BARGAINING WITH A VARIABLE POPULATION FOR GAMES WITH A REFERENCE POINT

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We consider axiomatic models of bargaining defined over a domain of problems containing different numbers of agents, define a concept called restricted monotonicity with respect of changes in the number of agents, and show that a solution due to Lahiri (1988), which satisfies monotonicity with respect to the disagreement point, meets the aforementioned requirement. Finally, we consider a class of solutions which are defined with respect to a reference point (concept due to Thomson (1981)) and show that this class satisfies our axiom of restricted monotonicity.
1. **Introduction:** There is an infinite population \( I \) of agents, indexed by the positive integers. Arbitrarily finite subsets of \( I \) may be confronted by a problem. The family of these subsets is denoted \( \sum^p \). Given \( P \in \sum^p \), \( \sum^p \) is the class of problems that the group \( P \) may conceivably face. Each \( (S, d) \in \sum^p \) is an ordered pair where \( S \) is a subset of \( IR^p_+ \), the non-negative portion of the \(|P|\)-dimensional Euclidean space with coordinates indexed by the members of \( P \), and \( d \in S \). Each point of \( S \) represents the von Neumann-Morgenstern utilities achievable by the members of \( P \) through some joint action. It is assumed that

1. \( S \) is a compact subset of \( IR^p_+ \) containing at least one strictly positive vector;
2. \( S \) is convex;
3. \( S \) is comprehensive (i.e., if \( X, Y \in IR^p_+ \), \( X \in S \), and \( x \geq y \), then \( Y \in S \)).

We in addition sometimes invoke the assumption that

4. \( d > 0 \)

(Given \( x, y \in IR^p, x \geq y \) means that \( x_i \geq y_i \) for all \( i \in P \); \( x \geq y \) means that \( x \geq y \) and \( x \neq y \); \( x > y \) means that \( x_i > y_i \) for all \( i \in P \)).

Compactness of \( S \) is a technical assumption, made for convenience. Requiring that \( S \in IR^p_+ \) implies that an appropriate choice of a zero of the utilities has been made. Assuming that \( S \) contains strictly positive vector ensures that all agents are nontrivially involved in the bargaining. Convexity of \( S \) holds in particular if agents can randomize jointly, since utilities are von Neumann-Morgenstern utilities, but this property of \( S \) may hold even in situations where randomization is not permitted. Similarly, comprehensiveness holds in particular if, but not only if, utilities are freely disposable.
We will also consider the domain \( \tilde{\Sigma}^p \) of problems satisfying the following condition in addition to conditions 1 to 3:

5. If \( x, y \in S \) and \( x \succ y \), then there exists \( z \in S \) with \( z \succ y \).

Then, the undominated boundary of \( S \) contains no segment parallel to a coordinate subspace. Note that any element of \( \tilde{\Sigma}^p \) can be approximated in the Hausdorff topology by a sequence of elements of \( \tilde{\Sigma}^p \). Finally, we set

\[
\Sigma \equiv \bigcup_{P \in \mathcal{P}} \tilde{\Sigma}^p \quad \text{and} \quad \tilde{\Sigma} \equiv \bigcup_{P \in \mathcal{P}} \tilde{\Sigma}^p
\]

In the traditional formulation of the bargaining problem, it is typically assumed that a fixed number of agents are involved.

What has been presented above is a framework (originally due to Thomson (1982)) in which the bargaining problem with a variable population can be analysed.

A solution is a function \( F : \bigcup_{P \in \mathcal{P}} \tilde{\Sigma}^p \rightarrow \bigcup_{P \in \mathcal{P}} \mathbb{R}^p \) such that for each \( P \) and each \( (S, d) \in \tilde{\Sigma}^p \), a unique point \( F(S, d) \in S \) is associated.

2. Monotonicity: We first introduce some notation. Given \( P \) in \( \mathcal{P} \) and \( x_1, \ldots, x_k \) in \( \mathbb{R}^p \), \( \text{co}\{x_1, \ldots, x_k\} \) is the convex and comprehensive hull of these \( k \) points, that is, the smallest convex and comprehensive subset of \( \mathbb{R}^p \) containing them. Also, \( e_p \) is the vector in \( \mathbb{R}^p \) whose coordinates are all equal to one. Given \( i \) in \( P \), \( e_i \) is the \( i \)th unit vector. Given \( P, Q \) in \( \mathcal{Q} \) with \( P \subset Q \), \( y \), a point of \( \mathbb{R}^3 \), and \( T \), a subset of \( \mathbb{R}^Q \), \( y_p \) and \( T_p \) designate the projections on \( \mathbb{R}^p \) of \( y \) and \( T \), respectively.

