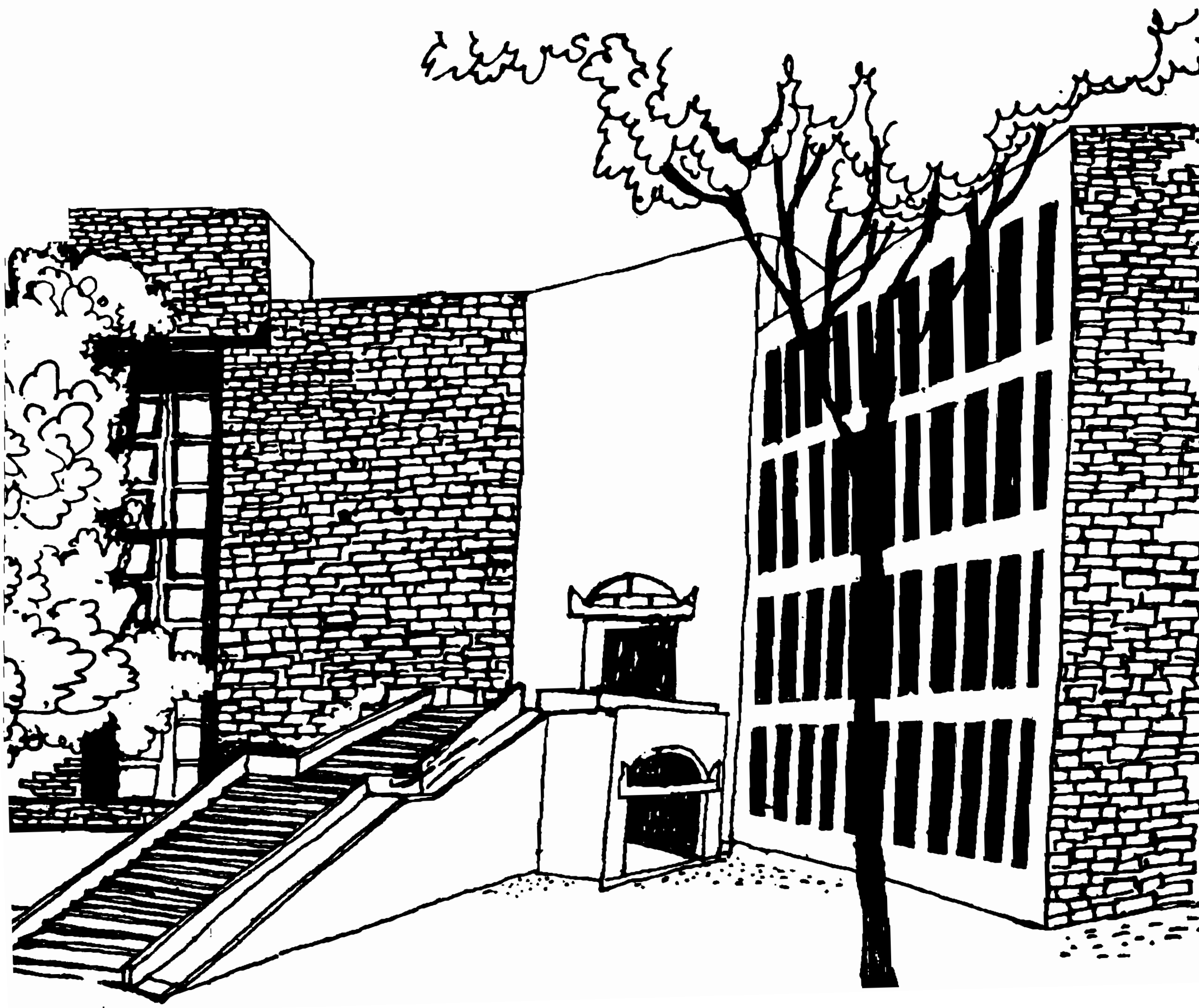
विद्याविनियोगाद्विकासः

IIM

AHMEDABAD

Working Paper



A CALCULUS APPROACH TO THE EXISTENCE OF MARKET EQUILIBRIA IN A DISTRIBUTION ECONOMY

By

Somdeb Lahiri

W P No. 1100
May 1993

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Acknowledgement

I would like to thank Professors A. McLenan and D.W. Kahn for helping me understand the underlying mathematics that this analysis appeals to. This paper was written while I was visiting the Management Science and Information Systems area of the Michael G. DeGroote School of Business. I would like to thank my hosts there for their kind hospitality and to Professor Prakash Abad in particular for making this visit possible. Although all those mentioned above have benefitted me through numerous discussions on this and related topics, none of them stands responsible for the errors that do remain in the exposition.

Abstract

In this paper we prove the existence of a market equilibria in distribution economies, without using any fixed-point theorems. Our method makes essential use of theorems in advanced calculus to establish the desired result.

1. **Introduction:**

In a series of papers written over a period of two years, Smale laid the foundations for the calculus approach to equilibrium analysis in a pure-trade economy. The entire work has been surveyed in Smale [1984]. Basically the paper uses two important theorems of advanced calculus, - the implicit function theorem and Sard's theorem to develop proofs of existence of Walrasian equilibria in a pure-trade economy.

It has been argued elsewhere (see Lahiri [1993]), that in some contexts, what is more relevant for economic analysis, is the concept of a distribution economy due to Malinvaud [1972]. This kind of economy forms the basis of the welfare economics of Dierker and Lenninghaus [1986] and fix-price analysis of Lahiri [1993]. A distribution economy is a multi-agent economy, with an aggregate initial endowment of consumption goods, which needs to be distributed among the agents. Each agent begins with an income, which for the purpose of our analysis, may be considered to be in the form of paper money (but is otherwise a payment from the production sector, in lieu of a numéraire commodity which the agents supply to the producers for the production of the bundle of consumption goods comprising the aggregate initial endowment). Each agent pays for his purchases with his income. As in all of economic theory, we assume that each agent maximizes his preferences subject to a budget constraint, so as to make his consumption choice. A market equilibrium occurs, when the vector of prices determining the budget constraints of the consumer, are such that all markets clear. Malinvaud [1972], uses the Brouwer's fixed point theorem to establish the existence of a market equilibrium. He also proceeds to provide proofs of the first and second fundamental theorems of welfare economics for a distribution economy.

In this paper we adopt the approach developed in Smale [1984] and proceed to provide proofs of the existence of a market equilibrium in a distribution economy, without appealing to any fixed point theorem argument. This, it is hoped, would further clarify the theoretical validity of a distribution economy and of a market equilibrium, in the contexts where it is obviously applicable.

2. Mathematics background:

In this section we reproduce from Smale [1984], the relevant theorems in calculus which are required in the rest of the paper. To begin with we have the following terminology:

Let f: $U \rightarrow \mathbf{R}^n$ be a function where U is some open subset of a Cartesian space $\mathbf{R}^k$, k and n being natural numbers. We will say that f is C^r if its r^{th} derivatives exist and are continuous. (Here r is a natural number.) For x in U, the derivative Df(x) (i.e. matrix of partial derivatives) is a linear map from $\mathbf{R}^k$ to $\mathbf{R}^n$. Then x is called a singular point if this derivative is not surjective ("onto"). The singular values are simply the images under f of all the singular points; and y in $\mathbf{R}^n$ is a regular value if it is not singular i.e. $f^{-1}(y)$ does not contain a singular point.

Theorem 1: (Implicit Function Theorem) If $y \in \mathbf{R}^n$ is a regular value of a C^1 map f: $U \rightarrow \mathbf{R}^n$, U open in $\mathbf{R}^k$, then either $f^{-1}(y)$ is empty or it is a submanifold V of U of dimension k-n.

Proof: See Spivak [1968].

Here V is a submanifold of U of dimension m = k-n if given $x \in V$, one can find a differentiable map h: $N(x) \rightarrow \sigma$ with the following properties:

(a) h has a differentiable inverse.

(b) $N(x)$ is an open neighborhood of x in U.

(c) σ is an open set containing 0 in $\mathbf{R}^k$.

(d) $h(N(x) \cap V) = \sigma \cap C$ where C is an affine subspace of $\mathbf{R}^k$ of dimension m.

Theorem 2: (Sard's Theorem) If $f: U \rightarrow R^n$, $U \subseteq \mathbf{R}^k$ is a differentiable of class C^r, $r > 0$ and $r > k\text{-}n$, then the set of singular values has measure zero.

Proof: See Spivak [1968].

Smale [1984] uses theorems 1 and 2 to prove:

Theorem 3: Let $f: D^l \rightarrow \mathbf{R}^l$ be a continuous map satisfying the boundary condition:

(B_D) if $x \in \delta D_l$ then $f(x)$ is not of the form μx for any $\mu > 0$.

Then there is $x^* \in D^l$ with $f(x^*) = 0$.

Here $D^l = \{x \in \mathbf{R}^l \,/\, \| x \| \leq 1\}$ and $\delta D^l = \{x \in D^l \,/\, \| x \| = 1\}$.

Theorem 3 is used by Smale [1984] to prove:

Theorem 4: Let ϕ: $\Delta_1 \rightarrow \Delta_0$ be a continuous map satisfying the boundary condition:

(B) $\phi(p)$ is not of the form $\mu(p\text{-}p_c)$, $\mu > 0$ for $p \in \delta\Delta_1$.

Then there is $p^* \in \Delta_1$ with $\phi(p^*) = 0$.

Here $\Delta_1 = \{p \in \mathbf{R}^l_+ \,/\, \Sigma p^i = 1\}$, $\delta\Delta_1 = \{p \in \Delta_1 \,/\, \text{some } p_i = 0\}$, $\Delta_0 = \{z \in \mathbf{R}^l \,/\, \Sigma z^i = 0\}$.

Notation: Given vectors x, y $\in \mathbf{R}^l$ let $x*y$ denote the vector $(x_1y_1, x_2y_2, \ldots, x_iy_i, \ldots, x_ly_l) \in \mathbf{R}^l$.

Let $q \in \mathbf{R}^l_{++} \equiv \{x \in \mathbf{R}^l \,/\, x_i > 0 \;\forall\, i = 1, \ldots, l\}$. Define

$\Delta_1(q) = \{p \in \mathbf{R}^l_+ \ / \ p*q \in \Delta_1\}$, $\delta\Delta_1(q) = \{p \in \Delta_1(q) \ / \text{ some } p_i = 0\}$.

We have the following theorem:

<u>Theorem 5</u>: Let $\overline{\phi}$: $\Delta_1(q) \rightarrow \Delta_0$ be a continuous map satisfying the boundary condition

(B′) $\overline{\phi}(p)$ is not of the form $\mu(p\text{-}s)$, $\mu > 0$ for $p \in \delta\Delta_1(q)$ where $s = (s_1, \ldots, s_l)$, $s_i = \frac{1}{q_i l}$ $\forall\, i = 1, \ldots, l$.

Then there exists $p^* \in \Delta_1(q)$ with $\overline{\phi}(p^*) = 0$.

<u>Proof</u>: Define ϕ: $\Delta_1 \rightarrow \Delta_0$ as follows:

$$\phi(p*q) = \overline{\phi}(p) \quad \forall \ / \ p \in \Delta_1(q)$$

It is easy to see that ϕ is well defined and that the boundary condition (B′) is implied and is implied by the boundary condition (B). Hence there exists $p^* \in \Delta_1(q)$ such that

$$0 = \phi(p^* * q) = \overline{\phi}(p^*)$$

Q.E.D.

We need to state and prove one more result before we close this section.

Let Z: $\Delta_1(q) \setminus \{0\} \rightarrow \mathbf{R}^l$ be a function satisfying

(i) $p \cdot Z(p) = 0 \ \ \forall\, p \in \Delta_1(q) \setminus \{0\}$.

(ii) $Z^i(p) \geq 0$ if $p^i = 0$.

We shall call such functions <u>excess demand functions</u>.

<u>Theorem 6</u>: If an excess demand function Z: $\Delta_1(q) \setminus \{0\} \rightarrow \mathbf{R}^l$ is continuous, then there exists $p^* \in \Delta_1(q)$ $\{0\}$ such that $Z(p^*) = 0$.

<u>Proof</u>: Define a map $\overline{\phi}$: $\Delta_1(q) \rightarrow \Delta_0$ by

$$\bar{\phi}(p) = \left(\frac{Z^i(p)}{q_i}\right)_{i=1}^{l} - \left(\sum_i \frac{Z^i(p)}{q_i}\right)(p*q)$$

Observe $\sum_i \bar{\phi}^i(p) = 0$, since $p*q \in \Delta_l \; \forall \; p \in \Delta_l(q)$.

Further $\bar{\phi}$ is continuous. If $p \in \delta\Delta_l(q)$, $p^i = 0$ and so $\bar{\phi}^i(p) = \dfrac{Z^i(p)}{q_i} \geq 0$. Thus (B′) of theorem (5) is satisfied since $s_i > 0 \; \forall \; i \in \{1, \ldots, l\}$. Hence by Theorem 5 there is $p^* \in \Delta_l(q)$ with $\bar{\phi}(p^*) = 0$ or

$$\left(\frac{Z^i(p^*)}{q_i}\right)_{i=1}^{l} = \left(\sum_i \frac{Z^i(p^*)}{q_i}\right)(p*q)$$

Take dot product on both sides with $\left(\frac{Z^i(p^*)}{q_i}\right)_{i=1}^{l}$ to obtain using condition (i) of the definition of an excess demand function that $\left\| \left(\frac{Z^i(p^*)}{q_i}\right)_{i=1}^{l} \right\| = 0$ or that $Z(p^*) = 0$ since $q \in \mathbf{R}^l_{++}$. This proves the theorem.

Q.E.D.

3. <u>Existence of market equilibria in a distribution economy:</u>

We assume that there are l perfectly divisible goods in the economy, the aggregate initial endowment of which is given by a vector $w \in \mathbf{R}^l_{++}$. We further assume that there are 'm' consumers in the economy, where m is a natural number and that the consumers are indexed by $i \in \{i, \ldots, m\} \equiv M$. Each consumer i begins with a strictly positive income $w_i > 0$. Consumer i's preferences over consumption bundles are represented by a utility function u_i: $\mathbf{R}^l_+ \to \mathbf{R}$ which is assumed to be continuous, strictly increasing and strictly quasi-concave (i.e. x, $y \in \mathbf{R}^l_+$, x ≥

y, $x \neq y \Rightarrow u_i(x) > u_i(y)$; $x, y \in \mathbb{R}^l_+$, $x \neq y \Rightarrow u_i(tx + (1-t)y) > \min\{u_i(x), u_i(y)\}$ $\forall\, t \in (0,1)$).

An allocation for this economy is a vector $x \in (\mathbb{R}^l_+)^m$. An allocation x is said to be a feasible allocation if $\sum_{i=1}^{m} x^i = \omega (x^i \in \mathbb{R}^l_+ \;\forall\, i \in M)$.

An allocation price pair $(\hat{x},\hat{p}) \in (\mathbb{R}^l_+)^m \times (\mathbb{R}^l_+ \setminus \{0\})$ is said to be a market equilibrium if

(i) $\hat{x}$ is a feasible allocation.

(ii) $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in \mathbb{R}^l_+ \,/\, \hat{p}\cdot x \leq w_i\}$ $\forall\, i = 1, \ldots, m$.

(iii) $\hat{p}\cdot\omega = \sum_{i=1}^{m} w_i$.

Let $V = \{x \in \mathbb{R}^l_+ \,/\, x \leq (1+m)\omega\}$.

An allocation price pair $(\hat{x},\hat{p}) \in (\mathbb{R}^l_+)^m \setminus (\mathbb{R}^l_+ \setminus \{0\})$ is said to be a V constrained market equilibrium if

(i) $\hat{x}$ is a feasible allocation.

(ii) $\hat{x}^i$ maximizes $u_i(x)$ on the set $\{x \in V \,/\, \hat{p}\cdot x \leq w_i\}$.

(iii) $\hat{p}\cdot\omega = \sum_{i=1}^{m} w_i$.

Note $l + m \geq 2$, since l and n are both natural numbers. Hence we have the following theorem:

Theorem 7: $(\hat{x},\hat{p})$ is a market equilibrium if and only if it is a V-constrained market equilibrium.

Proof: It is obvious from the definitions, that if $(\hat{x},\hat{p})$ is a market equilibrium it is a V-constrained market equilibrium. Thus suppose $(\hat{x},\hat{p})$ is a V-constrained market equilibrium, but not a market equilibrium. Thus there exists $i \in M$ and $x > (1+m)\omega$ with $\hat{p}\cdot x \leq w_i$ and $u_i(x)$

$> u_i(\hat{x}^i)$. Since $\sum_{i=1}^{m} \hat{x}^i = \omega, \hat{x}^i \leq \omega$. Further $(1+m)\omega >> \omega$ since $\omega \in \mathbf{R}^l_{++}$ and $1+m \geq 2$. Hence there exists $t \in (0,1)$ such that $(1+m)w \geq tx + (1-t)\hat{x}^i$ and by strict quasi-concavity $u_i(tx + (1-t)\hat{x}^i) > u_i(\hat{x}^i)$. Further $\hat{p}\cdot(tx + (1-t)\hat{x}^i) \leq w_i$, contradicting that $(\hat{x},\hat{p})$ is a V-constrained market equilibrium and proving the theorem.

Q.E.D.

Let $q \in \frac{1}{\sum_{i=1}^{m} w_i}(\omega)$; $q \in \mathbf{R}^l_{++}$. Define x^i: $\Delta_1(q) \setminus \{0\} \to \mathbf{R}^l_+$ as follows: $x^i(p)$ maximizes $u_i(x)$ on the set $\{x \in V / \hat{p}\cdot x \leq w_i\}$.

By the assumptions on our preferences, x^i is well defined $\forall\ i \in M$, continuous and satisfies $\hat{p}\cdot x^i(p) = w_i\ \forall\ i \in M,\ \forall\ p \in \Delta_1(q)\setminus\{0\}$.

Define Z: $\Delta_1(q) \setminus \{0\} \to \mathbf{R}^l$ as follows:

$$Z(p) = \sum_{i \in m} x^i(p) - w$$

Clearly Z is continuous and since $p \in \Delta_1(q)\setminus\{0\}$, $p\cdot Z(p) = 0$. Further if $p_j = 0$ for some $p \in \Delta_1(q)\setminus\{0\}$, then since preferences are increasing, $x^i_j(p) = (1+m)\omega_j\ \forall\ i \in M$. $\therefore\ Z_j(p) = m(1+m)w_j - w_j = (ml+m^2-1)w_j > 0$ since $ml + m^2 > 1$.

Thus Z as defined above is an excess demand function. By Theorem 6 above, there exists $\hat{p} \in \Delta_1(q)\setminus\{0\}$ such that $Z(\hat{p}) = 0$. Let $\hat{x}^i = x^i(\hat{p})\ \forall\ i \in M$. We thus have:

<u>Theorem 8</u>: For the distribution economy defined above there exists a market equilibrium.

<u>Proof</u>: $(\hat{x},\hat{p})$ obtained above is a V-constrained market equilibrium. Thus by Theorem 7, $(\hat{x},\hat{p})$ is a market equilibrium.

Q.E.D.

Remark 1: We have obtained the above theorem under the strong assumption that utility functions of the consumers are strictly quasi-concave. In most of economic theory however, results are validated under the assumption that the utility functions are semi-strictly quasi-concave i.e. $\forall$ i ϵ M $\forall$ x, y ϵ $\mathbf{R}^l_+$, $u_i(x) \neq u_i(y) \Rightarrow u_i(tx + (1-t)y) > \min \{u_i(x), u_i(y)\}$ $\forall$ t ϵ (0,1). In this case, $\forall$ i ϵ {1, ..., M}, the x^i's obtained above are not continuous functions but are convex-valued correspondences from $\Delta_1(q) \setminus \{0\}$ to $\mathbf{R}^l_+$ with closed graphs. Thus Z as defined in this section turns out to be a convex-valued correspondence from $\Delta_1(q) \setminus \{0\}$ to $\mathbf{R}^l_+$ with a closed graph. In this case we need to appeal to Lemmas A IV·2 and A IV·3 of Hildenbrand and Kirman [1988] to obtain an approximating sequence of continuous 'excess demand functions' and then appeal to the compactness of $\Delta_1(q)$ to obtain a zero of Z which belongs to $\Delta_1(q) \setminus \{0\}$. It is easily seen that Theorem 7 continues to hold if strict quasi-concavity is replaced by semi-strict quasi-concavity. The following Lemma leads to a possible further weakening of our assumptions.

