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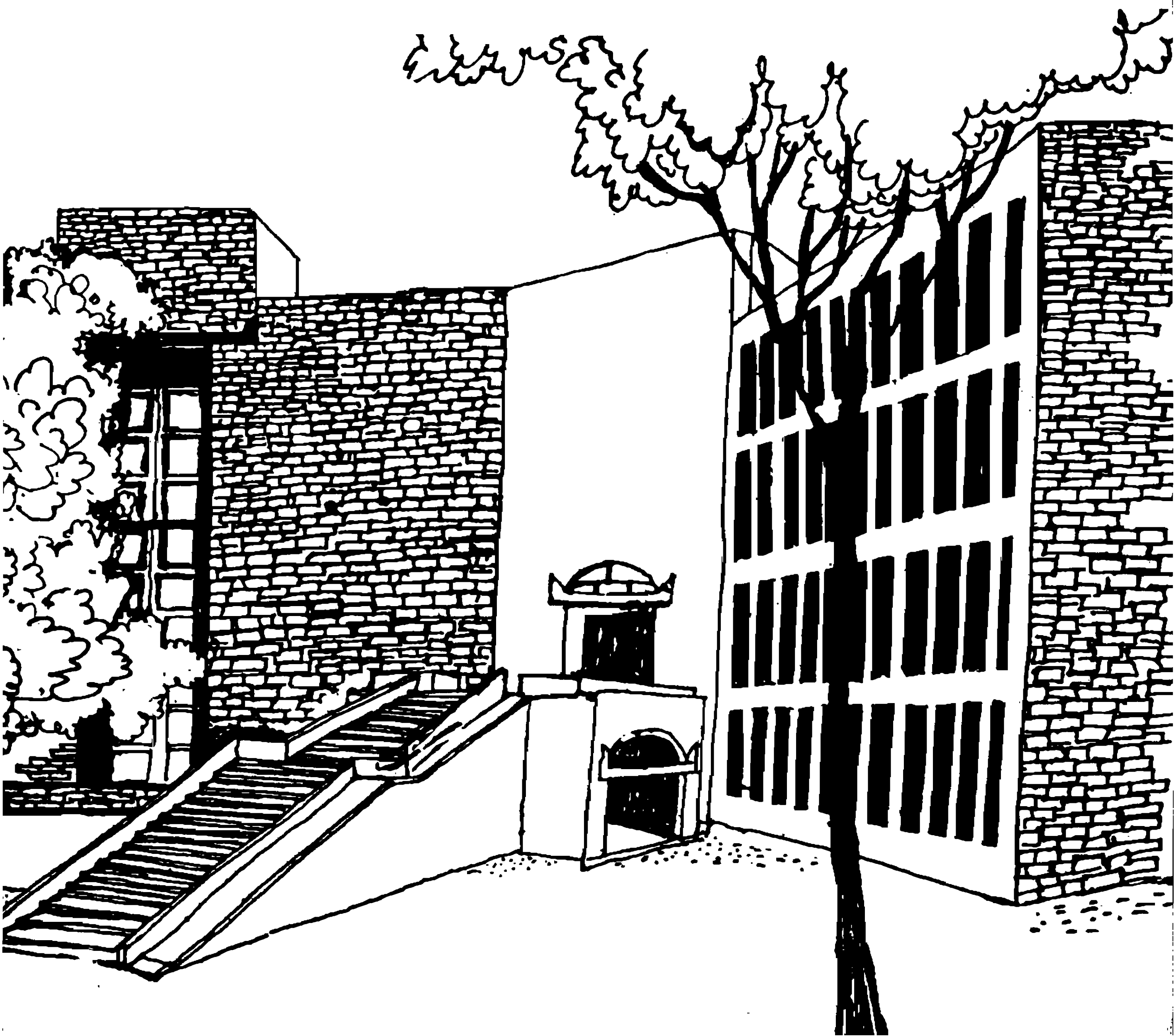

VALID INEQUALITIES AND FACETS OF THE MULTI PRODUCT CAPACITATED LOT-SIZING PROBLEM WITH CHANGEOVER COSTS

By
T. L. Magnanti
&
S. Trilochan Sastry

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
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Valid Inequalities and Facets for the Multi Product Capacitated Lot-Sizing Problem with Changeover Costs

T.L.Magnanti

S.Trilochan Sastry

Subject Category	Descriptive Phrase
1.Programming, integer	Inequalities for multi product scheduling
2.Inventory/Production	Multi item production planning with product changeover costs

Abstract

The polyhedral structure of various versions of the single item lot-sizing problem have been studied previously. These include the uncapacitated and capacitated versions of the problem, with and without changeover costs. However, the polyhedral structure of the multi item problem has not been studied in detail. In this paper we desrcibe several classes of inequalities and facets for the multi item capacitated lot-sizing problem with changeover costs. Some of these facets are valid for the uncapacitated problem as well. We also solve the separation problem for some inequalities.