We will try to formally pose the question, whether it is possible for a solution to assign greater utility to an agent initially present, after the departure of some agents with equally valid claims on the
fixed resources, then the utility he had been assigned originally.

The standard economic problem motivating this kind of analysis is
that of dividing fairly a bundle of goods among a group of agents.

The number of agents involved in the division is allowed to vary
while the resources at their disposal remain fixed.

The following axiom proposed by Thomson (1982, 1983) relates solution
outcomes across cardinalities.

**Monotonicity with respect to changes in the number of agents (MON):**

For all $P, Q \in \mathcal{P}$ with $P \subseteq Q$, for all $(T, d) \in \sum^Q$, if $S = \{ x \in T / x \in Q \cdot P \geq d \}_{P}$

then $F(S, d) \geq F(T, d)$.

We view this assumption as being worded extremely liberally in that it
is assumed that the possibilities open to $P$ in the absence of $Q \cdot P$
includes all the possibilities available to $P$ in the presence of $Q \cdot P$.

It is our feeling that it is somewhat more natural to view the situation
from a different perspective. In the absence of $Q \cdot P$, $P$ is more likely
to be restricted to only those possibilities which are consistent with
the disagreement payoff vector of $Q \cdot P$. In a way we view $Q$ as the
original combine. When bargaining breaks down and $Q \cdot P$ decide not to
participate in the bargaining process, they receive their disagreement
payoff. The coalition $P$ now bargains amongst itself for feasible
alternatives which are consistent with the disagreement point of $Q \cdot P$.

We thus define:

**Restricted Monotonicity With Respect to changes in the number of
agents (R.MON):** For all $P, Q \in \mathcal{P}$ with $P \subseteq Q$, for all $(T, d) \in \sum^Q$, if

$S = \{ x \in T / x \in Q \cdot P = d \}_{Q \cdot P}$

then $F(S, d) \geq F(T, d)$.

Our first theorem postulates the equivalence of (MON) and (R.MON).
Theorem 1: Let $\Sigma$ be the set of all bargaining problems satisfying conditions 1 to 3, $\Sigma = \bigcup \Sigma^P$. Let $(T, d) \in \Sigma^P$. Then both $(S_1, d_p)$ and $(S_2, d_p) \in \Sigma^P$ where $P \subseteq Q$ and

$$S_1 = \left\{ x \in T | x_{Q \setminus P} \geq d_{Q \setminus P} \right\}, S_2 = \left\{ x \in T | x_{Q \setminus P} = d_{Q \setminus P} \right\}.$$

Also $S_1 = S_2$

Proof: Clearly $S_2 \subseteq S_1$. To show $S_1 \subseteq S_2$, let $x \in S_1$

$$x \in T \text{ and } x_{Q \setminus P} \geq d_{Q \setminus P}.$$

By property 3, $(x, x_{Q \setminus P}) \geq (x, d_{Q \setminus P}) \in IR^+_+$ implies,

$$(x, d_{Q \setminus P}) \in T.$$

$$x \in S_2$$

Hence $S_1 = S_2$

Now, since $(S_2, d_p)$ satisfies conditions (1) to (3) both $(S_1, d_p)$ and $(S_2, d_p) \in \Sigma^P$.

Q.E.D.

Note Condition 3 is essential in establishing this equivalence.

It is very easy to construct examples where $S_1 \neq S_2$ if Condition 3 is violated.

Some axioms which play a vital role in our subsequent characterization are:

Weak Pareto-Optimality (WPO): For all $P \in \mathcal{P}$, for all $(S, d) \in \Sigma^P$, for all $y \in IR^+_+$, if $y > F(S, d)$, then $y \notin S$.

Pareto-Optimality (PO): For all $P \in \mathcal{P}$, for all $(S, d) \in \Sigma^P$, for all $y \in IR^+_+$, if $y \geq F(S, d)$, then $y \notin S$.

Individual Rationality (IR): For all $P \in \mathcal{P}$, for all $(S, d) \in \Sigma^P$,

$f(S, d) > d$. 

We denote by $\mathcal{WPO}(S)$ and $\mathcal{IR}(S)$, respectively, the sets of points that are weakly Pareto-optimal, Pareto-optimal and individually rational for $S$ (observe that $\mathcal{PO}(S) \subset \mathcal{WPO}(S)$).

