a <u>utility profile</u>.

Let $c: G \to \mathbb{R}^l_+$ be a function which denotes for each vector of public goods, the amount of each private good, required for its production. c can be conceived as a generalized cost function or an <u>input requirement function</u>. We assume that such a function exists.

A distribution problem is a pair [u,c] where u is a utility profile and c is an input requirement function for the production of public goods.

**Remark 1** : Individual preferences are defined over entire allocations (the domain of a utility function is $\mathbb{R}_+^{ln}\times G$). This assumption leaves room for externalities in consumption.

Conventions

(i) Vector inequalities are $\gg, >$ and $\geq$.

(ii) $0_m$ will denote the m-dimensional vector of zeroes, $(0,\ldots,0)$.

Following Villar (1988) and Nieto (1991) we introduce the notion of egalitarian allocations.

**Definition 1** :- An allocation $(\bar{x},g)\in\mathbb{R}_+^{ln}\times G$ is called egalitarian for a given profile u if, for all agents i, j, $u_i(\bar{x},g)>u_j(\bar{x},g) \Rightarrow \bar{x}_i=0_l$.

Interpretation : When allocation $(\bar{x},g)$ takes place, either every individual is equally well off or, if someone is better off, then she/he receives nothing (of all private goods).

The set of egalitarian allocations for a given distribution problem is denoted by E[u,c].

We say that $(\tilde{x},g)\in E[u,c]$ is strictly egalitarian if, $u_i(\tilde{x},g) = u_j(\tilde{x},g)$ for all i, j.

**Definition 2** : An allocation (x,g) in [u,c] is Pareto optimal (or efficient) if there is no allocation (x',g'), with $\sum_i x'_i+c(g')\leq\sum_i x_i+c(g)$, such that $u(x',g')>u(x,g)$.

The set of Pareto-optimal allocations for [u,c] is denoted by P[u,c].

If there is no (x',g') with $\sum_i x_i+c(g')\leq\sum_i x_i+c(g)$, such that $u(x',g')\gg u(x,g)$ then we call (x,g) weakly Pareto-optimal (or weakly efficient). The set of weakly Pareto-optimal allocations for [ u,c,] is denoted by WP[ u,c].

Consider now the following assumptions :

A.1 :- $u_i: \mathbb{R}_+^{nl} x G \to \mathbb{R}$ is a continuous function, for all i.

A.2 :- Let $(x,g),(y,g') \in \mathbb{R}^{nl} xG$ be such that $x>y$ and $g=g'$. If $x_i=y_i$, we have $u_i(x,g) \leqq u_i(y,g')$, $i=1,2,...,n$.

A.3 :- For any $v \in \mathbb{R}_+^n$ there exists $(x,g) \in \mathbb{R}_+^{nl} xG$ such that $u(x,g) \geqq v$.

Assumption A.1 is standard; it implies that the sets of preferred alternatives are opan.

Assumption A.2 says that when an allocation changes to a situation whre some agents get more goods whilst others get the same, the latter will not be happier.

Assumption A.3 says that more public good is preferred to less.

Assumption A.4 says that any predetermined vector of utility values can be reached, provided there are enough amounts of goods.

Let $v \in \mathbb{R}^n$ be a given vector of utility values. The problem of finding the amounts of goods and their corresponding distribution so that those utility levels are actually reached, can be formalized as the search for a solution to the following system :

(i) $u(x,g) \geqq v$,

(ii) $u_i(x,g) > v_i$ implies $x_i = o_i$,

(iii) $x \geqq o_{ln}$.

A solution to system (1) gives us a distribution of goods such that all agents achieve their target utility levels, with one proviso: if some agent ends up with utility greater than his/her component $\mathcal{V}_i$, then she must receive no goods.

In addition to the above assumptions on preferences, we make the following assumption on the input requirement function :

A.4 :- $c: G \to \mathbb{R}^l_+$ is continuous.

The following theorem ensures the existence of weakly Pareto-optimal solutions :

**Theorem 1** :- Under assumptions A.1, A.2, A.3 and A.4 system (1) has a weakly Pareto-optimal solution, $(x^*, g^*) \in \mathbb{R}^{nl}_+ \times G$, for any $\mathcal{V} \in \mathbb{R}^n$.

Proof :- Denote by $X(\mathcal{V}) = \{(x,g) \in \mathbb{R}^{nl}_+ \times G / u(x,g) \geqq \mathcal{V}\}$ and consider the following program :

$$\sum_{j=1}^{l}\left(\sum_{i=1}^{n} x_{ij}\right) + \sum_{j=1}^{l} c_j(g) \to \min$$

s.t.

