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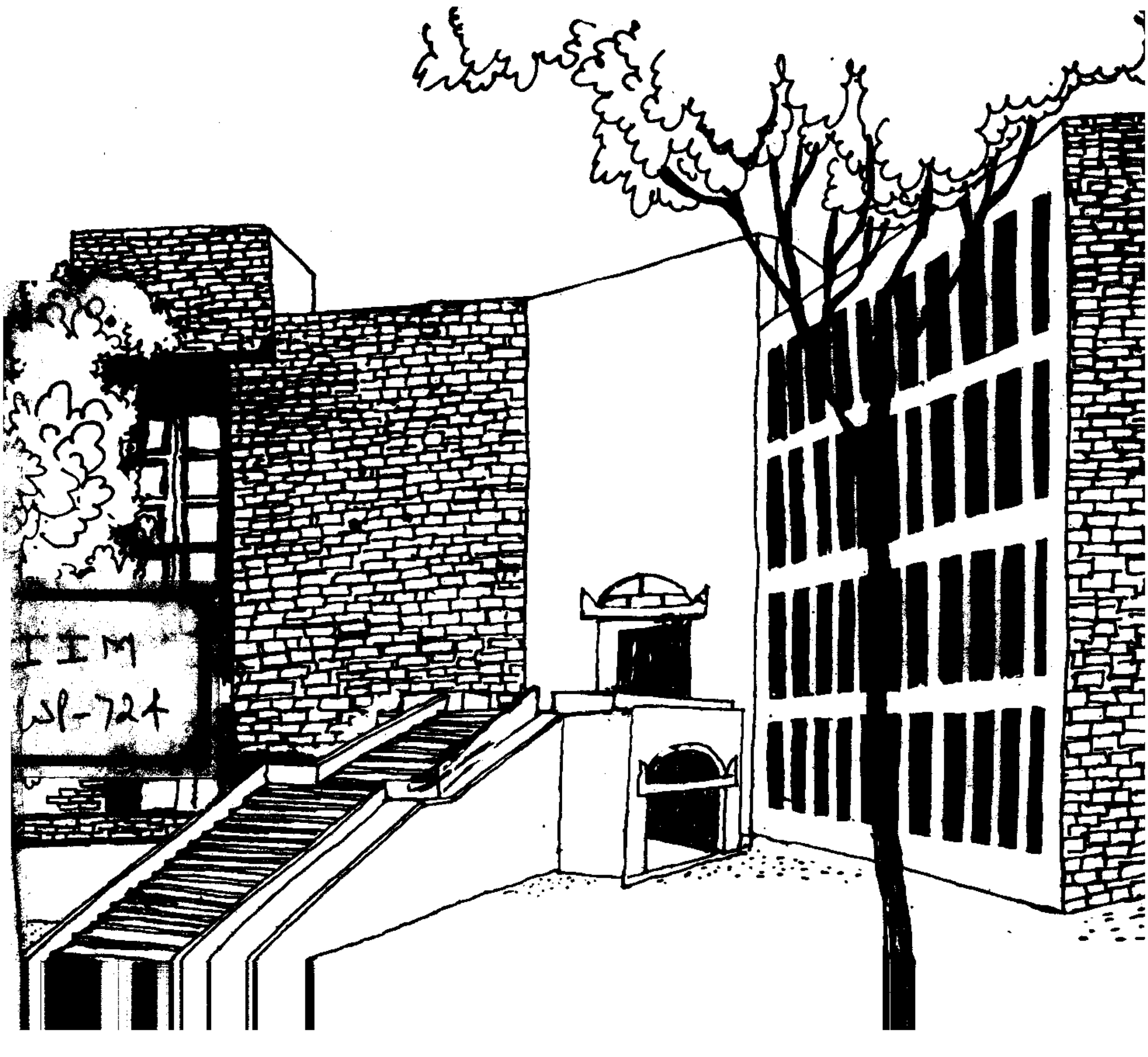

MONOTONICITY WITH RESPECT TO THE DISAGREEMENT POINT AND A NEW SOLUTION TO NASH'S BARGAINING PROBLEM: A NOTE

By

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ABSTRACT

We propose a solution to the bargaining problem which responds appropriately to certain changes in the disagreement point, for a fixed feasible set. If d_i increases, while for $j \neq i$ d_j remains constant, our solution recommends an increase in agent i's payoff, in agreement with intuition. This solution also satisfies the more conventional requirements which are usually imposed, e.g. individual rationality, Pareto optimality, Symmetry and Invariance With Respect to Affine Transformation. It is shown that our solution is the only monotone solution which satisfies these properties.

1. Introduction:- A 2-person bargaining problem is a pair (S,d) of a subset S of $\mathbb{R}^2$ and of a point $d \in S$. $\mathbb{R}^2$ is the utility space, S is the feasible set, and d is the disagreement point. If the agents unanimously agree on a point x of S, they obtain x. Otherwise, they obtain d.

The approach to the problem which we shall consider was first taken by John Nash (1950). He considered a framework which permitted a unique feasible outcome to be selected as the "solution" of a given bargaining problem. This was opposed to earlier approaches within the game theoretic tradition: the von Neumann-Morgenstern (1944) solution to the bargaining problem concides with Edgeworth's (1881) "contract curve", and is equal to the entire set of individually rational, Pareto optimal outcomes. Given a class of 2-person bargaining problems, a solution is a function F associating with every (S,d) in the class a point $F(S,d) \in S$, representing the compromise reached by the agents. In some contexts, F(S,d) may be interpreted as the compromise recommended to the agents by some impartial arbitrator.

In this paper we introduce a solution concept which responds appropriately to changes in d, for fixed S. A detailed investigation of this property, known as monotonicity of bargaining solutions with respect to the disagreement point for the more well known solutions is available in Thomson (1987). Given some agent i, suppose that d_i increases while d_j remains constant for $j \neq i$. Since d_i represents agent i's fallback position, one would expect agent i's final payoff to increase (or atleast not to decrease). The Nash solution was shown by Thomson (1987) to behave in this way on the class of problems considered by Nash and the Kalai-Smorodinsky and Egalitarian solutions also do on the subclass obtained by requiring utility to be freely disposable. Interesting results relating monotonicity of the disagreement point to various axioms proposed by Thomson and Myerson (1980), specifying how solutions should respond to certain changes in the geometry of S, for fixed d, have recently been established by Livne (1985) for two person bargaining problem.

Before, we propose our solution let us note why the monotonicity property we are interested in, is important. The monotonicity property is particulary relevant to situations in which each agent has some control over the position of the disagreement point. Owen (1982) argues that a natural choice for d_i is the maximum value of agent i, in the strategic form game underlying the bargaining problem under consideration. It is natural to investigate whether the intuitive idea that an agent would benefit from an increase in his maximum value, would, ceteris paribus, be helpful to him.

In economics the monotonicity property has independent appeal. Shubik (1982) suggests bargaining solutions as solutions to problems of fair division. Consider a group of individuals, each endowed with a bundle of resources, his initial endowment, and equipped with a concave utility function defined over his consumption space. Under standard assumptions on the utility functions, the image in utility space of the set of possible divisions among all consumers of the resources they jointly own satisfies the assumptions usually imposed on the feasible set of bargaining theory. In this economic context, it is natural to take the image in utility space of the list of initial endowments as the disagreement point. The monotonicity requirement is that agents be rewarde from starting out with larger endowments, which seems quite natural, especialy when they have exerted effort in producing them. It may in addition be required, that agent i be the only one to gain when d_i increases. If some other agent j also were to gain, then the burden on some third agent k would necessarily be greater, making the eventual acceptance of the compromise less likely.

Imagine that agent i, perhaps out of altrmistic feelings for agent j or in repayment of a debt, transfers part of his initial resources to him. Then since the solution that is being propmsed satisfied the additional monotonicity requirement, agent j will gain and agent i will lose, so that no "transfer paradox", (the phenomenon well known in international trade theory) will occur. Agent j is betteroff as agent i wished, and agent i pays some price for this improvement in agent j's welfare, which seems only natural.

Finally, since the disagreement point reflects the bargaining power of the players any increase in the disagreement payoff of player i should lead to an increase in his bargaining strength and consequent diminution in the bargaining strength of layer j≠i. The arbitrated outcome should be consistent with the changed bargaining strength of the players.

The nature of the solution we propose, and which is intented to satisfy the monotonicity requirement is the following: let $Z=(Z_1,Z_2)$ denote the vector of smallest payoffs that each player could concievably receive. The arbitrated value of the bargaining problem for the subclass of games we are considering corresponds to the unique maximal feasible payoff lying on the straight line joining Z to the disagreement point. Simple as the solution looks it does incorporate the idea that the ratio of the increase in the payoffs over the disagreement point, resulting from arbitration, should be equal to the ratio of the increase in the payoffs resulting from the disagreement solution over the smallest payoffs. The rewards of bargaining are therefore consistent with the rewards of the control exercised in arriving at the disagreement solution. It turns out that this solution satisfies the monotonicity requirement, which is elaborated below.

