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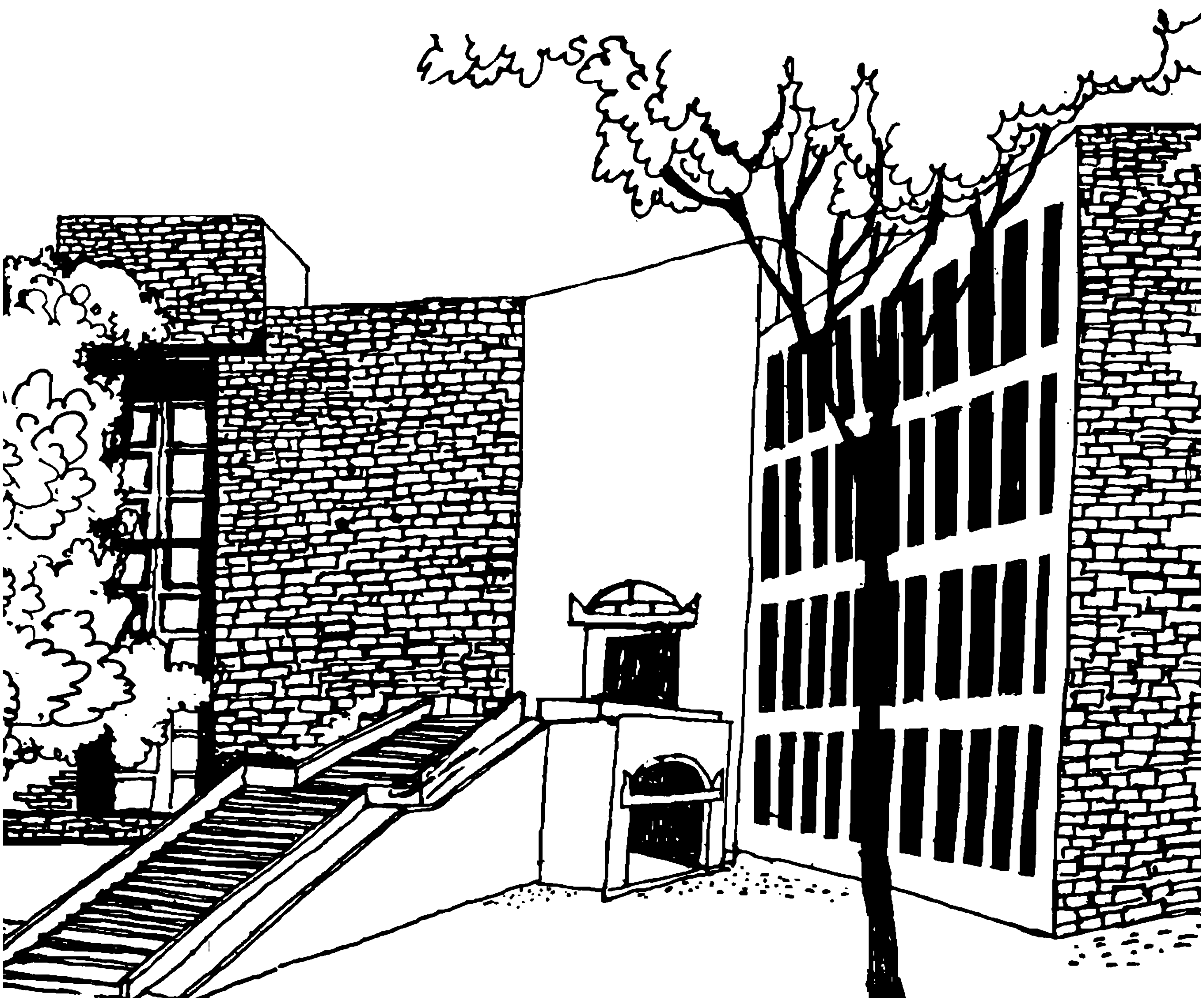

GROUP DECISION THEORY AND PRODUCTION PLANNING PROBLEMS

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Abstract

In the present work it is argued that a group decision problem can be viewed as a problem in output choice of a regulated firm and conversely.

Having developed the above isomorphism, we turn to a related problem: that of characterizing solutions to production planning problems which are non-decreasing in the cost constraint. Such solutions are called monotone solutions. We establish in this paper that monotone solutions to production planning problems are essentially continuous functions of the cost constraint.

Introduction :- There are two distinct branches of enquiry which have academic traditions in their own right : (a) the theory of multi-criteria decision making in operations research; (b) the theory of a regulated firm in economics. The two theories have developed quite independently with each having its own academic rationale.

