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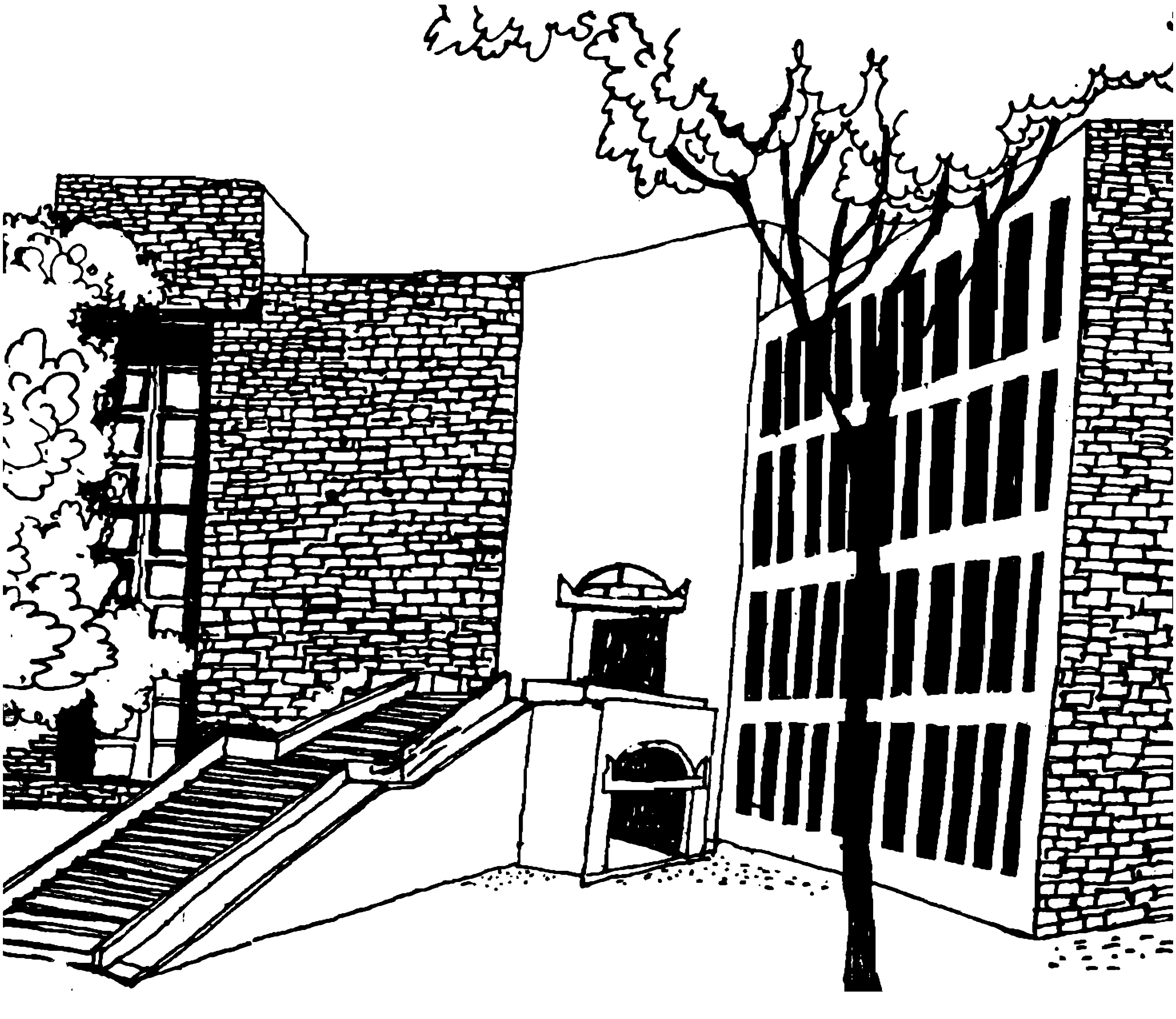

THE LOGARITHMIC RELATIVE EGALITARIAN SOLUTION: AN AXIOMATIC CHARACTERIZATION

By

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ABSTRACT

In this paper we propose and axiomatically characterize the Logarithemic Relative Egalitarian Solution for social choice problems.

1. Introduction:

Suppose we are given a pure exchange economy consisting of a finite number of consumers and a finite number of goods. Suppose that the utility function of each consumer is Cobb-Douglas. A common method of finding the demand function enroute to obtaining the Walrasian equilibrium (from equal income) price-allocation pair, is to take a suitable power of each utility function and then solve the consumer's maximization problem. Since Walrarsian equilibrium is invariant under exponential (infact any monotone) transformations of the utility function, the solution to the transformed problem is identical to the solution to the original problem.

In a **recent** paper (Lahiri (1994)) we showed that the set of all problems considered in classical bargaining theory (as for instance in Peters (1993)) is isomorphic to the set of simple distribution problems consisting of a finite number of agents each with a concave, continuous, non-constant and non-decreasing utility function for a single commodity. Hence the isomorphism extends easily to distribution problems dealing with a fixed yet finite number of commodities. Thus it would be desirable to know exactly which solution (or solutions) remains invariant under exponential transformations of the utility functions, in the context of welfarist social choice theory.

In this paper we propose and axiomatically characterize a solution which satisfies such a non-linear invariance property.

2. Preliminaries: As in Moulin (1988) we consider a class of choice problems defined thus: a choice problem is a non-empty set $S \subset \mathbb{R}^n_+$ ($n \in \mathbf{N}$, n fixed) satisfying the following properties:

i) S is compact, convex, comprehensive i.e. $x \in S$, $o \leq y \leq x \Rightarrow y \in S$

ii) $\exists\ x \in S$ such that $x_i > o \ \forall i \in \{1, \ldots, n\}$

iii) S satisfies minimal transferability i.e. $x \in S$, $x_i > o$ for some $i \in \{1, \ldots, n\} \Rightarrow \exists\ y \in S$ with $y_i < x_i$ and $y_j > x_j\ \forall\ j \neq i$, $j \in \{1, \ldots, n\}$

In the above $\mathbf{N}$ denotes the set of natural numbers.

Let Σ^n denote the class of choice problems defined above. A domain D^n is any nonempty subset of Σ^n

A choice function on D^n is a function $F: D^n \rightarrow \mathbb{R}^n_+$ such that $F(S) \in S\ \forall\ S \in D^n$.

The following axioms will be required of the choice function we propose to define:

Axiom 1: $\forall\ S \in D^n$, $F(S) \in P(S) = \{x \in S / y \geq x,\ y \neq x \Rightarrow\ y \notin S\}$

Axiom 1 is called <u>Efficiency</u>

Axiom 2: $\forall\ S \in D^n$ if, $\pi(S) = S \forall\ \pi : \{1, \ldots, n\} \Rightarrow \{1, \ldots, n\}$ which are 1-1, then $F_i(S) = F_j(S)\ \forall\ i, j \in \{1, \ldots, n\}$. (Note: For $x \in \mathbb{R}^n_+$, $y = \pi(x) \in \mathbb{R}^n_+$ with $y_{\pi(i)} = x_i$; $\pi(S) = \{\pi(x) / x \in S\}\ \forall S \subset \mathbb{R}^n_+$.

Axiom 2 is called <u>Symmetry.</u>

Axiom 3: $\forall S \in D^n$, $\forall \alpha \in \mathbb{R}^n_{++}$ with $\alpha_i \leq 1$ $\forall i \in \{1, \ldots, n\}$, $F(S^\alpha) = (F(S))^\alpha$ if $S^\alpha \in D^n$

Here, $x^\alpha = (x^{\alpha_1}, \ldots, x^{\alpha_n})$ $\forall x \in \mathbb{R}^n_+$; $S^\alpha = \{x^\alpha / x \in S\}$

Axiom 3 is called <u>Exponential Invariance.</u>

Axiom 4: $\forall S \in D^n$, $T \in D^n$, $S \subset T$, $u(S) = u(T) \Rightarrow F(S) \leq F(T)$. Here, for $S \in D^n$ and $i \in \{1, \ldots, n\}$, $u_i(S) = \max\{x_i / x \in S\}$; $u(S) = (u_1(S), \ldots, u_n(S))$ is called the utopia point for S.

Axiom 4 is called <u>Restricted Monotonicity</u>

We shall consider the following domain D^n on which we shall define our proposed solution:

$D^n = \{S \in \Sigma^n$ / either $u_i(S) > 1$ $\forall i \in \{1, \ldots, n\}$ or $u_i(S) < 1$ $\forall i \in \{1, \ldots, n\}\}$

We define the choice function $L : D^n \rightarrow \mathbb{R}^n_+$ as follows:

$L(S) = x$ such that $x \in P(S)$ and

$$\frac{\log x_i}{\log u_i(S)} = \frac{\log x_j}{\log u_j(S)} \quad \forall i, j \in \{1, \ldots n\}$$

It is easy to see that on the domain D^n, the choice function L is well defined.

We shall now proceed to an initial axiomatic characterisation of the choice function L, which may be called the Logarithmic Relative Egalitarian Choice Function.

3. A First Characterization Theorem:

Theorem 1: The only choice function on D^n to satisfy Axioms 1 to 4 is L.