This paper is a sequel to Sastry (1991), where we discussed several valid inequalities and facets for the single item capacitated lot-sizing problem with changeover costs. We showed that the inequalities are valid, identified conditions under which they are tight and proved they are facets. We also solved the separation problem for a large class of valid inequalities and presented some computational results to show the utility of these inequalities.

This paper extends that work to the multi product version of the problem. The inequalities and facets described here are substantial generalizations of those described for the single item problem. The arguments used to establish validity of the inequalities, and proofs that inequalities are facets are however similar. In many cases we have omitted detailed proofs, which have been provided in the earlier paper.

A lot of work has been done on the polyhedral structure of various versions of the of the single item problem (see for instance Barany, Van Roy and Wolsey (1984), Pochet (1988), Wolsey (1989), Leung, Magnanti and Vachani (1989) and Magnanti and Vachani (1990)). A detailed literature review is available in Magnanti and Vachani (1990). However not much work has been done on studying inequalities and facets for the multi item version of the problem.

In the next section we briefly describe the problem. We then describe several classes of valid inequalities and solve the separation problem using a heuristic approach. We also describe some *trivial facets*.

1. Problem Formulation

We describe a single machine, multi-product, production planning model. Let T denote the finite time horizon over which the facility is scheduled, P the number of products and d_{pi} the demand for product p in period i. We assume a constant capacity and follow a discrete production policy, i.e, we either do not produce at all or produce to capacity in each time period. This policy is reasonable when it is expensive to run the facility at less than full capacity, or when demand is high and the facility is capacity constrained. As shown in Magnanti and Vachani (1990), we can assume without loss of generality that capacity in each period is 1 unit and that demand is either 0 or 1.

We assume that the relevant costs for each product p in period i are the changeover cost K_{pi}, the fixed cost or the setup cost s_{pi}, and the inventory holding cost h_{pi}. Let z_{pi}, y_{pi} and w_{pi} denote the changeover, setup and production variables respectively. We assume that demands are nonnegative, initial production $w_{p0}=0$, and that there is no starting or ending inventory. Let $d_{ik} = \Sigma_{t=i}^{k} d_t$, denote the total demand in periods i through k. The Changeover Cost Scheduling Problem (CSP) can be formulated as follows:

(CSP) Minimize $U = \Sigma_{p=1}^{P} \Sigma_{i=1}^{T}\{h_{pi}w_{pi} + s_{pi}y_{pi} + K_{pi}z_{pi}\}$ (1)

subject to

$\Sigma_{j=1}^{i} w_{pj} \geq \Sigma_{j=1}^{i} d_{pj}$	for all p, i	(2)
$\Sigma_{j=1}^{T} w_{pj} = n_p$	for all p	(3)
$w_{pi} - y_{pi} \leq 0$	for all p, i	(4)

$z_{pi}+y_{p,i-1}-y_{pi} \geq 0$	for all p, i	(5)
$\Sigma_{p=1}^{P} y_{pi} \leq 1$	for all i	(6)
$\Sigma_{p=1}^{P} z_{pi} \leq 1$	for all i	(7)
$w_{pi},\ y_{pi},\ z_{pi} \geq 0$	and integer	(8).

Let CSP(L) denote the linear programming relaxation of CSP and let F(CSP) denote the set of feasible integer solutions for CSP. Constraints (2) and (3) are the demand constraints. Constraints (4) ensure that we can produce only if the machine is set up. Constraints (5) ensure that if the machine is set up for product p in period i (i.e., $y_{pi}=1$) but not in period i-1 then the changeover variable z_{pi} equals 1. Constraints (6) ensure that we produce only one product in any period. Magnanti and Vachani (1990) give a detailed formulation with all the underlying assumptions.

We describe different classes of valid inequalities for the multi item problem. One class is a *trivial* generalization of the single item inequalities.The other classes are substantially different and cannot be derived from the single item inequalities.

Lemma Let WUWZUYUYZUZ $\subseteq \{1, \ldots t_q\}$ and let $W_p=W$, $WZ_p = WZ$, $Y_p = Y$, $YZ_p = YZ$, $Z_p = Z$.

