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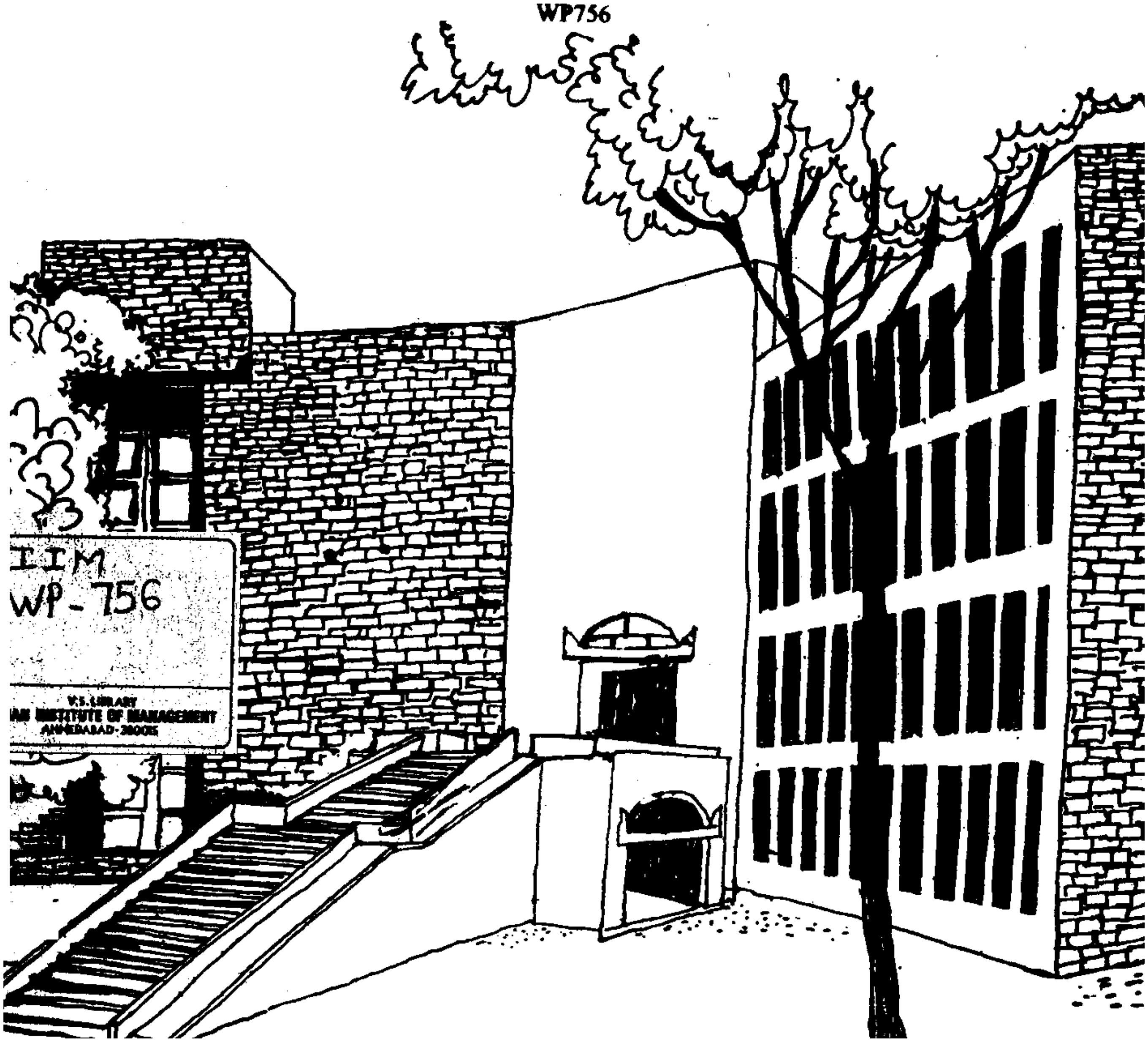

THE MAX-MIN SOLUTION FOR VARIABLE THREAT GAMES

By

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ABSTRACT

In this paper we obtain general inequality properties that max-min strategic solutions to Variable Threat Games satisfy under a set of very plausible assumptions.