The theory of multi-criteria decision making or group decision theory is concerned with the following problem: given a feasible set of attribute vectors, a vector of aspiration levels and a vector of targets, select an efficient feasible vector of attributes, so that no attribute is less than the corresponding aspiration level. Keeney and Raiffa (1976), French (1986), provide general discussions of the multi-criteria decision making problems; Bossert (1992) studies the same problem under the heading of bargaining problems with claims.

The theory of a regulated firm on the other hand focuses on a production process where the techno-economic considerations of production are summarized in a cost-function, where for each vector of outputs, the cost function gives us the actual costs (with a possible mark-up for a profit margin) that the producer incurs in producing that vector of outputs. The production process is constrained in the expenses it could possibly make. The problem of the producer is to choose a vector of outputs given two pieces of information (i) a cost function, (ii) a cost constraint. There may be additional pieces of information in the form of a minimum vector of outputs which the firm is required to produce and a target output vector, which may or may not be set exogenously.

Although the two theories have been developed independently, it will be argued in the present work that for a large class of group decision problems, these two theories are "isomorphic" i.e. a group decision problem can be viewed as a problem in output choice of a regulated firm and conversely. This result is quite appealing, since all the results and concepts that bear on group decision making can now be made to hold for problems of production planning by a regulated firm. Conversely, since a production planning problem is defined in terms of a function, a real number and two vectors, it is structurally more simple to comprehend than a group decision problem.

Having developed the above isomorphism, we turn to a related problem: that of characterizing solutions to production planning problems which are non-decreasing in the cost constraint. Such solutions are called monotone solutions. We establish in this paper that monotone solutions to production planning problems are essentially continuous functions of the cost constraint. A preliminary investigation of this property can be found in Lahiri (1992).

2. Production Planning By A Regulated Firm :- We assume that there is a regulated firm producing n-different commodities in non-negative amounts. Thus $\mathbb{R}^n_+$ is the output space of the firm. A cost function for the firm is a function $C:\mathbb{R}^n_+ \to \mathbb{R}$ which is unbounded above, strictly increasing [i.e. $x,y \in \mathbb{R}^n_+$, $x \geqq y$, $x \neq y \Rightarrow C(x)>C(y)$], quasi-convex [i.e. $x,y \in \mathbb{R}^n_+$ and $t \in [0,1] \Rightarrow C(tx+(1-t)y) \leqq \max\{C(x),C(y)\}$], continuous and satisfies $C(0)=0$. Costs are measured in units of money or a numeraire commodity if one such is available. Let $\mathcal{C}$ be the set of all cost functions. A cost-constraint is a non-negative real number $c \geqq 0$. A cost-constraint imposes a ceiling on the costs that the firm can incur in the production process. An aspiration level for the firm is a vector $\bar{y} \in \mathbb{R}^n_+$ such that $C(\bar{y}) \leqq c$. An aspiration level for the firm denotes the minimum quantity of each output the firm requires to produce. A target for the firm is a vector $\bar{x} \in \mathbb{R}^n_{++}$ which denotes a (possibly exogenously set) goal for the firm in

executing its production plans. The target may or may not satisfy $C(\bar{x}) \leq c$. A <u>production planning problem</u> for the firm is an element $(C,c,\bar{y},\bar{x}) \in \mathcal{C} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_{++}$ such that $\bar{x} >> \bar{y}$. A <u>production planning problem</u> $(C,c,\bar{y},\bar{x})$ is said to be <u>regular</u> if $c>0$ and $C(\bar{y})<c$. Here for $x,y \in \mathbb{R}^n_+$, $x>>y \Leftrightarrow x_i > y_i \ \forall i=1,\ldots,n$. Thus if $\mathcal{P}$ denotes the set of all production planning problems and $\mathcal{P}^0$ denotes the set of all regular production planning problems then

$$\mathcal{P} \equiv \{(C,c,\bar{y},\bar{x}) \in \mathcal{C} \times \mathbb{R}_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_{++} / \bar{x}>>y\}$$

$$\mathcal{P}^0 \equiv \{(C,c,\bar{y},\bar{x}) \in \mathcal{P} / C(\bar{y})<c\}.$$

Clearly for a regular production planning problem $(C,c,\bar{y},\bar{x})$, $c>0$. Note that in the above $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n / x_i \geq 0 \ \forall i=1,\ldots,n\}$ and $\mathbb{R}^n_{++}\{x \in \mathbb{R}^n_+ / x_i > 0 \forall i=1,\ldots,n\}$, where $\mathbb{R}^n$ is the n-dimensional Euclidean space.