Lemma 1: Let f: $\mathbf{R}^l_+ \to \mathbf{R}$ be continuous, strictly increasing and quasi-concave i.e. x, y ϵ $\mathbf{R}^l_+$ $\Rightarrow f(tx + (1-t)y) \geq \min \{f(x), f(y)\}$ $\forall$ t ϵ [0,1]. Then f is semi-strictly quasi-concave.

Proof: Suppose not. Then there exists x, y ϵ $\mathbf{R}^l_+$ with $f(x) > f(y)$ and $f(t^0x + (1-t^0)y) = f(y)$ for some $t^0 \in [0,1]$. Let $\underline{\epsilon} = (\epsilon_1, \ldots, \epsilon_l) \in \mathbf{R}^l_+ \setminus \{0\}$ such that $f(x) > f(x-\underline{\epsilon}) > f(y)$. This is possible by continuity of f and the fact that f is strictly increasing. By quasi-concavity $f(t^0(x-\underline{\epsilon}) + (1-t^0)y) \geq f(y)$. On the other hand $t^0(x-\underline{\epsilon}) + (1-t^0)y) < t^0x + (1-t^0)y$ and hence since f is strictly increasing $f(t^0(x-\underline{\epsilon}) + (1-t^0)y) < f(t^0x + (1-t^0)y) = f(y)$ which contradicts what we have obtained above and proves the lemma.

Q.E.D.

Remark 2: Our analysis above does not explicitly recognize the fact that all income held by consumers is in the form of a numéraire e.g. the units of 'standard labour' a consumer is endowed with. However, once we recognize the fact that for a certain level of production of produced consumption goods, 'aggregate leisure' is an output of the production process (say $\hat{L}$), then we can define q as $\frac{W}{\left(\sum_{i\in M} w_i - \hat{L}\right)}$ and obtain prices as above such that the market for produced consumption goods clear. As an easy consequence of this, we will obtain the simple fact that the labour market clears as well. Hence there is no loss of generality in our analysis above so long as $\hat{L} < \sum_{i\in M} w_i$.

4. Conclusion:

In the above analysis, we have proved the existence of a market equilibrium for a distribution economy, without using any fixed point theorem argument, but by resorting to a simple calculus approach. The framework of a distribution economy rests on the existence of a unique commodity in terms of which all costs and values are measured e.g. a labor theory of value. In the process of proving our result we have appealed to four significant, yet simple mathematical results, which already exist in the literature. Thus the contribution of this analysis to extending the mathematical frontiers of economics is at best marginal. What however is significant, is that these results can be used to establish the existence of market equilibria in distribution economies - a framework of analysis which is thus robust and often more realistic, than what exists in the literature.

References:

1. E. Dierker and J. Lenninghans [1986]. "Surplus Maximization and Pareto Optimality", Ch. 9 in W. Hildenbrand and A. Mas-Colell (ed.) Contributions to Mathematical Economics, in honor of Gérard Debreu.

2. W. Hildenbrand and A.P. Kirman [1988]. "Equilibrium Analysis", North-Holland.

3. S. Lahiri [1993]. "Fix-Price Equilibria in Distribution Economies", mimeo.

4. E. Malinvaud [1972]. "Lectures on Micro-Economic Theory", North-Holland.

5. S. Smale [1984]. "Global Analysis and Economics", Ch. 8 in K.J. Arrow and M. Intrilligator (ed.) Handbook of Mathematical Economics, Vol. 1, North-Holland.

6. M. Spivak [1968]. "Calculus on Manifolds", Addison-Wesley Publishing Company.