$(x,g) \in X(\mathcal{V})$

Since $X(\mathcal{V})$ is non-empty (by A.3), closed and bounded from below, Weirstass' Theorem ensures that program(P) will have a solution $(x^*, g^*) \in \mathbb{R}^{nl}_+ \times G$. By construction this solution satisfies (i) and (iii) in (1); let us show that it verifies (ii) as well.

Suppose $u_i(x^*, g^*) > \mathcal{V}_i$ and $x^*_i > 0$ for some i; without loss of generality let this happen for i=1,2,...,h, whilst for i=h+1,...,n either $u_i(x^*, g^*) = \mathcal{V}_i$ or $u_i(x^*, g^*) > \mathcal{V}_i$ and $x^*_i = 0_l$. Then define a vector $z \in \mathbb{R}^{nl}_+$ $(z = [z_1, \dots, z_n],\ z_i \in \mathbb{R}^l_+$ for all i), as follows:

$z_i = x^*_i$, for i=2,3,...,n

$z_1 < x_1$ , with $u_1(z, g^*) \geqq \mathcal{V}_1$

(we can always do this since $u_1$ is continuous and $x^*_1 > 0$). As a result, we have $z < x^*$ and, by A.2,

$u_i[z,g^*] \geqq u_i(x^*,g^*) \geqq v_i$, $i=2,3,\ldots,n$.

Furthermore $u_1[z,g^*] \geqq v_1$ by construction and

$$\sum_{j=1}^{l}\left(\sum_{i=1}^{n} z_{ij}\right) < \sum_{j=1}^{l}\left(\sum_{i=1}^{n} x^*_{ij}\right),$$

contradicting the minimality of $(x^*,g^*)$.

Let $w^* = \sum_{i=1}^{n} x_i^* + c(g^*)$, $w^* \in \mathbb{R}^l_+$, and consider a utility vector $v' \in \mathbb{R}^n$ such that $v' >> v$ and for some $i$, $x_i^* > 0$. We have to show that $v'$ is not feasible by distributing a bundle of goods equal to (or smaller than) $w^*$.

Define $X(v') = \{(x,g) \in \mathbb{R}^{nl}_+ \times G / u(x,g) \geqq v'\}$, and let $(x',g')$ be an allocation in $X(v')$. Suppose, $\sum_i x_i' + c(g') \leqq w^*$ (otherwise $x^*$ would be weakly Pareto-optimal); since $X(v') \subseteq X(v)$ we cannot have $\sum_{i=1}^{n} x_i' + c(g') < \sum_{i=1}^{n} x_i^* + c(g^*)$. Therefore assume $\sum_{i=1}^{n} x_i' + c(g') = \sum_{i=1}^{n} x_i^* + c(g$ and let $u_i(x^*,g^*) < u_i(x',g')$ for some $i$, with $x_i^* > 0$; in this case $x_i' > 0$ as well (otherwise $(x^*,g^*)$ would not be minimal : take $y_i = 0$, $y_j < x_j'$, $j \neq i$ such that $u_j(y,g') > v_j \forall j$). Now construct a vector $z \in \mathbb{R}^{nl}_+$ as follows :

$z_j = x_j'$ for all $j \neq i$

$z_i < x_i'$ with $u_i(z,g') \geqq u_i(x^*,g^*)$

By A.2, $u_j(z,g') \geqq v_j' > v_j$ for all $j \neq i$, and $u_i(z,g') \geqq v_i$ by construction. Furthermore,

$z < x'$, and hence

$$\sum_{j=1}^{l}\left(\sum_{i=1}^{n} z_{ij}\right) < \sum_{j=1}^{l}\left(\sum_{j=1}^{n} x_{ij}\right)$$

$$\Rightarrow \sum_{j=1}^{l}\left(\sum_{i=1}^{n} z_{ij}\right) + \sum_{j=1}^{l} c_j(g') < \sum_{j=1}^{l}\left(\sum_{i=1}^{n} x'_{ij}\right) + \sum_{j=1}^{l} c_j(g')$$

$$= \sum_{j=1}^{l}\left(\sum_{i=1}^{n} x_{ij}\right) + \sum_{j=1}^{l} c_j(g^*),$$

whilst, $(z,g')\in X(\mathcal{V})$. This contradicts the minimality of $x^*$.