A <u>solution</u> to production planning problems is a function $F:\mathcal{P} \to \mathbb{R}^n_+$ such that $\forall (C,c,\bar{y},\bar{x}) \in \mathcal{P}$, $C[F(C,c,\bar{y},\bar{x})] \leq c$ and $F(C,c,\bar{y},\bar{x}) \geq \bar{y}$. A solution $F:\mathcal{P} \to \mathbb{R}^n_+$ is said to be <u>efficient</u> if $\forall (C,c,\bar{y},\bar{x}) \in \mathcal{P}$, $C[F(C,c,\bar{y},\bar{x})]=c$. A solution $F:\mathcal{P} \to \mathbb{R}^n_+$ is said to be <u>monotone</u> if $\forall (C,c,\bar{y},\bar{x})$, $(C,c',\bar{y},\bar{x}) \in \mathcal{P}$ with $c>c'$ we have $F(C,c,\bar{y},\bar{x}) \geq F(C,c',\bar{y},\bar{x})$ and $F(C,c,\bar{y},\bar{x}) \neq F(C,c',\bar{y},\bar{x})$.

Monotonicity implies that if the cost constraint is relaxed, other things equal, no output is produced less and some output is actually produced more. Efficiency means that the cost constraints are met with equality by the production process. These two properties seem quite reasonable in the case of production planning for a regulated firm.

3. <u>Group Decision Problems</u> :- A general group decision problem with n-attributes will for our purposes be defined to be an

ordered triplet $(S,\bar{y},\bar{x}) \in 2^{\mathbb{R}^n_+} \times \mathbb{R}^n_+ \times \mathbb{R}^n_{++}$ such that $\bar{y} \in S$ and $\bar{x} \gg \bar{y}$. S is referred to in the literature as a feasible set of attribute vectors; $\bar{y}$ is generally conceived of as a status-quo point; $\bar{x}$ is called a target point or a claims point. Let Σ_n denote the class of all group decision problems. We shall assume the following about members of Σ_n :Given $(S,\bar{y},\bar{x}) \in \Sigma_n$.

<u>Assumption 1</u> :- S is a non-empty compact, convex subset of $\mathbb{R}^n_+$ satisfying comprehensiveness i.e. $x \in S$, $0 \leqq y \leqq x \Rightarrow y \in S$.

<u>Assumption 2</u> :- S satisfies minimal transferability i.e. $y \in S$, $y_i > 0 \Rightarrow \exists x \in S$ with $x_i < y_i$ and $x_j > y_j$ $\forall j \neq i$.

Assumption 1 is standard. Assumption 2 which can be found in Moulin (1988), basically says that the north-eastern boundary of S does not contain a segment which is parallel to any of the axes. It is really like a regularity assumption for group decision making problems and one that proves extremely convenient in establishing equivalence of weak and strong optimality properties.

4. <u>Equivalence of Production Planning and Group Decision Problems</u>:-

Let $(C,c,\bar{y},\bar{x}) \in \mathcal{P}$, as defined in section 2. Let $S(C,c)=\{y \in \mathbb{R}^n_+ / C(y) \leqq c\}$.

<u>Proposition 1</u> :- $\forall (C,c,\bar{y},\bar{x}) \in \mathcal{P}$, $(S(C,c),\bar{y},\bar{x}) \in \Sigma_n$.

<u>Proof</u> :- $S(C,c) \neq \emptyset$, since $0 \in S(C,c)$.

Further, since C is continuous, S(C,c) is closed and since C is unbounded above S(C,c) is bounded. Hence S(C,c) is compact. Let $x,y \in S(C,c)$. Then $C(x) \leqq c, C(y) \leqq c$. By quasiconvexity of C,

$C(tx+(1-t)y)\leq c\ \forall t\in[0,1]$. Thus $S(C,c)$ is convex.

Let $x\in S(C,c)$ with $x_i>0$. Let $0\leq y_i<x_i$. Since C is strictly increasing $C(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)<C(x)\leq c$. By continuity of C and since C is strictly increasing there exists $\epsilon>0$ such that $c\geq C(x_1+\epsilon,\ldots,x_{i-1}+\epsilon,y_i,x_{i+1}+\epsilon,\ldots,x_n+\epsilon)>C(x_1,\ldots,x_{i-1},y_i,x_{i+1},\ldots,x_n)$. Let $y_j=x_j+\epsilon$ for $j\neq i$ and $y=(y_j)^n_{j=1}$. Thus $y\in S(C,c)$. Hence $S(C,c)$ satisfies minimal transferability.