If $\Sigma_{i\in W \cup WZ}\alpha_i w_i+\Sigma_{i\in Y \cup YZ}\beta_i y_i+\Sigma_{i\in WZ \cup YZ \cup Z}\gamma_i z_i \geq \delta$ is a valid single item inequality, with demands in periods t_k for k=1, ... q, then

$\Sigma_{i\in W_p \cup WZ_p}\alpha_i w_{pi}+\Sigma_{i\in Y_p \cup YZ_p}\beta_i y_{pi}+\Sigma_{i\in WZ_p \cup YZ_p \cup Z_p}\gamma_i z_{pi} \geq \delta$ is valid for the multi item problem, with demands for item p occuring in periods t_k, k=1, ... q.

Proof

Consider any feasible solution to the multi item problem. It must satisfy the demand in periods t_1, t_2,... t_q for item p. Further, the single item inequality is satisfied by this feasible solution. The result follows.

We now extend our results to obtain new valid inequalities for the multi item problem. To motivate the discussion we first consider the two item problem.

Suppose demands for items 1 and 2 occur in periods $t_{1,1}$ and $t_{2,1}$ respectively. Consider the following inequality:

$$w_{2,1}+\Sigma^{t_{2,1}}_{i=2} z_{2i}+\Sigma^{t_{1,1}}_{i=3}w_{1i} \geq 1.$$

If we produce item 1 in any of the periods 3 through $t_{1,1}$, the inequality is satisfied. If we produce item 1 in period 1, then $\Sigma^{t_{2,1}}_{i=2}z_{2i} \geq 1$ because we need to turn on the machine for item 2 at least once in the interval {2, ... , $t_{2,1}$}. If we produce item 1 in period 2, then $w_{2,1}+\Sigma^{t_{2,1}}_{i=3}z_{2i} \geq 1$ because we either produce item 2 in period 1 or turn on the machine at least once in the interval {3, ... $t_{2,1}$}. Hence the inequality is valid.

In fact, we can partition the interval {3, ... $t_{1,1}$} into subsets W_1, Y_1 and Z_1. As in the single item case, we impose the condition that if $i \in W_1$, then $i+1 \notin Z_1$. Similarly, we can partition the interval {1, ... , $t_{2,1}$} into subsets W_2, Y_2 and Z_2. The following example illustrates this: demand for item 1 occurs in period 12, and demand for item 2 occurs in any period after that, say period 15. We can write down the following valid inequality:

Example 1.

$$\Sigma^{3}_{i=1}w_{2i}+\Sigma^{7}_{i=4}z_{2i}+\Sigma^{9}_{i=8}w_{2i}+y_{2,10}+\Sigma^{15}_{i=11}z_{2i}$$
$$+\Sigma^{2}_{i=1}w_{1i}+w_{1,5}+y_{1,6}+\Sigma^{12}_{i=7}z_{1i}\geq 1.$$

We can describe these inequalities as follows:

$$\Sigma_{i\in W_2}w_{2i}+\Sigma_{i\in Y_2}y_{2i}+\Sigma_{i\in Z_2}z_{2i}+\Sigma_{i\in W_1}w_{1i}+\Sigma_{i\in Y_1}y_{1i}+\Sigma_{i\in Z_1}z_{1i}\geq 1,$$

and impose the following conditions:

(i) W_2, Y_2 and Z_2 partition the interval $\{1, \ldots, t_{2,1}\}$.

(ii) If period $i\in W_2$, then $i+1\notin Z_2$ except for exactly one period j in the interval $\{1, \ldots, t_{2,1}\}$ where $j\in W_2$, and $j+1\in Z_2$.

(iii) W_1, Y_1 and Z_1 partition the set $\{1, \ldots, t_{1,1}\}\setminus\{j, j+1\}$.

(iv) If period $i\in Z_1$, then $i-1\in Y_1\cup Z_1$.

Note that we obtain similar inequalities if we partition the interval $\{1, \ldots, t_{1,1}\}$ into W_1, Y_1 and Z_1 and the set $\{1, \ldots, t_{2,1}\}\setminus\{j, j+1\}$ into W_2, Y_2 and Z_2 for some $j\in W_1$ and $j+1\in Z_1$.