**Symmetry (SY):** For all $p \in \mathcal{P}$, for all $(S,d) \in \sum^{\mathcal{P}}$, if for all one-one function $\forall : p \rightarrow p$, $S = \{x \in IR^{p} : \exists x \in S \text{ such that } x_{i}(i) = x_{j}(j)\}$, and $d_{i} = d_{j}$ for all $i, j \in p$, then for all $i, j \in p$, $F_{i}(S, d) = F_{j}(S', d)$.

A related condition is

**Anonymity (AN):** For all $p, p' \in \mathcal{P}$ with $|p| = |p'|$, for all one-one function $\forall : p \rightarrow p'$, for all $(S,d) \in \sum^{\mathcal{P}}$, $(S', d') \in \sum^{\mathcal{P}}$, if $S' = \{x \in IR^{p'} : \exists x \in S \text{ such that } x_{i}(i) = x_{j}(j)\}$ and $d_{i}' = d_{j}'$ for all $i \in p$, then for all $i \in p$, $F_{i}(S', d') = F_{i}(S, d)$.

**Scale Invariance (S.INV):** For all $p \in \mathcal{P}$, for all $(S,d)$ and $(S', d') \in \sum^{\mathcal{P}}$, for all $a \in IR^{p}$, if $S' = \{x \in IR^{p} : \exists x \in S, x_{i} = a_{i}x_{i}\}$, and $d_{i}' = a_{i}d_{i}$, then for all $i \in p$, $F_{i}(S', d') = a_{i}F_{i}(S, d)$.

**Continuity (CONT):** For all $p \in \mathcal{P}$, for all sequences $(S, d)$ of elements of $\sum^{\mathcal{P}}$, if $S \in S$, $d \rightarrow d$ and $(S,d) \in \sum^{\mathcal{P}}$, then $F(S, d) \rightarrow F(S)$. (In this definition, convergence of $S$ to $S$ is evaluated in the Hausdorff topology).

These axioms are standard. $\mathcal{WPO}$ states that it is infeasible to make all agents simultaneously better off, and $\mathcal{PO}$ that it is infeasible to make an agent better off without hurting some other agent. $\mathcal{IR}$ states that all agents are better off after bargaining than they would be at their disagreement payoff. $\mathcal{SY}$ states that if a problem is invariant under all permutations, then all agents should get the same amount. $\mathcal{AN}$ states that the names of the agents do not matter; only the geometrical structure of the problem at hand is relevant. $\mathcal{S.INV}$ states that the solution is independent of the choice of
particular members of the equivalence classes of utility functions representing
the agents' von Neumann-Morgenstern preferences. CONT states that small
changes in the data defining the problem cause only small changes in
solution outcomes.

Monotonicity With Respect to the Disagreement Point and \((R, \text{MON})\):

Let \(\sum_x^p\) be the subset of \(\sum_x^p\) consisting of all games satisfying conditions
1 to 4 and \(\tilde{\sum}_x^p\) be the subset of \(\sum_x^p\) consisting of all games satisfying
Conditions 1 to 5. Thus, \(\tilde{\sum}_x^p = \sum_x^p \cap \tilde{\sum}_x^p\).

We consider the following solution \(M: \sum_x^p \rightarrow UIp\) where
\[
\sum_x^p = \bigcup_{p \in P} \sum_x^p
\]
defined thus:

\[
\forall (S, d) \in \sum_x^p, \quad \frac{M_i(S, d)}{d_i} = \frac{M_j(S, d)}{d_j}
\]
for all \(i, j \in P\).

and \(X = \frac{X_i}{d_i} = \frac{X_j}{d_j}\) for all \(i, j \in P\), \(X \in S\) implies \(M(S, d) \succ X\).

In Lahiri (1988) it has been shown that \(M\) satisfies and is the only solution to
satisfy the following conditions:

**Axiom 1**: WRP

**Axiom 2**: IR

**Axiom 3**: S_INV

**Axiom 4**: SY

**Axiom 5**: (Monotonicity with respect to the Disagreement Point):

Let \((S, d)\) and \((S', d')\) belong to \(\sum_x^p\) and satisfy:

a) for some \(i \in P\), \(d_j = d'_j \forall j \in P - \{i\}; \quad d_i \leq d'_i\)

b) \(S \subseteq S'\).