Therefore $(S(C,c),\bar{y},\bar{x})\in\Sigma_n$.

Q.E.D.

<u>Proposition 2</u> :- $\forall(S,\bar{y},\bar{x})\in\Sigma_n$, there exists $(C,c,\bar{y},\bar{x})\in\mathcal{P}$ such that $S=S(C,c)$.

<u>Proof</u> :- Given $(S,\bar{y},\bar{x})\in\Sigma_n$ define the Pareto optimal set of S as follows :

$PO(S)=\{y\in S/\nexists x\in S \text{ with } x>y\}$.

There thus exists a function $\emptyset:[0,a_1]\times\ldots\times[0,a_{n-1}]\to[0,a_n]$ where $a_i\geq 0\ \forall i=1,\ldots,n$ satisfying the following properties:

(i) $\emptyset$ is a strictly decreasing function;

(ii) $\emptyset$ is concave and continuous;

(iii) $PO(S)=\{x\in\mathbb{R}^n_+ / x_n=\emptyset(x_1,\ldots,x_{n-1}), x_i\in[0,a_i], i=1,\ldots,n\}$.

Let $c=\emptyset(o)$, $o\in\mathbb{R}^{n-1}_+$ and define $C:S\to\mathbb{R}$ as follows

$C(x)=x_n-\emptyset(x_1,\ldots,x_{n-1})+c$.

Clearly C as defined is continuous, convex and strictly increasing with $C(o)=o$ and $C(x)=c\ \forall x\in PO(S)$.

Let $x\in\mathbb{R}^n_+\setminus S$. Then, there exists a unique $\lambda(x)>0$ and a unique $y(x)$ belonging to $PO(S)$, such that $x=\lambda(x)y(x)$. Put, $C(x)=\lambda(x)C(y(x))$.

It is a routine check now to verify that $C:\mathbb{R}^n_+ \to \mathbb{R}$, is quasi-convex, continuous, strictly increasing and unbounded above with $C(o)=0$. Further $S=S(C,c)$. This proves the proposition and the necessary equivalence.

Q.E.D.

5. <u>Monotonic Solutions To Production Planning Problems</u> :- A preliminary investigation of the continuity property of monotonic solutions for production planning problems can be found in Lahiri (1992). In this section we establish a similar continuity property of monotonic solutions to production planning problems as defined in this paper.

Let $\mathcal{B}(C,\bar{y},\bar{x})=\{c\in\mathbb{R}_+ \,/\, (C,c,\bar{y},\bar{x})\in\mathcal{P}\}$.

Clearly $C(\bar{y})\in\mathcal{B}(C,\bar{y},\bar{x})$. Hence $(C,\bar{y},\bar{x}) \neq \emptyset$.

<u>Lemma 1</u> :- $\mathcal{B}(C,\bar{y},\bar{x})$ is a left closed interval which is unbounded above.

<u>Proof</u> :- Let $c,c'\in\mathcal{B}(C,\bar{y},\bar{x})$. Thus $C(\bar{y})\leqq c$ and $C(\bar{y})\leqq c'$.

$\therefore\ \forall t\in[0,1],\ C(y)\leqq tc+(1-t)c'$.

$\therefore\ tc+(1-t)c'\in\mathcal{B}(C,\bar{y},\bar{x})\ \forall t\in[0,1]$.

$\therefore\ (C,\bar{y},\bar{x})$ is convex.

Let $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{R}_+$ such that $C(\bar{y})\leqq c_n \forall\, n\in\mathbb{N}$. If $\lim_{n\to\infty} c_n=c$, then $C(\bar{y})\leqq c$. Thus $c\in\mathcal{B}(C,\bar{y},\bar{x})$ and so $\mathcal{B}(C,\bar{y},\bar{x})$ is closed in $\mathbb{R}_+$.

Since $c<C(\bar{y}) \Rightarrow c\notin\mathcal{B}(c,\bar{y},\bar{x})$, $\mathcal{B}(C,\bar{y},\bar{x})$ is bounded below.

Thus $\mathcal{B}(C,\bar{y},\bar{x})$ is a left closed interval.

Since $c>C(\bar{y}) \Rightarrow c\in\mathcal{B}(C,\bar{y},\bar{x})$, $\mathcal{B}(C,\bar{y},\bar{x})$ is unbounded above.

Q.E.D.

