# Working Paper
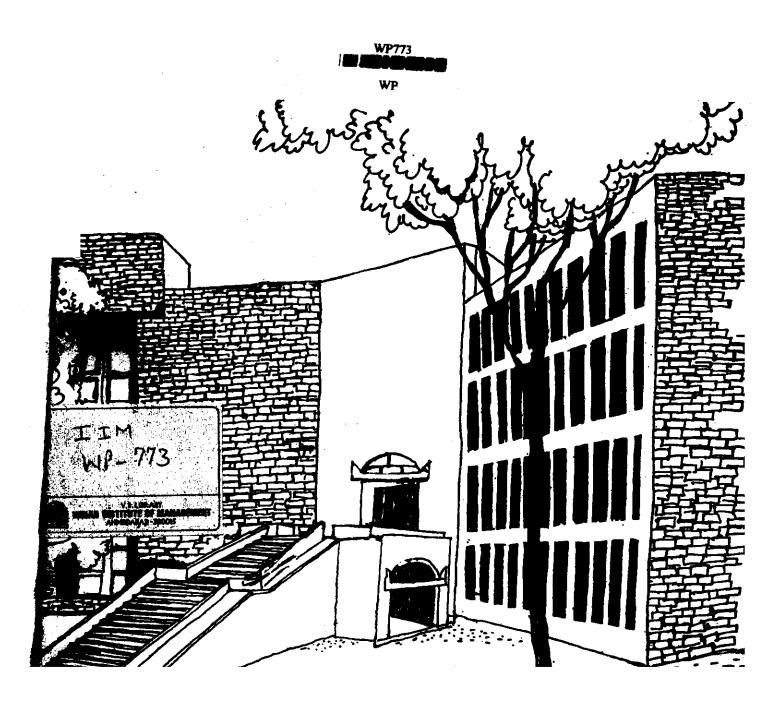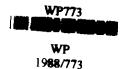
BARGAINING WITH A VARIABLE POPULATION FOR
GAMES WITH A REFERENCE POINT

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## ABSTRACT

We consider axiomatic models of bargaining defined over a domain
of problems containing different numbers of agents, define a
concept called restricted monotonicity with respect of changes
in the number of agents, and show that a solution due to Lahiri
(1988), which satisfies monotonicity with respect to the dis-
agreement point, meets the aforementioned requirement. Finally,
we consider a class of solutions which are defined with respect
to a reference point (concept due to Thomson (1981)) and show that
this class satisfies our axiom of restricted monotonicity.

1. <u>Introduction:-</u>   There is an infinite populati n I of agents, indexed by the positive integers. Arbitrary finite subsets of I may be confronted by a problem. The family of these subsets is denoted   .

Given $P \in \mathcal{P}$, $\sum^P$ is the class of problems that the group P may conceivably face. Each $(S,d) \in \sum^P$ is an ordered pair where S is a subset of $IR^P_+$, the non-negative portion of the $|P|$-dimensional Euclidean space with coordinates indexed by the members of P, and $d \in S$. Each point of S represents the von Neumann-Morgenstern utilities achievable by the members of P through some joint action. It is assumed that

1.  S is a compact subset of $IR^P_+$ containing at least one strictly positive vector;

2.  S is convex;

3.  S is comprehensive (i.e., if $X, Y \in IR^P_+$, $X \in S$, and $x \geq y$, then $Y \in S$).

We in addition sometimes invoke the assumption that

4.  $d > o$

(Given x, y $\in IR^P$, $X \geq Y$ means that $x_i \geq y_i$ for all $i \in P$; $x \geq y$ means that $x \geq y$ and $X \neq y$; $x > y$ means that $x_i > y_i$ for all $i \in P$).

Compactness of S is a technical assumption, made for convenience. Requiring that $S \subseteq IR^P_+$ implies that an appropriate choice of a zero of the utilities has been made. Assuming that S contains strictly positive vector ensures that all agents are nontrivially involved in the bargaining. Convexity of S holds in particular if agents can randomize jointly, since utilities are von Neumann-Morgenstern utilities, but this property of S may hold even in situations where randomization is not permitted. Similarly, comprehensiveness holds in particular if, but not only if, utilities are freely disposable.

We will also consider the domain $\widetilde{\Sigma}^P$ of problems satisfying the following condition in addition to conditions 1 to 3:

5. If $x, y \in S$ and $x \geq y$, then there exists $z \in S$ with $z > y$. Then, the undominated boundary of $S$ contains no segment parallel to a coordinate subspace. Note that any element of $\Sigma^P$ can be approximated in the Hausdorff topology by a sequence of elements of $\widetilde{\Sigma}^P$ . Finally, we set

$$\Sigma \equiv \bigcup_{P \in \mathcal{O}} \Sigma^P \quad \text{and} \quad \widetilde{\Sigma} \equiv \bigcup_{P \in \mathcal{O}} \widetilde{\Sigma}^P$$

In the traditional formulation of the bargaining problem, it is typically assumed that a fixed number of agents are involved. What has been presented above is a framework (originally due to Thomson (1982)) in which the bargaining problem with a variable population can be analysed.

A <u>solution</u> is a function $F : \bigcup_{P \in \mathcal{O}} \Sigma^P \to \bigcup_{P \in \mathcal{O}} IR^P_+$ such that for each $P$ and to each $(S,d) \in \Sigma^P$ , a unique point $F(S,d) \in S$ is associated.

2. <u>Monotonicity</u>: We first introduce some notation. Given $P$ in $\mathcal{O}$ and $x_1, \ldots, x_k$ in $IR^P$, $cch\{x_1, \ldots, x_k\}$ is the <u>convex</u> and <u>comprehensive hull</u> of these $k$ points, that is, the smallest convex and comprehensive subset of $IR^P$ containing them. Also, $e_p$ is the vector in $IR^P$ whose coordinates are all equal to one. Given $i$ in $P$, $e_i$ is the ith unit vector. Given $P$, $Q$ in $\mathcal{O}$ with $P \subset Q$, $y$, a point of $IR^Q$, and $T$, a subset of $IR^Q$, $y_p$ and $T_p$ designate the projections on $IR^P$ of $y$ and $T$, respectively.

We will try to formally pose the question, whether it is possible for a solution to assign greater utility to an agent initially present, after the departure of some agents with equally valid claims on the

fixed resources, than the utility he had been assigned originally. The standard economic problem motivating this kind of analysis is that of dividing fairly a bundle of goods among a group of agents. The number of agents involved in the division is allowed to vary while the resources at their disposal remain fixed.

The following axiom proposed by Thomson (1982,1983) relates solution outcomes across cardinalities.

<u>Monotonicity with respect to changes in the number of agents (MON)</u>:
For all $P$, $Q \in \mathcal{P}$ with $P \subset Q$, for all $(T,d) \in \sum^Q$ if $S = \left\{ x \in T/x_{Q \smallsetminus P} \geq d_{Q \smallsetminus P} \right\}_P$
then $F_P(S,d_P) \geq F_P(T)$.

We view tris assumption as being worded extremely liberally in that it is assumed that the possibilities open to $P$ in the absence of $Q \smallsetminus P$ includes all the possibilities available to $P$ in the presence of $Q \smallsetminus P$. It is our feeling that it is somewhat more natural to view the situation from a different perspective. In the absence of $Q \smallsetminus P$, $P$ is more likely to be restricted to only those possibilities which are consistent with the disagreement payoff vector of $Q \smallsetminus P$. In a way we view $Q$ as the original combine. When bargaining breaks down and $Q \smallsetminus P$ decide not to participate in the bargaining process, they receive their disagreement payoff. The coalition $P$ now bargains amongst itself for feasible alternatives which are consistent with the disagreement point of $Q \smallsetminus P$. We thus define:

<u>Restricted Monotonicity With Respect to changes in the number of agents (R.MON)</u>: For all $P$, $Q \in \mathcal{P}$ with $P \subset Q$, for all $(T,d) \in \sum^Q$, if $S = \left\{ x \in T/x_{Q \smallsetminus P} = d_{Q \smallsetminus P} \right\}_P$, then $F_P(S,d_P) \geq F_P(T,d)$.

Our first theorem postulates the equivalence of (MON) and (R.MON).

<u>Theorem 1</u> : Let $\sum$ be the set of all bargaining problems satisfying conditions 1 to 3, $\sum = \bigcup_{P \in \mathcal{P}} \sum^P$ . Let $(T, d) \in \sum^{()}$. Then both $(s^1, d_p)$ and $(s^2, d_p) \in \sum^P$ where $\quad P \subseteq Q$ and

$$s^1 = \left\{ X \in T/X_{Q \setminus P} \geq d_{Q \setminus P} \right\}_P, \quad s^2 = \left\{ X \in T/X_{Q \setminus P} = d_{Q \setminus P} \right\}_P. \text{Also } s^1 = s^2$$

<u>Proof</u> : Clearly $s^2 \subseteq s^1$. To show $s^1 \subseteq s^2$, let $x_p \in s^1$

$$\therefore X \in T \text{ and } X_{Q \setminus P} \geq d_{Q \setminus P}.$$

By property 3, $(X_p, X_{Q \setminus P}) \geq (X_p, d_{Q \setminus P}) \in IR^Q_+$ implies,

$$(X_p, d_{Q \setminus P}) \in T.$$

$$\therefore X_p \in s^2$$

Hence $s^1 = s^2$

Now, since $(s^2, d_p)$ satisfies conditions (1) to (3) both $(s^1, dp)$ and $(s^2, d_p) \in \sum^P$.

Q.E.D.

Note Condition 3 is essential in establishing this equivalence.

It is very easy to construct examples where $s^1 \neq s^2$ if Condition 3

is violated.

Some axioms which play a vital role in our subsequent characterizatioi

are :

<u>Weak Parebo-Optimality (WPO)</u> :   For all $P \in \mathcal{P}$, for all $(S,d) \in \sum^P$ ,

for all $Y \in IR^P_+$, if $Y > F(S,d)$, then $y \notin S$.

<u>Pareto-Optimality (PO)</u> : For all $P \in \mathcal{P}$, for all $(S,d) \in \sum^P$ , for all

$Y \in IR^P_+$, if $Y \geq F(S,d)$, then $Y \notin S$.

<u>Individual - Rationality (IR)</u> : For all $P \in \mathcal{P}$, for all $(S,d) \in \sum^P$,

$F(S,d) > d$.

$PO(S)$

We denote by WPO $(S)$, and $IR(S)$, respectively, the sets of points that are
Weakly Pareto-optimal, Pareto optimal and Individually rational for S
(observe that $PO(S) \subset WPO(S)$).

**Symmetry (SY):** For all $P \in \mathcal{P}$, for all $(S,d) \in \Sigma^P$, if for all one-one function
$\gamma : P \to P$, $S = \left\{ X' \in IR^P_+ / \exists X \in S \text{ such that } \forall i \in P, X'_{\gamma(i)} = X_i \right\}$, and $d_i = d_j$
for all $i, j \in P$, then for all $i, j \in P$, $F_i(S,d) = F_j(S',d)$.

A related condition is

**Anonymity (AN):** For all $P$, $P' \in \mathcal{P}$ with $|P| = |P'|$, for all one-one function
$\gamma : P \to P'$, for all $(S,d) \in \Sigma^P$, $(S',d') \in \Sigma^{P'}$, if $S' = \left\{ X' \in IR^P_+ / \exists X \in S \text{ such that } \forall i \in P, X'_{\gamma(i)} = X_i \right\}$ and $d'_{\gamma(i)} = d_i \forall i \in P$, then for all $i \in P$, $F_{\gamma(i)}(S',d') = F_i(S,d)$.

**Scale Invariance (S.INV):** For all $P \in \mathcal{P}$, for all $(S,d)$ and $(S', d')$ and $(S'',d'') \in \Sigma^P$,
for all $a \in IR^P_{++}$, if $S' = \left\{ X' \in IR^P / \forall i \in P, X'_i = a_i X_i \right\}$, and $d'_i = a_i d_i$,
$i \in P$, then for all $i \in P$, $F_i(S', d') = a_i F_i(S,d)$.

**Continuity (CONT):** For all $P \in \mathcal{P}$, for all sequences $(S^k, d^k)$ of elements of
$\Sigma^P$, if $S^k \in S$, $d^k \to d$ and $(S,d) \in \Sigma^P$, then $F(S^k, d^k) \to F(S)$. (In this
definition, convergence of $S^k$ to S is evaluated in the Hausdorff topology).

These axioms are standard. WPO states that it is infeasible to make all
agents simultaneously better off, and PO that it is infeasible to make an
agent better off without hurting some other agent. IR states that all agents
are better off after bargaining than they would be at their disagreement payoff.
SY states that if a problem is invariant under all permutations, then all
agents should get the same amount. AN states that the names of the agents
do not matter; only the geometrical structure of the problem at hand is
relevant. S.INV states that the solution is independent of the choice of

particular members of the equivalence classes of utility functions representing the agents' von Neumann-Morgenstern preferences. CONT states that small changes in the data definining the problem cause only small changes in solution outcomes.

Monotonicity with Respect to the disagreement point and (R.MON):-

Let $\sum_{*}^{P}$ be the subset of $\sum^{P}$ consisting of all games satisfying conditions 1 to 4 and $\widetilde{\sum}_{*}^{P}$ be the subset of $\widetilde{\sum}^{P}$ consisting of all games satisfying Conditions 1 to 5. Thus, $\widetilde{\sum}_{*}^{P} = \sum_{*}^{P} \cap \widetilde{\sum}^{P}$.

We consider the following solution $M : \sum_{*} \rightarrow \bigcup_{P \in \mathcal{P}} IR^{P}$ where

$$\sum_{*} = \bigcup_{P \in \mathcal{P}} \sum_{*}^{P}$$

defined thus:

$\forall (S,d) \in \sum_{*}^{P}$, $\quad \dfrac{M_i(S,d)}{d_i} = \dfrac{M_j(S,d)}{d_j}$ for all i, $j \in P$

and $\dfrac{X_i}{d_i} = \dfrac{X_j}{d_j}$ for all i, $j \in P$, $X \in S$ implies $M(S,d) \geq X$.

In Lahiri (1988) it has been shown that M satisfied and is the only solution to satisfy the following conditions:

Axiom 1 : WPO

Axiom 2 : IR

Axiom 3 : S.INV

Axiom 4 : SY

Axiom 5 : (Monotonicity with respect to the disagreement point):

Let (S,d) and (S', d') belong to $\sum_{*}^{P}$ and satisfy:

a) for some $i \in P$, $d_j = d'_j \; \forall \; j \in P \setminus \{i\}$; $d_i \leq d'_i$

b) $S \subseteq S'$.

Then $F_i(S',d') \geq F_i(S,d)$.

If in addition $S = S'$, then $F_j(S',d') \leq F_j(S,d)$ for $j \in N \setminus \{i\}$, with $F(S',d')$ $\neq F(S,d)$ if $d \neq d'$.

We first show the following:

<u>Theorem 2:-</u>    $M$ satisfies (AN).

<u>Proof</u> :- Let $P$, $P' \in \mathcal{P}$ with $|P| = |P'|$ and let $\Upsilon : P \to P'$ be a one-one function. Let $(S,d) \in \sum_{\ast}^P$ and define,

$$S' = \left\{ X' \in \mathbb{R}^{P'} / \exists X \in S \text{ such that } \forall\ i \in P,\ X'_{\Upsilon(i)} = X_i \right\}$$

and $d' = (d'_j)_{j \in P'}$ with $d'_{\Upsilon(i)} = d_i$.

$$\frac{X'_{\Upsilon(i)}}{d'_{\Upsilon(i)}} = \frac{X_i}{d_i} \qquad \forall\ i \in P$$

$$\frac{M_{\Upsilon(i)}(S',d')}{d'_{\Upsilon(i)}} = \frac{M_{\Upsilon(j)}(S',d')}{d'_{\Upsilon(j)}} \qquad \forall\ i, j \in P.$$

and $(X'_{\Upsilon(i)})_{i \in P}$ with $\dfrac{X'_{\Upsilon(i)}}{d'_{\Upsilon(i)}} = \dfrac{X'_{\Upsilon(j)}}{d'_{\Upsilon(j)}}$ $\forall\ i, j \in P$

implies $M_{\Upsilon(i)}(S',d') > X'_{\Upsilon(i)}$ $\forall\ i \in P$.

$\therefore\ M_{\Upsilon(i)}(S',d') > X_i$ $\forall\ i\ P$

whenever, $\dfrac{X_i}{d_i} = \dfrac{X_j}{d_j}$ $\forall\ i,\ j \in P$

Further, $\dfrac{M_{\Upsilon(i)}(S',d')}{d_i} = \dfrac{M_{\Upsilon(j)}(S',d')}{d_j}$ $\forall\ i,\ j \in P$

$\therefore\ M_{\Upsilon(i)}(S',d') = M_i(S,d)$ $\forall\ i \in P$.

<div align="right">Q.E.D.</div>

Hence M satisfies WPC, AN, S.INV; that it satisfies CONT. follows by metho similar to the ones used in the proof of Theorem 3.3 by Jansen and Tijs (1983). We next show that M satisfies R.MON.

**Theorem 3** :—  M satisfies R.MON.

**Proof** :— Let $P$, $Q \in \mathcal{P}$ with $P \subset Q$. Let $(T,d) \in \sum^Q$ and

$$S = \left\{ x \in T / x_{Q \setminus P} = d_{Q \setminus P} \right\} \, P. \qquad \text{So } M_j(T,d) = \frac{d_j}{d_i} M_i(T,d) \quad \forall \, j \in Q$$

Observe, $(M(S,d_p), \, d_{Q \setminus P}) \in WP(T)$.

Suppose $(M(S,d_p), \, d_{Q \setminus P}) \notin WP(T)$.

there exists $Y \in T \ni Y > (M(S,d_p), \, d_{Q \setminus P})$

Consider the point $(Y, \, d_{Q \setminus P})$.

Clearly $Y \geq (Y, d_{Q \setminus P}) \geq (M(S,d_p), \, d_{Q \setminus P})$.

Since $T$ is comprehensive, $(Y, \, d_{Q \setminus P}) \in T$.

By the definition of $S$, $Y \in S$.

But $Y > M(S,d_p)$.

This contradicts that $M(S,d_p) \in WP(S)$.

Hence $(M(S,d_p), \, d_{Q \setminus P}) \in WP(T)$.

Clearly $M_j(S,d_p) = \frac{d_j}{d_i} M_i(S,d_p) \quad \forall \, i, \, j \in P$.

Now, $M_{Q \setminus P}(T,d) > d_{Q \setminus P}$.

Suppose $M_i(T,d) > M_i(S,d_p)$ for some $i \in P$.

$\therefore M_j(T,d) = \frac{d_j}{d_i} M_i(T,d) > \frac{d_j}{di} M_i(S,d_p) = M_j(S,d_p) \quad \forall \, j \in P.$

$$\therefore \; M(T,d) \; = \; (M_p(T,d), \; M_{Q \sim p}(T,d)) > (M(S,d_p), \; d_{Q \sim p}),$$

contradicting that $(M(S,d_p), \; d_{Q \sim p}) \in$ WPO $(T)$.

Hence $M_j(T,d) \leq M_j(S,d_p) \quad \forall \; j \in P.$

$M_p(T,d) \leq M(S,d_p)$ and M satisfies R.MON as was required to be proved.

Q.E.D.

In Thomson (1983) it is shown that the Kalai-Smorodinsky (1975) solution also satisfies WPO, IR, S.INV, AN and R.MON. We will thus look for a general characterization of all solutions satisfying the above conditions on $\sum_*^P$ .

4. <u>Bargaining Problems With a reference point</u> :-

Thomson (1981) observes that for many solutons to bargaining problems, one oftenresorts to what is known as a <u>reference function,</u> which is a function singling out, for each bargaining problem, a point of the utility space summarizing its essential features and facilitating the evaluation of the relative bargaining strength of the players. Formally we have the following:

<u>Definition 1:</u> A <u>reference function</u> g : $\sum_* \to \bigcup_{P \in \mathcal{P}} \mathbb{R}^P$ associates to every bargaining problem $(S,d) \in \sum_*^P$ a point $g \; (S,d) \in$ $IR^P$.

The role of the reference function is essentially informational. In the evaluation of the their respective bargaining strengths, the players summarize what they see as the main features of the bargaining problem by focusing on particular outcomes, like the outcomes that are most favourable to each of them (see Kalai and Smorodinsky (1975); Rosenthal (1976); Roth (1977)). Observe that the definition of a reference function developed above, does not require that g (S,d) be a feasible alternative. In fact, the ideal point of Kalai and Smorodinsky, is fea ible only in the trivial case in which it is the only Pareto-optimal outcome.

We shall consider reference functions g which satisfy the following properties:

__Property 1__  (S.INV): For all $P \in \mathcal{P}$, for all $(S,d)$ and $(S',d') \in \Sigma_*^P$, for all $a \in IR_{++}^P$, if $S' = \left\{ x' \in IR^P / \forall \ i \in P, \ x'_i = a_i x_i \right\}$, $d' = (d'_i)_{i \in P}$ where $d'_i = a_i d_i$, $i \in P$, then for all $i \in P$, $g_i(S',d') = a_i g_i(S,d)$.

__Property 2__ (Modified Multilateral Stability or $(m^2.STAB)$): For all $P$, $Q \in \mathcal{P}$ with $P \subset Q$, for all $(S,d') \in \Sigma_*^P$, $(T,d) \in \Sigma_*^Q$, if $S = \left\{ x \in T/X_{Q \setminus P} = d_{Q \setminus P} \right\}$ and $d' = d_P$, then $g(S,d') = g_P(T,d)$.

__Property 3__: For all $P \in \mathcal{P}$ and $(S,d) \in \Sigma_*^P$ the following is true:

$\quad$ __either__ $g_i(S,d) > d_i \quad \forall \ i \in P$

$\quad$ __or__ $g_i(S,d) < d_i \quad \forall \ i \in P$

__Property 4__ :- (AN): For all $P$, $P' \in \mathcal{P}$ with $|P| = |P'|$, for all one-one function $\gamma : P \rightarrow P'$, for all $(S,d) \in \Sigma_*^P$, $(S',d') \in \Sigma_*^{P'}$, if $S' = \left\{ x' \in IR_+^P / \exists \ x \in S \text{ such that } \forall \ i \in P, \ x'_{\gamma(i)} = x_i \right\}$ and $d'_{\gamma(i)} = d_i \ \forall \ i \in P$, then for all $i \in P$, $g_{\gamma(i)}(S',d') = g_i(S,d)$.

Given a reference function $g : \Sigma_* \rightarrow \bigcup_{P \in \mathcal{P}} IR^P$  satisfying Properties 1,2,3 and 4 we consider a solution to bargaining problems in $\Sigma_*$ defined thus:

Let $m_g : \Sigma_* \rightarrow \bigcup_{P \in \mathcal{P}} IR^P$  be given by

$m_g(S,d) = d + \bar{\alpha}(S,d) \ (g(S,d) - d)$

where $\bar{\alpha}(S,d) \in IR$ satisfies $d + \bar{\alpha}(S,d) \ (g(S,d) - d) \in WPO \ (S)$, for all $(S,d) \in \Sigma_*^P$.

It is easily observed that $M_g$ is well-defined, satisfies Conditions 1,2,3,4. We shall now show that $M_g$ satisfies (R.MCN).

<u>Theorem 4</u> :- Let $g$ be a reference function satisfying Properties 2 and 3. Then $M_g$ satisfies R.MCN.

<u>Proof</u> :- Let $P$, $Q \in P$ with $P \subset Q$, $(T,d) \in \sum_{*}^{Q}$ and $S = \left\{ x \in T / x_{Q \sim p} = d_{Q \sim p} \right\} P$.