2. Generalizing the inequalities to the multi item problem.

We can generalize these inequalities to p items with demands in periods $t_{1,1}$, $t_{2,1}, \ldots, t_{p,1}$; $t_{1,2}$, $t_{2,2}, \ldots, t_{p,2}$; ...., t_{1,q_1}, $t_{2,q_2}, \ldots, t_{p,q_p}$. We say that there is a *junction* due to item r at period i_r if $i_r\in W_r$ and $i_r+1\in Z_r$ for some product r. We have exactly one junction for each item $2\leq r\leq p$, in the demand interval $\{t_{r,q_r-1}+1, \ldots, t_{r,q_r}\}$. For items $1\leq r\leq p-1$, we impose the following conditions:

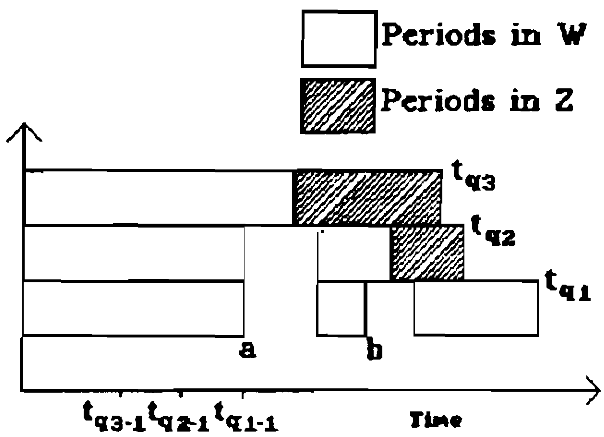
(i) we partition the set $\{t_{r,q_{r-1}},\ldots,t_{r,q_r}\}\setminus\bigcup^p_{k=r+1}\{i_k,\ i_k+1\}$ into subsets W_r, Y_r and Z_r and skip the periods $\bigcup^p_{k=r+1}\{i_k,\ i_k+1\}$.

(ii) if period $i\in Z_r$, then $i-1\in Y_r\cup Z_r$ for all $i\notin i_r+1$.

We show that the following inequalities are valid:

$$\Sigma^p_{r=1}\Sigma^{t_{r,q_{r-1}}}_{i=1}w_{ri}+\Sigma^p_{r=1}[\Sigma_{i\in W_r}w_{ri}+\Sigma_{i\in Y_r}y_{ri}+\Sigma_{i\in Z_r}z_{ri}]$$
$$\geq \Sigma^p_{r=1}(q_r-1)+1 \qquad (VI)$$

Figure 1 shows a three item inequality



$$\Sigma^a_{i=1}w_{3i}+\Sigma^{t_{q3}}_{i=a+1}z_{3i}+\Sigma^{a-1}_{i=1}w_{2i}+\Sigma^{b}_{i=a+1}w_{2i}+\Sigma^{t_{q2}}_{i=b+1}z_{2i}+\Sigma^{a-1}_{i=1}w_{1i}+$$
$$\Sigma^{b-1}_{i=a+1}w_{1i}+\Sigma^{t_{q1}}_{i=b+1}w_{1i} \geq q_1+q_2+q_3-2$$

Figure 1

Theorem 1. *The inequalities (VI) are valid.*

Proof.

For any product r, the terms $\Sigma^{t_{r,q_r-1}}_{i=1}w_{ri}$ add up to at least (q_r-1) in any feasible solution. Hence the terms $\Sigma^p_{r=1}\Sigma^{t_{r,q_r-1}}_{i=1}w_{ri}$ add up to at least $\Sigma^p_{r=1}(q_r-1)$ in any feasible solution. Therfore, we need to show that the terms in the last demand intervals add up to at least

1. If $\Sigma^{p}_{r=1}\Sigma^{t_{r,q_r-1}}_{i=1}w_{ri} = \Sigma^{p}_{r=1}(q_r-1)+1$, the inequality is valid. Otherwise $\Sigma^{t_{r,q_r-1}}_{i=1}w_{ri} = (q_r-1)$ for each product r.

The terms $\Sigma_{i\epsilon W_p}w_{pi}+\Sigma_{i\epsilon Y_p}y_{pi}+\Sigma_{i\epsilon Z_p}z_{pi}$ where W_p, Y_p and Z_p partition the interval $\{t_{p,q_{p-1}}+1, \ldots, t_{p,q_p}\}$ add up to 1 unit unless we turn on the machine in period i_p and produce in period i_p+1.

If we do, then we cannot produce any other product in periods i_p, i_p+1. Similarly, the terms $\Sigma_{i\epsilon W_r}w_{ri}+\Sigma_{i\epsilon Y_r}y_{ri}+\Sigma_{i\epsilon Z_r}z_{ri}$ where W_r, Y_r and Z_r partition the interval $\{t_{r,q_r-1}+1,\ldots,t_{r,q_r}\}\setminus\cup^{p}_{k=r+1}\{i_k, i_k+1\}$ add up to 1 unit unless we turn on the machine in period i_r and produce in period i_r+1, for $2 \le r \le p-1$. If each of these add up to zero, then we turn on the machine for product r in period i_r, and produce in period i_r+1. Thus we cannot produce product 1 in any of the periods $\cup^{p}_{k=2}\{i_k, i_k+1\}$. So we must produce product 1 in one of the periods in

$S_1 = \{t_{1,q_1-1}+1,\ldots,t_{1,q_1}\}\setminus\cup^{p}_{k=2}\{i_k, i_k+1\}$.