Given a solution $F:\mathcal{P}\to\mathbb{R}^n_+$ define $g:\mathcal{B}(C,\bar{y},\bar{x})\to\mathbb{R}^n_+$ as follows:

$g(c)=F(C,c,\bar{y},\bar{x})\ \forall c\in \mathcal{C}(C,\bar{y},\bar{x})$.

Q.E.D.

We may now assert the following proposition:

Proposition 3 :- Suppose $F:\mathcal{P}\to\mathbb{R}^n_+$ is a solution to production planing problems which satisfies the following properties:

(i) F is efficient

(ii) F is monotonic

(iii) $C(\bar{x})=c \Rightarrow F(C,c,\bar{y},\bar{x})=\bar{x}\ \forall(C,c,\bar{y},\bar{x})\in\mathcal{P}$

(iv) $C(\bar{y})=c \Rightarrow F(C,c,\bar{y},\bar{x})=\bar{y}\ \forall(C,c,\bar{y},\bar{x})\in\mathcal{P}$.

Let $\mathcal{C}(C,\bar{y},\bar{x})=[c_1,+\infty)$

Then $g:\mathcal{C}(C,\bar{y},\bar{x})\to\mathbb{R}^n_+$ is a continuous function such that $g(c_1)=\bar{y}$ and $\bar{x}$ belongs to the range of g. Further $c>c' \Rightarrow g(c)\geqq g(c')$, $g(c)\neq g(c')$ whenever $c,c'\in\mathcal{C}(C,\bar{y},\bar{x})$.

Proof :- The fact that $c>c' \Rightarrow g(c)\geqq g(c'), g(c)\neq g(c')$ whenever $c,c'\in\mathcal{C}(C,\bar{y},\bar{x})$ follows from the definition of g and the monotonicity of F.

It is easy to see that $c_1=C(\bar{y})$.

Hence by property (iv) $g(c_1)=\bar{y}$.

Let $c=C(\bar{x})>c_1$. Then by property (iii) $g(c)=\bar{x}$. Hence $\bar{x}$ belongs to the range of g.

Since $c,c'\in\mathcal{C}(C,\bar{y},\bar{x})$, $c>c' \Rightarrow g(c)\neq g(c'), g(c)\geqq g(c')$, g is also 1-1. Further (i) implies $C[g(c)]=c\ \forall c\in\mathcal{C}(C,\bar{y},\bar{x})$.

Suppose towards a contradiction g is not continuous. Then there exists a sequence $\{c_m\}_{m\in\mathbb{N}}\subseteq\mathcal{C}(C,\bar{y},\bar{x})$ such that $\lim_{m\to\infty} c_m=c$, but $\lim_{m\to\infty} g(c_m)\neq c$. Without loss of generality, we may assume that the sequence $\{c_m\}_{m\in N}$ is monotonic; in particular we may assume that it is monotonically increasing, since a similar argument applies if it is monotonically decreasing. Hence there exists $\epsilon>0$, such that $\|g(c)\| \geq \|g(c_m)\| + \epsilon$ for infinitely many m's. We may assume without loss of generality that it is true $\forall m \in N$. However by property (i),

$$C(g(c)) - C(g(c_m)) = c - c_m$$

$$\therefore \lim_{m\to\infty} C(g(c_m)) = C(g(c))$$

But $\|g(c)\| \geq \|g(c_m)\| + \epsilon \; \forall m \in N$, g is monotonic increasing and C is strictly increasing and continuous implies that there exists $\delta>0$ such that $C(g(c)) \geq C(g(c_m)) + \delta \; \forall m \in N$, which contradicts $\lim_{m\to\infty} C(g(C_m)) = C(g(c))$ and proves continuity of g.

Q.E.D.

6. <u>Conclusion</u> :- In this analysis we have established the equivalence of two distinct areas of research : (i) group decision theory; and (ii) theory of output choice by a regulated firm. What do we propose to gain from that?

The primary gain from such an equivalence result, is that axiomatic characterization theorems for solutions to group decision problems can now be made available for the theory of output choice by a regulated firm. Output decisions in a regulated firms can now be made on the basis of desirable properties that these choices should satisfy rather than on ad hoc guidelines which are often decided on the basis of political considerations. Thus the main advantage is that of being able to develop well defined criteria of output choice.

Finally, in the above analysis we turn to the inherent continuity property satisfied by monotonic solutions to production planning problems. This tells us that if a solution is monotonic and discontinuous, then it is of necessity violating efficiency. Efficiency, as we define it, is in the case of a regulated firm, the ability to meet planned expenditures and thus a desirable consequence to insist upon. Thus, so is continuity of the observed solution.

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