1. Introduction :- In a pure bargaining problem between a group of two participants there is a set of feasible outcomes, anyone of which will be the result if it is specified by the unanimous agreement of all the participants. In the event that no unanimous agreement is reached, a given disagreement outcome obtains. Thus, a 2 - person bargaining problem is a pair (H,d) of a subset H of IR^2 and of a point $d \in H$. IR^2 is the utility space, H is the feasible set, and d is the disagreement (or threat)point . If the agents unanimously agree on a point x of H, they obtain x. Otherwise, they obtain d. Given a class of 2-person bargaining problems, a solution is a function F associating with every (H,d) in the class a point $F(H,d) \in H$, representing the compromise reached by the agents. In come contexts, $F(H,d)$ may alternatively be interpreted as the compromise recommended to the agents by some impartial arbitrator.

In Lahiri (1988), we analysed a model in which binding agreements are possible, but in which each player has considerable scope for action in the absence of an agreement, and in which the decision of each player affects both of them. The earliest known exposition is the work of Nash (1953). We view the above situation as in Owen (1982), which combines arbitration with a non-cooperative game. Suppose each of the two players has a strategy set, S_i, for player i, and assume that $p_i(s_1,s_2)$ is the payoff function for player i in the absence of an agreement. Thus the game $(\{1,2\}, S, P)$ is a default non-cooperative game that the two players must play if they cannot agree. There is no threat point as such. Suppose further that $H \subseteq IR^2$ consists of all pay-off points that the two players can reach by means of binding agreements. H would naturally contain as a subset all those points attainable in the default game. Such games are called <u>variable threat games</u>. Then $(\{1,2\}, S, F(H,P(.)))$, defines the associated non-cooperative game which combines both the default non-cooperative game as well as the two-pwerson bargaining problem.

In Lahiri (1988) we obtained a saddle point property that equilibrium threat strategies must satisfy, under a set of very plausible assumptions. The solution concept chosen for that analysis was the Nash non-cooperative equilibrium concept. Hence equilibrium threat strategies are self-enforceable. In this paper, we will consider somewhat different behavioural assumptions. Each agent is assumed to exhibit a form of risk averse behaviour characterized by the maximin postulates. Given every one of his possible strategies, he will determine the most unfavourable (to him) configuration of the other agents' strategies, and will pick his own strategy so as to maximize the pay-off to him of this worst outcome. The above behaviour is much more general than the Nash behaviour, in the sense that maximin equilibria continue to exist even in situations where Nash equilibria do not. Even if the pay-off function $P_i(.,.)$ of agent i is concave in his own strategy (a rather common place requirement for the existence of Nash equilibria), the composite function $F_i(H,P(.))$ may fail to be so.

2. <u>The Model</u>:- The class of games studied in this paper is based on a pair (H,d) and by the rule that the players will attain any single pay off point in H that they jointly agree on. In the absence of an agreement, they attain 'd'.

Definition 1 :- The pair $\Gamma = (H,d)$ is a two-person fixed threat bargaining game if $H \subseteq IR^2$ is compact, convex with non-empty interior, $d \in H$, and H contains at least one element U, such that $U \gg d$. The requirement that H has nonempty interior, precludes the possibility that H is merely a positively sloped line segment.

We make the following blanket assumption on H :-

Assumption 1 :- If $x, y \in H$, $x \neq y$, $y \leq x$, then $t x + (1-t) y$ belong to int. (H) $\forall t \in (0,1)$.

Note that assumption 1 is strictly weaker than assuming that H is strictly convex. We could without further injury require Assumption 1 to hold only for all $x \in H$ for which there does not exist U belonging to H with $U_1 > X_1$ and $U_2 > X_2$ where $U = (U_1, U_2)$ and $X = (X_1, X_2)$.