2. <u>The Model</u>:- We will consider a class of problems defined below:

Let $W = \{(S,d)/S \subseteq R^2,$ S is convex, compact, and $\exists\, x \in S$ with $x > d\}$

There $x = (x_1, x_2) \geqq (y_1, y_2) = y$ means $x_i \geqq y_i$, $i = 1,2$

$x = (x_1, x_2) \geq (y_1, y_2) = y$ means $x \neq y$, $x_i \geqq y_i$, $i = 1,2$

$x = (x_1, x_2) > (y_1, y_2) = y$ means $x_i \geq y_i$, $i = 1,2$

Let $\bar{W} = \{(S,d) \in W/\text{if } x \in S \text{ and } d \leqq y \leqq x, \text{ then } y \in S\}$

and $\bar{\bar{W}} = \{(S,d) \in \bar{W}/\exists\ u \in S \text{ such that } d > u\}$

We shall refer to games $(S,d) \in \bar{W}$ as <u>comprehensive names and</u> to games $(S,d) \in \bar{\bar{W}}$, as <u>proper comprehensive names</u>. Given $(S,d) \in W$, its <u>Nash</u> (1950) solution $N(S,d)$ is the point where the product $(x_1-d_1)(x_2-d_2)$ is maximized for $x \in S$ with $x \geqq d$; its <u>Kalai-Smorodinsky</u> (1975) solution outcome $K(S,d)$ is the maximal point of S on the segment joining d to $M(S,d)$, where for each, i, $M_i(S,d) = \max \{x_i/x \in S;\ x \geqq d\}$ $i = 1,2$, its Egalitarian (See Kalai(1977)) solution outcome $E(S,d)$ is the maximal point x of S with $x_1-d_1 = x_2 - d_2$.

In this paper we consider a solution $F: \bar{\bar{W}} \rightarrow R^2$ defined thus.

Let $Z(S) = (Z_1(S), Z_2(S))$, where $Z_i(S) = \min. \{x_i/x \in S\}$. Then $\forall\ (S,d) \in W$, $F(S,d)$ satisfies the following two conditions:

(a) $$\frac{F_2(S,d) - Z_2(S)}{F_1(S,d)-Z_1(S)} = \frac{d_2-Z_2(S)}{d_1-Z_1(S)}$$

(b) $$\frac{x_2-Z_2(S)}{x_1-Z_1(S)} = \frac{d_2-Z_2(S)}{d_1-Z_1(S)},\ x > F(S,d) \text{ implies } x \notin S.$$

The conditions defining the above bargaining solution are:

Condition 1:- $F(S,d) \geqq d$ for all $(S,d) \in \bar{\bar{W}}$

Condition 2:- Let $a_1, a_2 \in R_{++}$, $b_1, b_2 \in R$, and (S,d), $(S',d') \in \bar{\bar{W}}$ and define $d'_i = a_i d_i + b_i$, $i=1,2$ and $S' = \{X \in R^2 / x_i = a_i y_i + b_i,\ i=1,2,\ y \in S\}$
Then $F_i(S',d') = a_i F_i(S,d) + b_i$, $i=1,2$.

Condition 3:- If $(S,d) \in \bar{\bar{W}}$ satisfies $d_1 = d_2$ and $(x_1,x_2) \in S$ implies $(x_2,x_1) \in S$, then $F_1(S,d) = F_2(S,d)$.

Condition 4:- If $x > F(S,d)$ then $x \notin S$

Condition 5:- Let (S,d) and (S',d') satisfy (a) $Z(S) = Z(S')$, (b) $d_1 = d'_1$, $d_2 \leqq d'_2$ and (c) $S \subseteq S'$. Then $F_2(S',d') \geqq F_2(S,d)$. If in addition $S=S'$, then $F_1(S',d') \leqq F_1(S,d')$ with $F(S',d') \neq F(S,d)$ if $(d_1,d_2) \neq (d'_1,d'_2)$

Condition 1 stipulates individual rationality.

Condition 2, requires that the solution should be invariant to positive affine utility transformations. A positive affine transformation of utility is a transformation of the form $x_i = a_i y_i + b_i$, defined for all y_i, that moves the zero point of utility in an arbitrary way (b_i can be positive or negative), and changes the scale of units on which utility is measured (a_i must be strictly positive).

Condition 3 imposes symmetry: If the attainable set S is symmetric about a 45° line through the origin and $d_1 = d_2$, then $F_1(S,d) = F_2(S,d)$. This does not imply comparability of the two utility scales. Given symmetry if $F_1(S,d) \neq F_2(S,d)$ it would appear that one player was being favoured over the other.

Condition 4, weak Pareto optimality, states that $x \in S$ is Pareto optimal if there is no $x' \in S$ for which $x' > x$. That is, a point is not weakly Pareto optimal if there is another point that gives more to each player.

Condition 5, is the monotonicity condition we invoke in our analysis and which has been discussed earlier. A similar condition has been used by Moulin (1985) to characterize solutions to a class of problems with a somewhat different structure.

Our basic theorem is the following:

<u>Theorem 1</u>: The function $F(S,d)$ is well defined, satisfies Condition 1 to 5 and is the only function to satisfy these conditions.

<u>Proof</u>:- The function $F(S,d)$ is well defined, due to the convexity and compactness of S. That it satisfies condition 4 is apparent from its definition. Condition 1 follows from the fact, that (S,d) in definitely a comprehensive game (infact a proper comprehensive game) together with the fact that there exists $x \in S$ such that $x > d$. These two properties imply that d belongs to the interior of S and hence Condition 1 is satisfied, due to the definition of $F(S,d)$ and the definition of a Symmetric game.

To see that Condition 5 is satisfied, suppose that (S,d) and (S',d') are two games satisfying $Z(S) = Z(S')$, $d_1=d'_1$, $d_2 \leq d'_2$

$$\therefore \quad \frac{d'_2-Z_2(S')}{d'_1-Z_1(S')} = \frac{d_2-Z_2(S)}{d_1-Z_1(S)}$$

Let us call the ray joining d to Z (S,d) the <u>defining ray of d</u> and the ray joining d' to Z (S,d') the <u>defining ray of d'</u>. Thus the defining ray of d' has steeper slope than the defining ray of d. Hence the defining ray of d' intersects the upper right Pareto optimal frontier of S. Since $S \subseteq S'$ and the defining ray of both d and d' are positively sloped, we may conclude the following.

(a) If $\phi(d,S)$ is the ordinate of the point of intersection of the defiring ray of d with the upper right Pateto optimal frontier of S, then $\phi(d,S) = F_2(S,d)$

(b) If $\phi(d',S)$ is the ordinate of the point of intersection of the defiring ray of d' with the upper Pareto optimal frontier of S, then $\phi(d',S) \geq \phi(d$

(c) If $\phi(d',S')$ is the ordinate of the point of intersection of the defining ray of d' with the upper right Pareto optimal frontier of S', then,

$$\phi(d',S') = F_2(S',d')$$

and $$\phi(d',S') \geqq \phi(d',S)$$

Hence $$F_2(S',d') \geqq F_2(S,d)$$

If in addition $S \neq S'$, then $F_2(S',d') \geqq F_2(S,d)$ and the Pareto optimality of the solutions guarantee that $F_1(S',d') \leqq F_1(S,d)$. Further since $F(S',d')$ and $F(S,d)$ lie on distinct rays, $F(S',d') \neq F(S,d)$

To see that only $F(S,d)$ satisfies the various conditions, it is first shown to hold for games in which $d = 0$ and $Z_1(S) = Z_2(S) = 0$. Extension to be full class of games follows from the requirement that the solution satisfy invariance to affine transformation (Condition 2). Denote the true solution by $F^*(S,d)$ and let S' be the convex hull of the set of points $\{(0,1), (1,F_2(S,d)), (F_1(S,d),1), F(S,d)\}$. By conditions 1,3 and 4, $F^*(S',d) = F(S,d)$. But $S' \subseteq S$ and Condition 5 implies $F_i^*(S,d) = F_i(S',d)$, $i = 1,2$. This follows because the threat point of both games being equal, the requirement on the threat point Condition 5 easily follows. Hence $F^*(S,d) = F(S,d)$ as was required to be proved.

Q.E.D

3. <u>Conclusion</u>:- We have thus proved in this paper the existence of a bargaining solution which satisfies the monotonicity property. With a slight modification of Condition 5, incorporating the notion of an ideal point (See Roth 1979)) instead of the point of minimal expectation, another solution can be shown to exist which is maximal on the ray connecting the threat point to the ideal point. Our purpose however, would have been served, it it is realized that the monotonicity property w ich is so intuitive in the economics literature, arises very naturally in some solutions to bargaining games.

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