Since the partition of S_1 into W_1, Y_1 and Z_1 satisfies the feasibility condition for the single product inequalities, therefore $\Sigma_{i\epsilon W_1}w_{1i}+\Sigma_{i\epsilon Y_1}y_{1i}+\Sigma_{i\epsilon Z_1}z_{1i}\ge 1$. Hence the inequality is valid.

Example 2. Suppose we have three products, with the firt three demands in the following periods:

	Periods		
Product 1	5	10	20
Product 2	6	11	21
Product 3	7	12	22

The following inequality is valid:

$$\Sigma^{12}_{i=1}w_{3i}+\Sigma^{22}_{i=13}z_{3i}+\ldots.$$
$$+\quad \Sigma^{11}_{i=1}w_{2i}+\Sigma^{17}_{i=14}w_{2i}+\Sigma^{21}_{i=18}z_{2i}+\ldots.$$
$$+\quad \Sigma^{11}_{i=1}w_{1i}+\Sigma^{16}_{i=14}w_{1i}+\Sigma^{20}_{i=19}w_{1i} \qquad \geq 7.$$

3. Further generalizations

A general principle that the partitioning inequalities (PI) for the single item problem satisfied was that if we produce j times in any sequence of periods, then the lefthand side increases by j units. For the skip inequalities, we modified it so that if we produce j times in sequence, and j' of these are not skipped, then the lefthand side must increase by at least j'. However, for the multi-item problem, we do not require that the lefthand side increases by j'. We compensate by introducing terms for other products.

Before we extend the inequalities further, we introduce a few terms. Let i_r denote the first period of a junction for item r, i.e., $i_r \in W_r$ and $i_r+1 \in Z_r$. Suppose the junction occurs in demand interval j_r+1, i.e., in the interval $\{t_{r,j_r}+1,\ldots,t_{r,j_{r+1}}\}$. Let $l'(r)$ denote the number of consecutive periods in Z_r starting in period i_r+1, and let $l(r)=\min\{l'(r),\ q_r-j_r\}$ denote the *length* of the junction. We skip $l(r)+1$ consecutive periods for all items $k \leq r-1$ starting from period i_r. We call these skips as *compensating skips* to distinguish it from those skips present in single item skip inequalities. Let j_k+1 denote the demand interval in which period i_r falls for items $k \leq r-1$. We have exaclty one junction for each item 2 through p. Product 1 has no junctions. Let $CS_r=\{i_r,\ldots,i_r+l(r)\}$ denote the compensating skipped interval for all items $k\le r-1$.

For each item r, we partition the set $QR=\{1,\ldots,t_{r,qr}\}\setminus\cup_{k=r+1}^{p}CS_k$ into subsets W_r, Y_r, Z_r, YZ_r and WZ_r satisfying a modified form of the compensation condition for feasibility of single item partitioning inequalities (PI) for all periods in QR as described below.

Let c_{ri} denote the coefficient of z_{ri}, $m_r(i,i^*)$ the sum of the coefficients of y_{rt} for $t\in Y_r\cup YZ_r$ and $n_r(i,i^*)$ the number of terms in Z_r in the interval $\{i,\ldots,i^*\}$. Let $Y_r=\{i(r1),i(r2),\ldots,i(rm)\}$ and for $1\le s\le m$ let $i''(rs)=\min\{i'>i(rs):\ i'\in Z_r$ and $i'+1\in W_r\cup Y_r\cup_{k=r}^{p}CS_k\}$, i.e., $i''(rs)$ is the first period after $i(rs)$ for which $i''(rs)+1$ belongs to W_r or Y_r or to a compensating skipped interval. Let $B(rs)=\{i(rs),\ldots,i''(rs)\}$ for $1\le s\le m$ and $1\le r\le p$. Let $j_r(i)=\max\{j'_r\le i-1$: demand in period j'_r equals 1, and $j'_r\le i_r\}$.
Then the *multi item compensation condition (MIC)* can be described as follows:

MIC Condition. For any two periods $i^*\in Z_r$ and $i\le i^*$ in interval $B(rs)$ let $i''=\min\{i'\ge i:\ i'\notin WZ_r\}$ belong to demand interval j_r+1. Then

$$m_r(i,i^*)+c_{ri}$$
$$\ge \min\{q_r-j_r,\ m_r(i,i^*)+n_r(i,i^*),\ \Sigma_{k=1}^{p}(q_k-l(k)-j_k(i))\}.$$

Then the *multi item junction inequalities (JI)* can be written as follows:

$$\Sigma^{p}_{r=1}[\Sigma_{i\in W_{ri}}w_{ri}+\Sigma_{i\in Y_{ri}}y_{ri}+\Sigma_{i\in Z_{ri}}c_{ri}z_{ri}+\Sigma_{i\in YZ_r}(y_{ri}+c_{ri}z_{ri})+\Sigma_{i\in WZ_r}(w_{ri}+c_{ri}z_{ri}]$$
$$\geq \Sigma^{p}_{r=1}(q_r-l(r)) \qquad \text{(JI)}$$

where

(i) $\cup_{k=1}^{p}\{i_k,\ i_k+1\}\subseteq\{t_{r,j_r}+1,\ldots,t_{r,j_{r+1}}\}$ for all $1\leq r\leq p$, i.e., all junctions lie in the same demand interval for each item,

(ii) we have one junction per item

(iii) for each item r, $q_k-j_k\leq l(r)$ for all $k\leq r-1$,

(iv) for $i\in QR$, the subsets W_r, Y_r, Z_r, YZ_r and WZ_r satisfy the multi item compensation condition (MIC).

Theorem 2. *The multi item junction inequalities (JI) are valid.*

Proof

We can produce at most $\Sigma_{r=1}^{p}(l(r)+1)$ units in the compensating skipped intervals. If we turn on the machine for item r in period i_r, and produce item r in periods i_r+1 through $i_r+l(r)$, i.e., in the compensating skip interval CS_r, the contribution loss is at most $l(r)$. Suppose we produce some other item in the interval CS_r. Since $q_k-j_k\leq l(r)$ for all items $k\leq r-1$, we need to produce at most q_k-j_k units in the interval to meet the demand up to $t_{k,qk}$. The contribution loss is therefore at most $l(r)$. Notice that we skip periods in CS_r only for items $k\leq r-1$. It is easy to verify that therefore the total loss in contribution due to all the compensating skip intervals is at most $\Sigma_{r=1}^{p}l(r)$. Consider the rest of the periods. If for $i^*\in Z_r$ and $i\leq i^*$,

$$\min\{q_r-j_r,\ m_r(i,i^*)+n_r(i,i^*)\}\leq\Sigma_{k=1}^{p}(q_k-l(k)-j_k(i))$$

then the feasibility condition for the single item partitioning inequalities (PI) is satisfied. Hence any feasible solution will satisfy the inequality. Otherwise, since the total contribution up to period i is at least $\Sigma_{k=1}^{p} j_k(i)$, the remaining periods must contribute at least $\Sigma_{k=1}^{p}(q_k-l(k)-j_k(i))$. The inequality is therefore valid.

Example 3.

Suppose item 1 has demands in periods 10, 20, 30 and 40, and item 2 in periods 15, 25, 35 and 45. Hence $q_1=q_2=4$. If item 2 has a junction over the interval {11, 12, 13}, then $q_2-l(2)=2$. A valid inequality is

$$\Sigma^{11}_{i=1}w_{2i}+z_{2,12}+z_{2,13}+\Sigma^{40}_{i=14}w_{2i}+\Sigma^{10}_{i=1}w_{1i}+\Sigma^{55}_{i=14}w_{1i} \geq 6.$$

We have used the single item partitioning inequalities and combined them to include junctions and compensating skipped intervals. We can generalize the junction inequalities (JI) if we use the single item skip inequalities instead of the partitioning inequalities. Let b_r denote the number of periods skipped for item r up to period $t_{r,qr}$, i.e., those periods that are skipped but are not compensating skips, and let b_{rj} denote the number of periods skipped for item r up to period $t_{r,j}$. If period i is skipped for item r, then it is skipped for all items. Let S denote the set of all skipped periods and let $b=|S|$. The length of a junction $l(r)=\min\{l'(r), q_r-j_r\}$. We have exactly one junction per item.

For each item r, we partition the set $QR=\{1,\ldots,t_{qr}\}\backslash\{S\cup_{k=r+1}{}^{p}CS_k\}$ into subsets W_r, Y_r, Z_r, YZ_r and WZ_r satisfying a modified version of the skip condition for single item inequalities (SI) for periods in QR described below.

Let $W(rj)=\max((k_r-b_{rk}: k_r\leq j_r), 0)$ and let $u(ri)=q_r-b-W(rj)$. Let $j_r(i)=\max\{j'_r\leq i-1$: demand in period j'_r equals 1, and $j'_r\leq i_r\}$. Then the multi item skip condition (MIS) can be described as follows.

MIS Condition. For any two periods $i^*\in Z_r$ and $i\leq i^*$ in interval B(rs), let $i''=\min\{i'\geq i: i'\notin WZ_r\}$. Then $m_r(i,i^*)+c_{ri}\geq \min\{u(ri''), m_r(i,i^*)+n_r(i,i^*), \Sigma_{r=1}{}^{p}W(rj_r(i))\}$.

The multi item *junction skip inequalities (JSI)* can be described as follows.

$$\Sigma^{p}_{r=1}[\Sigma_{i\in W_{ri}}w_{ri}+\Sigma_{i\in Y_{ri}}y_{ri}+\Sigma_{i\in Z_{ri}}c_{ri}z_{ri}+\Sigma_{i\in YZ_r}(y_{ri}+c_{ri}z_{ri})+\Sigma_{i\in WZ_r}(w_{ri}+c_{ri}z_{ri}]$$
$$\geq \Sigma^{p}_{r=1}(q_r-l(r))-b. \qquad \text{(JSI)}$$

where

(i) $\cup_{k=1}{}^{p}\{i_k, i_k+1\}\subseteq\{t_{r,j_r}+1,\ldots,t_{r,j_{r+1}}\}$ for all $1\leq r\leq p$, i.e., all junctions lie in the same demand interval for each item,

(ii) we have one junction per item

(ii) for each item r, $q_k-j_k\leq l(r)$ for all $k\leq r-1$,

(iii) for $i\in QR$, the subsets W_r, Y_r, Z_r, YZ_r and WZ_r satisfy the multi item skip condition (MIS).

valid.

Since the theorem can be established by using arguments similar to those in theorem 2, we omit the proof.

Remark. If b=0, then $W(rj)=j_r$ and $u(ri)=q_r-j_r$ where i is in demand interval j_r+1. The skip junction inequalities therefore reduce to the junction inequalities.

Example 4.

Consider the following 3 item problem. Against each item, the periods in which demands occur are shown.

Item					
1	20	40	60	80	100
2	25	45	65	85	-
3	30	50	70	-	-

A valid inequality is

$$w_{1,2}+z_{1,3}+z_{1,4}+z_{1,5}+z_{1,6}+\Sigma^{100}{}_{i=7}w_{1i}\cdot\cdot\cdot\cdot$$
$$+w_{2,7}+z_{2,8}+z_{2,9}+z_{2,10}+z_{2,11}+\Sigma^{85}{}_{i=12}w_{2i}\cdot\cdot\cdot\cdot$$
$$+\Sigma^{70}{}_{i=12}w_{3i} \geq 3.$$

Items 1, 2 and 3 skip period 1. Hence b=1. Item 1 has a junction at period 2 of length 4, and item 2 has a junction at period 7 of length 4. It is easy to see that l(1)=4 and l(2)=4. The righthand side is therefore equal to $\Sigma_{r=1}{}^{3}q_r-b-l(1)-l(2)=12-9=3$.

Remark. For ease of notation items have been numbered arbitrarily from 1 through p for any subset of cardinality p from the set of items $\{1,2,\ldots,P\}$. We can therefore construct multi item skip inequalities for all subsets of items from $\{1,\ldots,P\}$.

4. Separation Problem.

We develop a heuristic to solve the separation problem for the multi item inequalities (MI) having exactly one junction in the last demand interval $\{t_{r,q_r-1}+1,\ldots,t_{q_r}\}$, assuming that the subset of items in the inequality is known and assuming an arbitrary numbering of the items from 1 through p. The heuristic proceeds by finding the junction for item p, then for item p-1 and so on until item 2.

We define $f(w,r,j,i)$ $\{f(y,r,j,i),\ f(z,r,j,i)\}$ as

$$f(w,r,j,i)=\min\{\{\Sigma_{t\in W_r} w_{rt}+\Sigma_{t\in Y_r} y_{rt}+\Sigma_{t\in Z_r} z_{rt}\}$$

where $i\in W_r$ and for $\{j,\ldots,i\}\subseteq\{t_{r,q_r-1}+1,\ldots,t_{q_r}\}$, the subsets W_r, Y_r, Z_r partition $\{j,\ldots,i\}\setminus\cup_{k=r+1}^{p}\{i_k,\ i_k+1\}$ and satisfy the conditions: (i) if $j\geq t_{r,q_r-1}+1$ then we have a junction at period j, i.e., $j\in W_r$ and $j+1\in Z_r$, (ii) for $i\geq j+2$, if $i\in Z_r$, then $i-1\in Y_r\cup Z_r$ and (iii) if $j\in Y_r$ then $j+1\in Z_r$. The last condition is imposed because otherwise the inequality is not tight since we can replace y_{rj} by w_{rj}.