Definition 2 : The set of two-person fixed threat bargaining games is denoted W.

Definition 3 :- A solution to $(H,d) \in W$ is a point $F(H,d) \in H$. A solution is defined $\forall (H,d) \in W$.

Given $(H,d) \in W$, its Nash (1950) solution outcome $N(H,d)$ is the point where the product $(x_1-d_1)(x_2-d_2)$ is maximized for $x = (x_1,x_2) \in H$, with $x \geqq d$, its Egalitarian (Kalai (1977)) solution outcome is the point $E(H,d)$ which is maximal along the ray $x_1-d_1 = x_2-d_2$; its Kalai -Somorodinsky (1975) solution outcome $K(H,d)$ is the maximal point of H on the segment connecting d to $M(H,d)$, where for each i, $M_i(H,d) \equiv \max \{x_i / x \in H; x \geqq d\}$.

Suppose that each of the players has a strategy set, S_i for player i, that is compact and convex, and assume that $P_i(s_i, s_2)$ is the pay-off for player 'i' in the absence of an agreement. Thus the game $(\{1,2\}, S, P)$ is a default non-cooperative game that the two players must play if they cannot agree. There is no threat point as such.

<u>Definition 4</u> :- $\Gamma = (N,S,P,H)$ is a variable threat two-person co-operative game where (N,S,P) is a two-person non-cooperative game that is played if no agreement is reached and the compact, convex set H is the co-operative attainable pay-off set. H contains $\{P(s) \in IR^2 / s \in S\}$.

The arbitration game proceeds by having the two players simultaneously choose strategies for the default game, $s_i \in S_i$, which are used to determine a threat point, $P(s)$. The arbitrated outcome to the game is the bargaining solution $F(H,P(s))$. Thus, the players are not interested in the pay-offs $P_i(s)$ for their own sakes; they care about the effect on the final outcome that is due to their choices of s_i. Potentially, any point on the Pareto optimal frontier of H could be an arbitrated outcome.

The set up outlined so far is identical to the one in Lahiri (1988). Let us now define the concept of a <u>max-min strategy</u>.

<u>Definition 5</u> :- $\bar{s} \in S$ is called a <u>max-min threat strategy</u> for the <u>variable threat game</u> (N,S,P,H) <u>equipped with the solution</u> F if

$$F_1(H,P(\bar{s}_1, s_2(\bar{s}_1))) \geq F_1(H,P(s_1, s_2(s_1))) \quad \forall s_1 \in S_1$$

$$F_2(H,P(s_1(\bar{s}_2), \bar{s}_2))) \geq F_2(H,P(s_1(s_2), s_2))) \quad \forall s_2 \in S_2$$

where

$$s_2(s_1) \in \left\{ s'_2 \in S_2 / F_1(H,P(s_1,s'_2)) = \min_{s_2 \in S_2} F_1 (H,P(s_1,s_2)) \right\} \quad (1)$$

and $$s_1(s_2) \in \left\{ s'_1 \in S_1 / F_2(H,P(s'_1,s_2)) = \min_{s_1 \in S_1} F_2 (H,P(s_1,s_2)) \right\} \quad (2)$$

3. <u>The Nash Fiber</u> :-

In this section we shall introduce the concept of a fiber and impose certain conditions on the solution F to a bargaining game.

<u>Condition 1</u> :- $F(H,d) \geqslant d \quad \forall (H,d) \in W$

<u>Condition 2</u> :- Let $(H,d) \in W$ and $u \in H$ with $u \neq F(H,d)$. Then either $F_1(H,d) > u_1$ or $F_2(H,d) > u_2$.

<u>Condition 3</u> :- Let (H,d) and $(H,d') \in W$ with $d'_i = d_i$ and $d'_j \geqq d_j$ for $j \neq i$. Then, $F_j(H,d') \geqq F_j(H,d)$. If in addition $d'_j > d_j$, then $F_j(H,d') > F_j(H,d)$.