Then \(F_i(S', d') \geq F_i(S, d)\).
If in addition \( S = S' \), then \( F_j(S', d') \leq F_j(S, d) \) for \( j \in N \setminus \{1\} \), with \( f(S', d') \neq f(S, d) \) if \( d \neq d' \).

We first show the following:

**Theorem 2.1** \( M \) satisfies (AV).

**Proof:** Let \( p, p' \in \mathcal{O} \) with \( \| p \| = \| p' \| \) and let \( \gamma : \mathcal{P} \to \mathcal{P}' \) be a one-one function. Let \( (S, d) \in \Sigma \), and define,

\[
S' = \left\{ x' \in \mathbb{R}^{p'} \mid \exists x \in S \text{ such that } \forall i \in p, x'_\gamma(i) = x_i \right\}
\]

and \( d' = (d'_j) \) \( j \in p' \) with \( d'_\gamma(i) = d_i \).

\[
\frac{x'_\gamma(i)}{d'_\gamma(i)} = \frac{x_i}{d_i} \quad \forall i \in p
\]

\[
\frac{M_\gamma(i) (S', d')}{d'_\gamma(i)} = \frac{M_\gamma(i) (S', d')}{d'_\gamma(i)} \quad \forall i, j \in p.
\]

and \( (X'_\gamma(i))_{i \in p} \) with \( \frac{x'_\gamma(i)}{d'_\gamma(i)} = \frac{x'_\gamma(j)}{d'_\gamma(j)} \quad \forall i, j \in p \)

implies \( M_\gamma(i) (S', d') > x'_\gamma(i) \quad \forall i \in p \).

\[
\Rightarrow M_\gamma(i) (S', d') > x_i \quad \forall i \in p
\]

whenever, \( \frac{x_i}{d_i} = \frac{x_j}{d_j} \quad \forall i, j \in p \)

Further, \( \frac{M_\gamma(i) (S', d')}{d_i} = \frac{M_\gamma(i) (S', d')}{d_j} \quad \forall i, j \in p \)

\[
\therefore M_\gamma(i) (S', d') = M_\gamma(S, d) \quad \forall i \in p.
\]

\( \therefore C.E.D. \)
Hence $M$ satisfies $WP$, $AK$, $S.INV$; that it satisfies $C.inT.$ follows by method similar to the ones used in the proof of Theorem 3.3 by Jensen and Tjä (1983). We next show that $M$ satisfies $R.MON$.

**Theorem 3:** $M$ satisfies $R.MON$.

**Proof:** Let $p, q \in \mathcal{P}$ with $p \subset q$. Let $(T, d) \in \sum \mathcal{P}$ and

$$S = \left\{ x \in T : x_{Q \setminus p} = d_{Q \setminus p}, p \right\}$$

So $m_j(T, d) = \frac{d_i}{d_i} m_i(T, d) \forall j \in Q$.

Observe, $(m_j(S, d_p), d_{Q \setminus p}) \in WP(T)$.

Suppose $(m_j(S, d_p), d_{Q \setminus p}) \notin WP(T)$.

there exists $y \in T \ni y \succ (m_j(S, d_p), d_{Q \setminus p})$.

Consider the point $(y, d_{Q \setminus p})$.

Clearly $y \succ (y, d_{Q \setminus p}) \succ (m_j(S, d_p), d_{Q \setminus p})$.

Since $T$ is comprehensive, $(y, d_{Q \setminus p}) \in T$.

By the definition of $S$, $y \notin S$.

But $y \succ m_j(S, d_p)$.

This contradicts that $m_j(S, d_p) \in WP(S)$.

Hence $(m_j(S, d_p), d_{Q \setminus p}) \in WP(T)$.

Clearly $m_j(S, d_p) = \frac{d_i}{d_i} m_i(S, d_p) \forall i, j \in Q$.

Now, $m_{Q \setminus p}(T, d) \succ d_{Q \setminus p}$.

Suppose $m_i(T, d) \succ m_i(S, d_p)$ for some $i \in Q$.

$$\therefore m(T, d) = \frac{d}{d_i} m_i(T, d) \succ d_i m_i(S, d_p) = m_j(S, d_p) \forall j \in P.$$
\[ \mathcal{M}(T,d) = (\mathcal{M}_p(T,d), \mathcal{M}_{d-p}(T,d)) \geq (\mathcal{M}(S,d_p), d_{d-p}) \]
contradicting that \((\mathcal{M}(S,d_p), d_{d-p}) \in \mathcal{WPO}(T)\).