To show, $M_g(S, d_p) \geq (M_g)_p$ $(T,d)$

$$M_g(S, d_p) = d_p + \bar{\alpha}(S, d_p) \left[ g(S, d_p) - d_p \right].$$

and $M_g(T,d) = d + \bar{\alpha}(T,d) \left[ g(T,d) - d \right]$

By property 2, $g(S, d_p) = g_p(T,d)$.

Since $T$ is comprehensive and convex, for the same reasons as in the proof of Theorem 3, $(M_g(S, d_p), d_{Q \sim p}) \in WP(T)$.

Now, $(M_g)_{Q \sim p} \ (T,d) > d_{Q \sim p}$

$(M_g)_i \ (T,d) > (M_g)_i(S, d_p)$ for some $i \in P$.

$\therefore d_i + \bar{\alpha}(T,d) \left[ g_i(T,d) - d_i \right] > d_i + \bar{\alpha}(S, d_p) \left[ g_i(T,d) - d_i \right]$.

$\therefore \left[ \bar{\alpha}(T,d) - \bar{\alpha}(S, d_p) \right] \left[ g_i(T,d) - d_i \right] > 0$.

<u>Case(1)</u>: $g_i(T,d) > d_i$.

By property 3, $g_j(T,d) > d_j$ $\forall j \in Q$.

Also, $\bar{\alpha}(T,d) > \bar{\alpha}(S, d_p)$

$(M_g)_j \ (T,d) = d_j + \bar{\alpha}(T,d) \left[ g_j(T,d) - d_j \right] > d_j + \bar{\alpha}(S, d_p) \left[ g_j(T,d) - d_j \right]$.

$(M_g)_j(S, d_p) \quad \forall j \in P$.

$\therefore M_g(T,d)$ weakly Pareto dominates $(M_g(S,d_p), d_{\backslash p})$, a contradiction.

$\therefore (M_g)_i(T,d) \le (M_g)_i (S,d_p)$ for all $i \in P$.

Hence $M_g$ satisfies R.MON.

__Case (ii)__ : $g_i(T,d) < d_i$

By property 3, $g_j (T,d) < d_j \quad \forall j \in Q$.

Also, $\bar{\alpha} (T,d) < \bar{\alpha} (S,d_p)$

Arguing as in Case (i) we get $M_g$ satisfies R.MON.

$$Q.E.D.$$

Note: (i) The only properties of g that were used to establish Theorem 4 were properties 2 and 3. Properties 1 and 4 in conjunction with properties 2 and 3 are used to establish that, $M_g$ satisfies Axioms 1 to 4 enunciated for bargaining solution on $\Sigma_*$.

(ii) The bargaining solution M defined earlier has reference function,

$g : \Sigma_* \to \bigcup_{P \in \mathcal{P}} \mathbb{R}^P$ defined by

$$g(S,d) \equiv 0 \quad \forall (S,d) \in \Sigma_*$$

g satisfies properties 1,2,3 and 4 of a reference function.

(iii) The bargaining solution K $: \Sigma_* \to \bigcup_{T \in \mathcal{P}} \mathbb{R}^T$ due to Kalai and Smorodinsky (1975) where $K(S,d)$ is the maximal feasible point on the segment connecting the disagreement point to the "ideal point" $a(S,d)$

where for each i $\in$ P, $a_i(S,d) = \max\left\{x_i / x \in S, \quad x \geq d\right\}$ has a : $\sum_* \rightarrow$ as a reference function, 'a' satisfies Properties 1 to 4 of a reference function.

(iii) For all $\beta$ in $[0,1)$, the following reference functions also satisfy Properties 1 to 4:

(a) $g(S,d) = \beta d + (1-\beta) \, a(S,d)$

(b) $g(S,d) = \beta d + (1-\beta) \, h(S,d)$

(c) $g(S,d) = \beta d$.

Hence $M_g$ satisfies R.MON for all these reference functions.

## 5. Conclusion:-

The intention of this paper was to show that a very large class of solution described with respect to a reference function satisfy "Monotonicity with respect to changes in the number of agents". We have also shown, in the course of our analysis that a solution discussed in Lahiri (1988) also satisfies this property. Since much of bargaining theory can be rewritten in the context of a variable population, our results tend to reflect the innate desirability of the solution suggested.

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