An auxilliary concept, that of a fiber (see Brito, Buoncristiani and Intrilligator (1977); Lahiri (1988)) is introduced below.

<u>Definition 6</u> :- The set $T_i(H,d;F;u)$, $i = 1,2$, is the set of all threat point d', leading to a non-conflict allocation at least as beneficial to player i as the Pareto optimal allocation u.

$$T_i(H,d;F;u) = \left\{ d' / d' \in H,\ F_i(H,d') \geqslant u_i \right\} \qquad (i = 1,2).$$

These sets are nested in the following sense : if $\bar{u}_i \quad \hat{u}_i$, then $T_i(H,d;F;\bar{u}) \subseteq T_i(H,d;F;\hat{u})$.

<u>Definition 7</u> :- The intersection of $T_1(H,d;F;u)$ and $T_2(H,d;F;u)$ in the set of all threat points leading to the non-conflict allocation u.

$\Pi(H,d;F;u) = T_1(H,d;F;u) \cap T_2(H,d;F;u)$ is called the <u>u-fiber</u>. Since it will not cause ambiguity the fiber will be denoted by $\Pi(H;d;F(H,d))$ or simply $\Pi(H;F(H,d))$.

<u>Condition 4:</u> Let $(H,d) \in W$ and u be a Pareto-optimal point of H. Then $\Pi(H,d;F,u)$ is a convex set containing more than one point. Such fibers are called Nash fibers.

The following results have been stated and proved in the last mentioned reference.

<u>Theorem 1:</u>- $\Pi(H,d;F;u)$ is a positively sloped straightline. If $F(H,d) \neq F(H,d')$, then $\Pi(H,d;F(H,d)) \cap \Pi(H,d';F(H,d')) = \emptyset$

Let us further impose the condition :

<u>Condition 5</u> :- $F(H,.) : H^{\circ} \longrightarrow H$ is continuous where

$$H^{\circ} = \{ d \in H / \exists\, u \in H \text{ with } u \gg d \}$$

As a consequence of this condition we obtain,

<u>Theorem 2</u> :- The Nash fiber $\Pi(H,d;F(H,d))$ is an interval with both endpoints lying on the boundary of H.

This theorem has also been stated and proved in Lahiri (1988). To establish the theorem we have in mind in the next section we also need to assume:

<u>Condition 6</u> :- $\forall\, (H,d) \in W$, $F(H,d)$ is an endpoint of $\Pi(H,d;F(H,d))$. This and all the other conditions mentioned above have been discussed extensively in Lahiri (1988). In the following section we discuss the properties satisfied by max-min threat strategies.

4. A General Property of Max-Min Threat Strategies:-

In this section we prove a general property satisfied by all max-min threat strategies.

Theorem 3 :- Let $\bar{s} \in S$ be a max-min strategy for the variable threat game (N,S,p,H) equipped with the solution F. Suppose F satisfies Conditions 1 to 6 and let $s_2(s_1)$, $s_1(s_2)$ satisfy equations (1) and (2) respectively. Then,

$$\frac{F_2(H,p(\bar{s}_1, s_2(\bar{s}_1))) - p_2(\bar{s}_1,s_2(\bar{s}_1))}{F_1(H,p(\bar{s}_1,s_2(\bar{s}_1))) - p_1(\bar{s}_1,s_2(\bar{s}_1))} \geqslant \frac{F_2(H,p(\bar{s}_1, s_2(\bar{s}_1))) - p_2(s_1,s_2(s_1))}{F_1(H,p(\bar{s}_1,s_2(\bar{s}_1))) - p_1(s_1,s_2(s_1))} \quad \forall s_1 (\in S_1)$$

and

$$\frac{F_2(H,p(s_1(\bar{s}_2), \bar{s}_2))) - p_2(s_1(\bar{s}_2), \bar{s}_2)}{F_1(H,p(s_1(\bar{s}_2), \bar{s}_2))) - p_1(s_1(\bar{s}_2), \bar{s}_2)} \leqslant \frac{F_2(H,p(s_1(\bar{s}_2),\bar{s}_2))) - p_2(s_1(s_2),s_2)}{F_1(H,p(s_1(\bar{s}_2),\bar{s}_2))) - p_1(s_1(s_2),s_2)}$$

$$\forall s_2 \in S_2.$$

Further,

$$F_1(H, p(s_1(\bar{s}_2), \bar{s}_2)) \geq F_1(H,p(\bar{s}_1,\bar{s}_2)) \geq F_1(H,p(\bar{s}_1, s_2(\bar{s}_1))$$

$$F_2(H,p(\bar{s}_1, s_2(\bar{s}_1)) \geq F_2(H,p(\bar{s}_1,\bar{s}_2)) \geq F_2(H,p(s_1(\bar{s}_2), \bar{s}_2)).$$

Proof :- Since $\bar{s} \in S$ is a max-min strategy,

$F_2(H, p(\bar{s}_1,s_2)) \geqslant F_2(H,p(\bar{s}_1,s_2(\bar{s}_1))) \geqslant F_2(H,p(s_1, s_2(s_1))) \; \forall \; s_1 \in S_1$ and $s_2 \in S_2$

and $F_1(H,p(s_1,\bar{s}_2)) \geqslant F_1(H,p(s_1(\bar{s}_2), \bar{s}_2)) \geqslant F_1(H,p(s_1(s_2),s_2)) \; \forall \; s_1 \in S_1$ and $s_2 \in S_2$.

Suppose $\Pi(H,p(s_1,s_2(s_1)))$ lies below $\Pi(H,p(\bar{s}_1,s_2(\bar{s}_1))$. Consider the projection of any point d $\Pi(H,p(\bar{s}_1,s_2(\bar{s}_1))$ onto the pay-off axis of player 2. It intersects $\Pi(H,p(\bar{s}_1,s_2(\bar{s}_1))$ at a point $d' = (d'_1,d'_2)$ where $d'_1 < d_1$ and $d'_2 = d_2$.

By monotonicity,

$F_1(H,P(\bar{s}_1, s_2(\bar{s}_1)) = F_1(H,d') < F_1(H,P(s_1,s_2(s_1))$ contradicting $\bar{s}$ is a max-min threat strategy. Hence $\Pi(H,P(s_1,s_2(s_1))$ lies on or above $\Pi(H,P(\bar{s}_1, s_2(\bar{s}_1))) \forall s_1 \in S_1$. By a symmetric argument $\Pi(H,P(s_1(s_2) s_2))$ lies on or below $\Pi(H, P(s_1(\bar{s}_2), \bar{s}_2)) \forall s_2 \in S_2$.

Since $P(s_1,s_2(s_1)) \in \Pi (H,P(s_1,s_2(s_1))) \forall s_1 \in S_1$, the slope of the line joining $P(s_1,s_2(s_1))$ to $F(H, P(\bar{s}_1,s_2(\bar{s}_1))$ is less than or equal to the slope of the line joining $P(\bar{s}_1, s_2(\bar{s}_1))$ to $F(H,P(\bar{s}_1,s_2(\bar{s}_1)))$. Similarly the slope of the line joining $P(s_1(s_2),s_2)$ to $F(H,P(s_1(\bar{s}_2), \bar{s}_2))$ is greater than or equal to the slope of the line joining $P(s_1(\bar{s}_2), \bar{s}_2)$ to $F(H,P(s_1(\bar{s}_2), \bar{s}_2))$. This proves the first part of our theorem.

Now $F_1(H,P(\bar{s}_1,\bar{s}_2)) \geq \underset{s_1 \in S_1}{\text{max}}\text{-}\underset{s_2 \in S_2}{\text{min}} F_1(H,P(s_1,s_2)) = F_1(H,P(\bar{s}_1,s_2(\bar{s}_1))$

and $F_2(H,P(\bar{s}_1,\bar{s}_2)) \geq \underset{s_2 \in S_2}{\text{max}} \underset{s_1 \in S_1}{\text{min}} F_2(H,P(s_1,s_2)) = F_2(H,P(s_1(\bar{s}_2), \bar{s}_2))$

The second inequality and the Pareto optimality of F implies,

$F_1(H,P(s_1(\bar{s}_2), \bar{s}_2)) \geq F_1(H,P(\bar{s}_1,\bar{s}_2))$.

Similarly we prove the other pair of inequalities.

Q.E.D.

A second theorem along these lines which follows from our conditions is the following :-

<u>Theorem 4</u> :- Let $\bar{s} \in S$ be a max-min strategy for the variable threat game (N,S,P,H) equipped with the solution F. Suppose F satisfies Condition 1 to 6 and let $s_2(s_1)$, $s_1(s_2)$ be as defined earlier. Then,

$$\frac{F_2(H,P(\bar{s}_1,s_2(\bar{s}_1))) - P_2(\bar{s}_1,s_2(\bar{s}_1))}{F_1(H,P(\bar{s}_1,\ s_2(\bar{s}_1))) - P_1(\bar{s}_1,s_2(\bar{s}_1))} \leqslant \frac{F_2(H,P(\bar{s}_1,s_2(\bar{s}_1))) - P_2(\bar{s}_1,\ s_2)}{F_1(H,P(\bar{s}_1,s_2(\bar{s}_1))) - P_1(\bar{s}_1,s_2)} \quad \forall\ s_2 \in$$

$$\frac{F_2(H,P(s_1(\bar{s}_2),\bar{s}_2))) - P_2(s_1(\bar{s}_2),\bar{s}_2)}{F_1(H,P(s_1(\bar{s}_2),\bar{s}_2))) - P_1(s_1(\bar{s}_2),\bar{s}_2)} \geqslant \frac{F_2(H,P(s_1(\bar{s}_2),\bar{s}_2)))-P_2(s_1,\bar{s}_2)}{F_1(H,P(s_1(\bar{s}_2),\bar{s}_2)))-P_1(s_1,\bar{s}_2)} \quad \forall\ s_1 \in S$$

Proof :- Follows immediately from a reasoning similar to the proof of theorem 3 and the fact that

$$F_2(H,P(\bar{s}_1,s_2)) \geqslant F_2(H,P(\bar{s}_1,s_2(\bar{s}_1))) \quad \forall\ s_2 \in S_2$$

$$F_1(H,P(s_1,\bar{s}_2)) \geqslant F_1(H,P(s_1(\bar{s}_2),\ \bar{s}_2)) \quad \forall\ s_1 \in S_1.$$

Q.E.D.

5. Conclusion :- It is very easy to conceive of economic contexts in which the above theory of variable threat games is applicable. Consider a two-person two good pure-exchange economy, where the only information available to an impartial arbitrator is the total available endowment between the agents and that the set of all utility functions representing the preferences of the agents, belong to a family which have a common image in the pay-off space. The arbitration procedure begins by each agent being asked to announce the utility of his current endowment and the arbitrator consequently allocates a pay-off possibility on the frontier of the space of feasible pay-offs. In this game there is considerable scope for strategic behaviour on the part of the agents, while announcing the utility derived from their initial endowment. Given the rather general nature of the conditions involved, our results find applicability for most bargaining solutions.

References :

1. Kalai, Ehud (1977): "Proportional Solutions To Bargaining Situations: Interpersonal Utility Comparisons", Econometrica 45, 1623-1630.

2. Kalai, Ehud and M. Smorodinsky (1975): "Other Solutions to Nash's Bargaining Problem", Econometrica 43, 513-518.

3. Lahiri, Somdeb (1988): "A General Saddle Point Property For Two Person Variable Threat Games", IIM-A, Working Paper No. 732.

4. Nash, John F (Jr.) (1953): "Two Person Co-operative Games", Econometrica 21: 128-140.

5. Owen, Guillermo (1982): "Game Theory", Academic Press Inc.