Hence \(\mathcal{M}_j(T,d) \leq \mathcal{M}_j(S,d_p) \forall j \in P\).

\(\mathcal{M}_p(T,d) \leq \mathcal{M}(S,d_p)\) and \(\mathcal{M}\) satisfies \(\text{R-MON}\) as was required to be proved.

Q.E.D.

In Thomson (1983) it is shown that the Kalai-Smorodinsky (1975) solution also satisfies \(\mathcal{WPO}, \text{IR, S.INV, AN and R-MON}\). We will thus look for a general characterization of all solutions satisfying the above conditions on \(\sum_{\mathcal{K}}^P\).

4. Bargaining Problems with a reference point:

Thomson (1981) observes that for many solutions to bargaining problems, one often resorts to what is known as a reference function, which is a function singling out, for each bargaining problem, a point of the utility space summarizing its essential features and facilitating the evaluation of the relative bargaining strength of the players. Formally, we have the following:

**Definition 1:** A reference function \(g : \sum_{\mathcal{K}}^P \rightarrow \bigcup_{\mathcal{K}}^P \mathbb{R}^P\) associates to every bargaining problem \((S,d) \in \sum_{\mathcal{K}}^P\) a point \(g(S,d) \in IR^P\).

The role of the reference function is essentially informational. In the evaluation of their respective bargaining strengths, the players summarize what they see as the main features of the bargaining problem by focusing on particular outcomes, like the outcomes that are most favourable to each of them (see Kalai and Smorodinsky (1975); Rosenthal (1976); Roth (1977)). Observe that the definition of a reference function developed above, does not require that \(g(S,d)\) be a feasible alternative. In fact, the ideal point of Kalai and Smorodinsky, is feasible only in the trivial case in which it is the only Pareto-optimal outcome.
We shall consider reference functions $g$ which satisfy the following properties:

**Property 1 (S.INV):** For all $p \in P$, for all $(S,d)$ and $(S',d') \in \sum^P_\lambda$, for all $a \in IR^P$ if $S = \{x_i \in IR^P / i \in P, x_i = a_i x_i\}'$, $d' = (d_i)'$, $i \in P$ where $d_i' = a_i d$, $i \in P$, then for all $i \in P$, $g_i(S',d') = g_i(S,d)$.

**Property 2 (Modified Multilateral Stability or (M^2,STAB)):** For all $p$, $q \in P$ with $p \subseteq q$, for all $(S,d') \in \sum^P_\lambda$, $(T,d) \in \sum^q_\lambda$, if $S = \{x_i \in T / x_{q,p} = d_{q,p}\}'$ and $d' = d_p$, then $g(S,d') = g_p(T,d)$.

**Property 3:** For all $p \in P$ and $(S,d) \in \sum^P_\lambda$ the following is true:

either $g_i(S,d) > d_i \quad \forall \ i \in P$

or $g_i(S,d) < d_i \quad \forall \ i \in P$

**Property 4:** (AN): For all $p$, $p' \in P$ with $|p| = |p'|$, for all one-one function $\tau : p \rightarrow p'$, for all $(S,d) \in \sum^P_\lambda$, $(S',d') \in \sum^P_\lambda$, if $S = \{x_i \in IR^P / x_{\tau(i)} = x_i\}'$ and $d'_{\tau(i)} = d_i \quad \forall \ i \in P$, then for all $i \in P, g_i(S',d') = g_i(S,d)$.

Given a reference function $g : \sum^P_\lambda \rightarrow \bigcup_{P \in P} IR^P$ satisfying Properties 1,2,3 and 4 we consider a solution to bargaining problems in $\sum^P_\lambda$ defined thus:

Let $M : g : \sum^P_\lambda \rightarrow \bigcup_{P \in P} IR^P$ be given by

$M_g(S,d) = d + \alpha (S,d) (g(S,d) - d)$

where $\alpha (S,d) \in IR$ satisfies $d + \alpha (S,d) (g(S,d) - d) \in WP0 (S)$, for all $(S,d) \in \sum^P_\lambda$. 
It is easily observed that \( M_g \) is well-defined, satisfies Conditions 1, 2, 3, 4.

We shall now show that \( M_g \) satisfies (R, MCN).

**Theorem 4** - Let \( g \) be a reference function satisfying Properties 2 and 3.

Then \( M_g \) satisfies R, MCN.

**Proof:** - Let \( p, q \in P \) with \( P \subseteq Q \), \( (T, d) \in \sum_{x \in \mathcal{Q}} \) and \( S = \left\{ x \in T/X_Q, p = d_{Q-p} \right\} \).