Now suppose $\mathcal{V}'\in\mathbb{R}^n$, $\mathcal{V}'>>\mathcal{V}$ and $x_i^*=0\ \forall i=1,\ldots,n$, and suppose there exists no solution $(x,g)$ to the program where $x>0$. In this case $(0,g^*)$ belongs to $\mathcal{G}$ the set of solutions to (1). Maximize $\sum_i u_i(0,g)$ (or any positive linear combination) subject to $(0,g)\in\mathcal{G}$, and $\sum_j c_j(g)\leqq\sum_j c_j(g^*)$. Since the feasible set is compact a solution to this program exists. Call this solution $(0,g')$.

Clearly $(0,g')$ is weakly Pareto-optimal, or else $\exists(\bar{x},\bar{g})\in\mathcal{G}$, $\bar{x}>0$, with $\sum_i \bar{x}_i+c(\bar{g})\leqq c(g')$ and $u_i(\bar{x},\bar{g})>u_i(0,g')\forall i$. Therefore, $\sum_i \bar{x}_i+c(\bar{g})\leqq c(g^*)$ and $(\bar{x},\bar{g})\in\mathcal{G}$, contradicting the non-existence of such a solution.

Q.E.D.

**3. Welfare-fair allocations** : Now suppose utility functions can be compared interpersonally in ordinal terms (that is, we assume that statements of the form: '$u_i(x,g)\geqq u_j(x,g)$' are meaningful).

**Definition 3** :- We shall say that an allocation, $(x^*,g^*)\in\mathbb{R}_+^{nl}\times G$, is <u>welfare-fair</u> if it is both egalitarian and weakly Pareto-efficient.

The following result obtains :

**Theorem 2** :- Let $w\in\mathbb{R}_+^l$ be a given bundle of goods. Under assumption A.1, A.2 and A.4, there exists a welfare-fair allocation $(x^*,g^*)$ such that $\sum_i x_i^*+c(g^*)\leqq W$.

<u>Proof</u> :- Define the following sets :

$$X(w)=\{(x,g)\in\mathbb{R}_+^{nl}\times G/\ \sum_{i=1}^{n} x_i+c(g)\leqq w\}$$

$$V(w)=\{\mathcal{V}\in\mathbb{R}^n/\mathcal{V}=u(x,g),\ x\in X(w)\}.$$

Then consider a function $f:V(w)\to\mathbb{R}$, defined by $f(\mathcal{V})=\min \mathcal{V}_i$ .

Since $X(w)$ is a nonempty compact set and $u$ is a continous transformation, $V(w)$ will also be a compact set. Let $v_i^*$ be the maximum of $f$ over $V(w)$ (which exists since $f$ is continuous), and call $v^* = v_i^*(1,1,\ldots,1)$, a vector in $\mathbb{R}^n$ all components of which are equal to $v_i^*$. Then applying Theorem 1 we get the desired result.

We may think of welfare-fair distributions as the result of maximizing a particular social welfare functional, the maximin rule.

__Corollary 1__ :- Let $w \in \mathbb{R}_+^{nl}$ be a given bundle of goods, and suppose assumptions A.1, A.2 and A.4 hold. Then the program

$$\max F[u(x,g)]$$

s.t.

$$\sum_{i=1}^{n} x_i + c(g) \leq w$$

has a solution and this solution is welfare-fair, when F is the maximin rule.

__Strict welfare-fair allocations__ :- Consider the following definition :

__Definition 4__ :- We shall say that an allocation $(x^*, g^*) \in \mathbb{R}_+^{nl} \times G$ is __strictly welfare-fair__ if it is both strictly egalitarian and weakly Pareto-optimal.

Before proving the existence of a strictly welfare-fair solution, let us take note of the following corollary of Theorem 1.

__Corollary 2__ : Under assumptions A.1, A.2, A.3 and A.4, let $u_i^\circ = \max\{u/o_u, g)\}$ i $g \in G$ and let $v \in \mathbb{R}^n$ be a vector of utility values such that $v \gg u^\circ = (u_1^\circ, \ldots, u_n^\circ)$. Then the equation syatem $u(x,g) = v$

has a weakly Pareto-optimal solution $(x^*, g^*) \in \mathbb{R}_+^{nl} \times G$, where G is the set of public good levels.

<u>Proof</u> :- Immediate from Theorem 1.

Using this Corollary and Theorem 2, we may assert the following :

**<u>Corollary 3</u>** : Let $w \in \mathbb{R}_+^l$ be a given bundle of goods, and suppose assumptions A.1, A.2 and A.4 hold. Let $u_i^o = \max u_i \ (o_{nl}, g)$ $g \in G$ and $\nu^*$ (as in Theorem 2) be such $\nu^* \gg u^o = (u_1^o, \ldots, u_n^o)$. Then there exists a strictly welfare-fair allocation of w.

**<u>References</u>** :-


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