For product r=p down to 1,

For $i=t_{q_r-1}+1$ to t_{r,q_r},

For $j=t_{q_r-1}$ to i,

If $i\geq j+2$ then

$f(w,r,j,i)=w_{ri}+\min\{f(w,r,j,i-1),\ f(z,r,j,i-1)\}$

$f(y,r,j,i)=y_{ri}+\min\{f(w,r,j,i-1),\ f(z,r,j,i-1)\}$

$f(z,r,j,i)=z_{ri}+\min\{f(y,r,j,i-1),\ f(z,r,j,i-1)\}$

If $i=j+1$ and $j\geq t_{q_r-1}+1$ then

$f(w,r,j,j+1)=\infty$, $f(y,r,j,j+1)=\infty$ and $f(z,r,j,j+1)=z_{rj}$.

If $i=j+1$ and $j=t_{q_r-1}$ then

$f(w,r,j,i)=w_{ri}$

$f(y,r,j,i)=y_{ri}$

$f(z,r,j,i)=z_{ri}$

We define $f(.,r,j,i)=\infty$ if $i<j+1$.

Once we have found $f'(.,r,j,i)$ we can find

$h(w,r,i)=\min\{f(w,r,t_{q_r-1},j)+f(w,r,j,i):t_{r,q_r-1}\leq j\leq t_{r,q_r}\}$

$h(y,r,i)=\min\{f(w,r,t_{q_r-1},j)+f(y,r,j,i):t_{r,q_r-1}\leq j\leq t_{r,q_r}\}$

$h(z,r,i)=\min\{f(w,r,t_{q_r-1},j)+f(z,r,j,i):t_{r,q_r-1}\leq j\leq t_{r,q_r}\}$

If $i=t_{r,q_r}\epsilon Y_r$, we can tighten the inequality by replacing y_{ri} by w_{ri}. We can therefore find

$H(r,t_q)=\Sigma^{p}{}_{r=1}\Sigma^{t_{r,q_r}}{}_{-1i=1}w_{ri}+\min\{h(w,r,t_{r,q_r}),\ h(z,r,t_{r,q_r})\}$, which would give the minimum contribution for item r. By backtracking we can find the junction i_r, i_r+1.

Finally, we can find

$F(q_1,q_2,\ldots,q_p)=\Sigma_{r=1}{}^{p}H(r,t_q)$.

If $F(q_1,q_2,\ldots,q_p)<\Sigma_{r=1}{}^{p}(q_r-1)+1$, then we have identified a violated inequality. For a given q_p the quantities q_r for $1\leq r\leq p-1$ are uniquely defined since we require that $t_{p,q_p-1}\leq t_{r,q_r}\leq t_{p,q_p}$. For a given q_p it takes $O(pT^2)$ to find a violated inequality if one exists. Hence the heuristic runs in $O(pT^3)$ time.

5. Trivial facets.

Magnanti and Vachani (1990) have described some trivial facets for the single item problem, and have mentioned that they are valid for the multi item problem also. We describe some more facets which are not included in the original formulation. They are not based on counting arguments for satisfying demand. The proof is straightforward and has been omitted.

Proposition 1. For any subsets Q and R partitioning the set of items {1,....,P}, the constraint $\Sigma_{r\in R}y_{ri}+\Sigma_{r\in Q}z_{ri} \le 1$ is a facet for $i\ge2$.

If $i=1$, then $z_{r1}\ge y_{r1}$, and since $\Sigma_r z_{ri}\le1$ is a valid inequality, the constraint $\Sigma_{r\in R}y_{ri}+\Sigma_{r\in}z_{ri} \le 1$ cannot be a facet.

Remark. Proposition 1 holds true for the uncapacitated multi item lot sizing problem also.

6.Future Directions.

The multi-item skip inequalities (MSI) can be generalized in several ways. We point out some of the directions in which we can proceed to discover new valid inequalities. We have restricted the number of junctions per item to one. It is possible to increase the number of junctions per item. Secondly, junctions are of the type $w_{ri}+z_{r,i+1}$. We can have more general junctions.

The separation problem for the inequalities needs to be solved for a more general version of the inequalities. Computational work using these inequalities would also provide information on their utility. Since the problem can be cast as a special case of the fixed charge network design problem (see for instance Magnanti and Vachani (1990)), these results could also yield insights for developing inequalities for the more general network design problem.

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