Proof: That L satisfies the four axioms is obvious. Thus suppose $S \in D^n$. By Axiom 3, we may assume $u_i(S) = u_j(S)$ $\forall i, j \in \{1, \ldots, n\}$. Let $a_i(S) = (0, \ldots, u_i(S), o, \ldots, o)$ $\forall i \in \{1, \ldots, n\}$, ie. the i^{th} coordinate of $a_i(S)$ is $u_i(S)$ and the other coordinates are zero. Observe L(S) has all coordinates equal.

Let T = convex hull $\{o, a_1(S), \ldots, a_n(S), L(S)\}$.

$T \subset S$ and $u(T) = u(S)$. It is easy to see that $T \in D^n$.

By Axiom 4, $F(T) \leq F(S)$ where F is any solution on D^n satisfying Axioms 1 to 4.

By Axioms 3 and Axiom 1, $F(T) = L(S)$

$\therefore$ $L(S) \leq F(S)$

But $L(S) \in P(S)$

Thus, $L(S) = F(S)$

4. Characterisation with a variable number of agents: The current trend in axiomatic choice theory is to obtain characterization for solutions defined on domains with a variable number of agents. The set of potential agents is now considered to be the set of natural numbers $\mathbb{N}$. Let $\mathcal{P}$ denote the set of all finite subsets of $\mathbb{N}$. For $M \in \mathcal{P}$, let $\mathbb{R}^M_+$ denote the set of all functions from M to $\mathbb{R}^n_+$.

Let $\Sigma^M = \{S \subseteq \mathbb{R}^M_+ /$

(i) $S \neq \emptyset$, S is compact, convex and comprehensive

(ii) $\exists \; x \in S$ with $x_i > o \; \forall i \in M$

(iii) S satisfies minimal transferability $\}$

Let $D^M \subset \Sigma^M$ be similarly defined $\forall M \in \mathcal{P}$ and let $D = \bigcup_{M \in \mathcal{P}} D^M$. Let $L : D \to \bigcup_{M \in \mathcal{P}} \mathbb{R}^M_+$ be defined as follows:

$\forall \; M \in \mathcal{P}$, $\forall S \in D^M$, $L(S) \in P(S)$ and

$$\frac{\log L_i(S)}{\log u_i(S)} = \frac{\log L_j(S)}{\log u_j(S)} \forall i, j \in M$$

We now invoke the following properties:

Property 1 (Efficiency): $\forall S \in D$, $F(S) \in P(S)$

Property 2 (Anonymity) : $\forall M, N \in \mathcal{P} \; \forall \; S \in D^M$, $T \in D^N$ if $\pi : M \to N$ be a 1-1 function and if $T = \pi(S)$ then $F(T) = \pi[F(S)]$.

Property 3 (Exponential Invariance): $\forall M \in \mathcal{P}$, $\forall S \in D^M$, $\forall \alpha \in \mathbb{R}^M_{++}$ with $\alpha_i \leq 1 \; \forall \; i \in M$, $F(S^\alpha) = [F(S)]^\alpha$

Property 4 (Monotonicity with respect to a variable number of agents): $\forall M, N \in \mathcal{P}$ with $M \subseteq N$ and $S \in D^n$, let $T = \{x_M / x \in S\}$

Here x_M is the vector in $\mathbb{R}^M_+$ whose i^{th} coordinate is $x_i \forall i \in M$)
Then, $F(T) \geq F_M(S)$.

In the above F is any function from D to $\bigcup_{M \in \mathcal{P}} \mathbb{R}^M_+$ such that $F(S) \in S \forall S \in D$.

We can now state a characterization theorem whose proof after appealing to property 3, follows the characterization theorem in Thomsom (1983) for the relative egalitarian solution.

Theorem 2: The only choice function on D to satisfy properties 1 to 4 is L.

In the above characterization theorem we appeal to monotinity with respect to a variable number of agents instead of restricted monotonicity.

5. Conclusion:

In this paper we propose the logarithmic relative egalitarian solution for choice problems and provide two different axiomatic characterization of the same. The solution we propose is not scale invariant but invariant under concave transformations which result by taking powers of the co-ordinates. There was a time when scale invariance was considered canonical in the relevant literature because choice problems were considered as consisting of vectors of von-Neumann Morgenstern utility levels, obtained form lotteries over underlying physical alternatives. With the recent interpretation in Lahiri (1994) of choice problems being generated from underlying distribution problems, scale invariance ceases to be a must. Invariance under alternative transformations (so long as they preserve the convexity of the choice set) become meaningful as well. This paper could be considered as a starting point in such a venture.

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