To show, \( M_g(S, d_p) \geq (\mathcal{M}_g)_{P}(T, d) \)

\[
M_g(S, d_p) = d_p + \alpha(S, d_p) \left[ g(S, d_p) - d_p \right].
\]

and \( M_g(T, d) = d + \alpha(T, d) \left[ g(T, d) - d \right] \)

By property 2, \( g(S, d_p) = g_p(T, d) \).

Since \( T \) is comprehensive and convex, for the same reasons as in the proof of

Theorem 3, \((M_g(S, d_p), d_{Q-p}) \in WP(T)\).

Now, \((M_g)_{Q-p}(T, d) \geq d_{Q-p} \)

\[
(M_g)_{i}^{1}(T, d) \geq (M_g)_{i}(S, d_p) \text{ for some } i \in P.
\]

\[
\therefore d_i + \alpha(T, d) \left[ g_i(T, d) - d_i \right] \geq d_i + \alpha(S, d_p) \left[ g_i(T, d) - d_i \right].
\]

\[
\therefore \left[ \alpha(T, d) - \alpha(S, d_p) \left[ g_i(T, d) - d_i \right] \right] > 0.
\]

**Case(i):** \( g_j(T, d) \geq d_j \forall j \in Q \).

Also, \( \alpha(T, d) \geq \alpha(S, d_p) \)

\[
(M_g)_{j}^{1}(T, d) = d_j + \alpha(T, d) \left[ g_j(T, d) - d_j \right] d_j + \alpha(S, d_p) \left[ g_j(T, d) - d_j \right] = (M_g)_{j}(S, d_p) \forall j \in P.
\]
\[ M(T,d) \text{ weakly Pareto dominates } (M(S,d), d_{\mathcal{P}}), \text{ a contradiction.} \]

\[ (M)_{i}(T,d) \leq (M)_{i}(S,d) \text{ for all } i \in \mathcal{P}. \]

Hence \( M \) satisfies R.MON.

**Case (ii):** \( q_{i}(T,d) \prec d_{i} \)

By property 3, \( q_{j}(T,d) \prec d_{j} \quad \forall j \in \mathcal{Q}. \)

Also, \( \alpha(T,d) \prec \alpha(S,d) \)

Arguing as in Case (i) we get \( M \) satisfies R.MON.

Q.E.D.

**Note:** (i) The only properties of \( g \) that were used to establish Theorem 4 were properties 2 and 3. Properties 1 and 4 in conjunction with properties 2 and 3 are used to establish that, \( M_{g} \) satisfies Axioms 1 to 4 enunciated for bargaining solution on \( \sum_{x} \).

(ii) The bargaining solution \( M \) defined earlier has reference function, \( g : \sum_{x} \rightarrow \bigcup_{P \in \mathcal{P}} \mathbb{R}^{P} \) defined by

\[ g(S,d) \equiv 0 \quad \forall (S,d) \in \sum_{x} \]

\( g \) satisfies properties 1, 2, 3 and 4 of a reference function.

(iii) The bargaining solution \( K : \sum_{x} \rightarrow \bigcup_{P \in \mathcal{P}} \mathbb{R}^{P} \) due to Kalai and Smorodinsky (1975) where \( K(S,d) \) is the maximal feasible point on the segment connecting the disagreement point to the "ideal point" \( a(S,d) \).
where for each \( i \) \( P_i \), \( a_i(S, c) = \max \left\{ x_1 / x \leq S, x \geq d \right\} \) has a \( \sum \) as a reference function. 'a' satisfies Properties 1 to 4 of a reference function.

(iii) For all \( \beta \) in \( (0, 1) \), the following reference functions also satisfy Properties 1 to 4:

(a) \( g(S, c) = \beta d + (1-\beta) a(S, c) \)
(b) \( g(S, c) = \beta d + (1-\beta) h(S, c) \)
(c) \( g(S, c) = \beta d \).

Hence \( M \) satisfies \( R, MDN \) for all these reference functions.

5. Conclusion

The intention of this paper was to show that a very large class of solution described with respect to a reference function satisfy "Monotonicity with respect to changes in the number of agents". We have also shown, in the course of our analysis that a solution discussed in Lahiri (1988) also satisfies this property. Since much of bargaining theory can be rewritten in the context of a variable population, our results tend to reflect the innate desirability of the solution suggested.
References:


