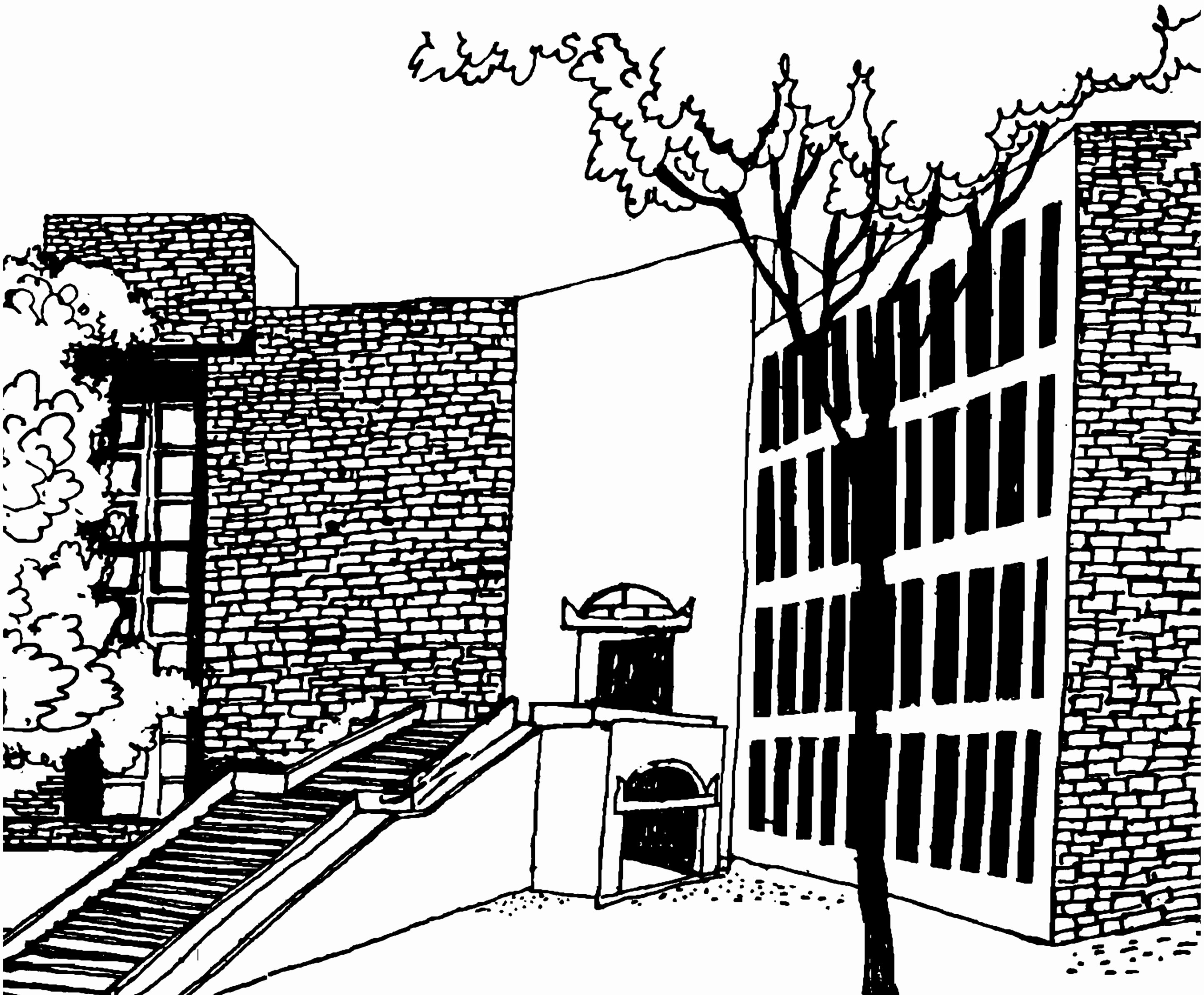
विद्याविनियोगाद्विकासः

IIM AHMEDABAD

Working Paper



A NOTE ON EXPANSION INDEPENDENCE IN MULTIATTRIBUTE CHOICE PROBLEMS

By

Somdeb Lahiri

W P No. 1120
July 1993

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Abstract

Two appealing independence properties have been used by us to characterize the egalitarian, utilitarian and relative egalitarian choice functions.

1. Introduction :- In Thomson (forthcoming) can be found an independence property for multiattribute choice problems called "expansion independence". The purpose of this note, is to show that along with efficiency and symmetry, this property uniquely characterizes the egalitarian choice function, and this property along with shift invariance characterizes the utilitarian choice function. A variant of this property characterizes the relative egalitarian choice function.

2. Multiattribute Choice Problems :- A multiattribute choice problem is an ordered pair (S,c) where $0 \in S \subseteq \mathbb{R}^n_+$ and $c \in \mathbb{R}^n_+$ for some $n \in \mathbb{N}$ (the set of natural numbers). The set S is called the set of feasible attribute vectors and the point c is called a target point.

We shall consider the following class $\mathcal{Q}$ of admissible **multiattribute** choice problems: $(S,c) \in \mathcal{Q}$ if and only if

(i) S is compact and convex

(ii) S satisfies minimal transferability: $x \in S$, $x_i > 0 \Rightarrow \exists y \in S$ with $y_i < x_i$ and $y_j > x_j \ \forall j \neq i$.

(iii) S is comprehensive: $x \in S$, $0 \leqq y \leqq x \Rightarrow y \in S$.

(Here for $x, y \in \mathbb{R}^n$, $x \geqq y$ means $x_i \geq y_i \ \forall i \in \{1, \ldots, n\}$; $x > y$ means $x \geqq y$ but $x \neq y$; $x >> y$ means $x_i \geqq y_i \ \forall i = 1, \ldots, n$).

A <u>domain</u> is any subset D of $\mathcal{Q}$.

A (<u>multiattribute</u>) <u>choice function</u> on $D (\subseteq \mathcal{Q})$ is a function $F: D \to \mathbb{R}^n$ such that $\forall (S,c) \in D$, $F(S,c) \in S$.

Let $F: D \to \mathbb{R}^n$ be a choice function. Three important properties often required of a choice function are the following:

(P.1) <u>Efficiency</u> :- $\forall (S,c) \in D$, $x \in S$, $x \geqq F(S,c) \Rightarrow x = F(S,c)$.

(P.2) <u>Symmetry</u> :- If $\forall$ permutation $\sigma: N \to N$, $\sigma(S) = S$ and $\sigma(c) = c$, then $F_i(S,c) = F_j(S,c) \ \forall \ i, j \in \{1, \ldots, n\}$. Here for $x \in \mathbb{R}^n_+$, $\sigma(x)$ is the vector in $\mathbb{R}^n_+$ whose ith coordinate is $x_{\sigma(i)}$ and $\sigma(S) = \{\sigma(x) : x \in S\}$.

(P.3) Scale Independence :- $\forall (S,c) \in D$, $\alpha \in \mathbb{R}^n_{++}$, $(\alpha.S, \alpha.c) \in D \Rightarrow$

$F(\alpha.S, \alpha.c) = \alpha.F(S,c)$.

Here $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n_+ / x_i > 0 \forall i=1,\ldots,n\}$: $\alpha.x = (\alpha_1 x_1, \ldots, \alpha_n x_n) \in \mathbb{R}^n_+$ for $x \in \mathbb{R}^n_+$ and $\alpha.S = \{\alpha.x / x \in S\}$.

An important domain, studied traditionally in axiomatic bargaining (and where our analysis) will be restricted is the following:

$\mathcal{L}_u = \{(S,c) \in \mathcal{L} / c = u(S)$ where $u_i(S) = \max\{x_i / x \in S\}\}$

Problems $(S,c) \in \mathcal{L}_u$ will be denoted simply by S.

Let $F: \mathcal{L}_u \to \mathbb{R}^n$ be a choice function. Two properties that we shall investigate separately are:

(P.4) Expansion Independence :- $\forall S,T \in \mathcal{L}_u$, $S \subseteq T$,

$F(S) \in P(T) \equiv \{x \in T / y \geq x \Rightarrow y = x\} \Rightarrow F(T) = F(S)$.

This property is due to Thomson (forthcoming).

(P.5) Restricted Expansion Independence :- $\forall S,T \in \mathcal{L}_u$,

$u(S) = u(T)$, $S \subseteq T$, $F(S) \in P(T) \Rightarrow F(T) = F(S)$.

In investigating the above two properties, we shall be characterizing the following two choice functions:

(a) $F_E : \mathcal{L}_u \to \mathbb{R}^n_+$ called the egalitarian choice function and defined as

$F_E(S) = \bar{\lambda} e$, where $\bar{\lambda} = \max\{\lambda \geq 0 / \lambda e \in S\}$, e being the vector in $\mathbb{R}^n$ with all coordinates being equal to one.

(b) $F_{RE} : \mathcal{L}_u \to \mathbb{R}^n_+$ called the relative egalitarian choice function and defined as

$F_{RE}(S) = \bar{\lambda}.u(S)$, where $\bar{\lambda} = \max\{\lambda \geq 0 / \lambda u(S) \in S\}$.

The two solutions have been discussed in Moulin (1988) for instance.

3. Characterization of the egalitarian choice function :-

Theorem 1 :- The only solution on $\mathcal{L}_u$ to satisfy efficiency, symmetry and expansion independence is the egalitarian choice function.

Proof :- It is easy to verify that F_E satisfies the above conditions. Conversely suppose $F: \mathcal{L}_u \rightarrow \mathbb{R}^n_+$ be any choice function satisfying the given properties. If $S=\{0\}$, then $F(S)=0=F_E(S)$. So assume $S \neq \{0\}$. Then $\bar{\lambda}=\text{Sup}\{\lambda \geq 0 / \lambda.e \in S\} > 0$ and $\lambda.e \in P(S)$. By minimal transferability $\forall i \in \{1,\ldots,n\}$, $\exists\, v^i \in S$ such that $v^i_i < \bar{\lambda}$, $v^i_j > \bar{\lambda}$ if $j \neq i$. Let $\alpha = \min_{1 \leq i \neq j \leq n} v^i_j$. Clearly $\alpha > \bar{\lambda}$. Define for $i \in \{1,\ldots,n\}$, $a^i \in \mathbb{R}^n_+$ such that $a^i_j = \alpha$ for $j \neq i$, $a^i_i = 0$ $a^i \leq v^i \; \forall$ i and hence by comprehensiveness, $a^i \in S$. Let T=convex hull $\{0, a^1, \ldots, a^n\}$. $\bar{\lambda}e) \bar{\lambda}e \in P(T)$ and hence by symmetry, $F(T)=\bar{\lambda}.e \in P(S)$. Thus by expansion independence, $F(S)=\bar{\lambda}.e=F_E(S)$.

Q.E.D

We are thus able to characterize the choice function F_E without Nash's Independence of Irrelevant Alternatives Assumption, which goes as follows:

$\forall S, T \in \mathcal{L}_u$, $S \subseteq T$, $F(T) \in S \Rightarrow F(S)=F(T)$.

Variants of this assumption have been severely criticized in the literature as for instance in Sen (1993).

4. Characterization of the utilitarian choice function :-

As in Lahiri (1993) we consider the domain

$\mathcal{L}^0_u = \{S \in \mathcal{L}_u : x, y \in P(S),\ t \in (0,1) \Rightarrow tx+(1-t)y \notin P(S)\}$.

On such domains we may define the utilitarian choice function $F_{ut} : \mathcal{L}^0_u \rightarrow \mathbb{R}^n_+$ as follows:

$$F_{ut}(S) = \arg\max_{x \in S} (\Sigma^n_{i=1} x_i)$$

It is easy to see that this choice function is well defined. In Lahiri (1993) along with Nash's Independence of Irrelevant Alternatives, efficiency, symmetry and a property called "shift invarnance", we uniquely characterized the utilitarian choice function. In this section we do the same by just replacing Nash's Independence of Irrelevant Alternatives by expansion independence.

Let $F: \mathcal{L}^0_u \rightarrow \mathbb{R}^n_+$ be a choice function

(F.6) Shift Invariance :- $\forall\, S \in \mathcal{L}^0_u, \forall\, a \in \mathbb{R}^n_+$, $a \leq F_{ut}(S)$, F((S-

$\{a\}) \cap \mathbb{R}^n_+) = F(S)-a$.

Theorem 2 :- The only choice function on $\mathcal{L}^0_u$ to satisfy efficiency, symmetry, shift-invariance and expansion independence is the utilitarian solution.

Proof :- That F_{ut} satisfies the above properties is clear. If $S=\{0\}$, then $F_{ut}(S)=F(S)=0$. Hence assume $S \neq \{0\}$ and $F: \mathcal{L}^0_u \to \mathbb{R}^n_+$ satisfies the above properties. Let $x^* = F_{ut}(S)$ and $\bar{\lambda} = \sup\{\lambda \geq 0 / x^* - \lambda e \in \mathbb{R}^n_+\}$. Let $a = x^* - \bar{\lambda} e$ and $T = (S - \{a\}) \cap \mathbb{R}^n_+$. Clearly $F_{ut}(T) = \bar{\lambda} e$. Let U be the largest symmetric set in $\mathcal{L}^0_u$, which is contained in T. By efficiency and symmetry and since $\bar{\lambda} e \in P(U)$ we have $F(U) = \bar{\lambda} e$. But $\bar{\lambda} e \in P(T)$ and by expansion independence $F(T) = \bar{\lambda} e$. By shift invariance $F(S) = x^* = F_{ut}(S)$.

Q.E.D.

5. Characterization of the relative egalitarian choice function:-

Theorem 3 :- The only choice function on $\mathcal{L}_u$ to satisfy efficiency, symmetry, scale independence and restricted expansion independence is the relative egalitarian solution.

Proof :- That F_{RE} satisfies the above mentioned properties is once again easy to verify.

Thus, let $F: \mathcal{L}_u \to \mathbb{R}^n_+$ be a choice function satisfying the above properties. If $S=\{0\}$, then $F(S)=0=F_{RE}(S)$. Hence, suppose $S \neq \{0\}$. By scale invariance, we may assume $u(S)=e$. Let $F_{RE}(S) = \bar{\lambda}.e$. Clearly $\bar{\lambda} > 0$. Consider the set T=convex hull $\{0, e_1, \ldots, e_n, \bar{\lambda}.e\}$ where e_i is the ith (unit) coordinate vector in $\mathbb{R}^n$. $T \subseteq S$, $u(T)=u(S)=e$. By symmetry and efficiency $F(T) = \bar{\lambda}.e \in P(S)$. By restricted expansion independence, $F(S)=F(T)=F_{RE}(S)$.

Q.E.D.

5. Conclusion

:- Two appealing independence properties, which exist in the literature, have in this paper been used to characterize, three different but well-known choice functions. More recent (yet different) characterizations of two of the above choice functions can be found in Livne (1989). The earliest known analysis of the expansion independence property can be found in Thomson and Myerson (1980).

References :-

1. Lahiri, S. (1993) : "Revealed Preference and Utilitarianism in Multiattribute Choice Problems", mimeo.

2. Livne, Z.A. (1989) : "Axiomatic Characterizations of the Raiffa and the Kalai-Smorodinsky Solutions to the Bargaining Problem", Operations Research 37, 972-980.

3. Moulin, H. (1988) : "Axioms of Cooperative Decision Making", Cambridge University Press.

4. Sen, A.K. (1993) : "Internal Consistency of Choice", Econometrica, 61(3) , 495-521.

5. Thomson, W. (forthcoming) : "Bargaining Theory : The Axiomatic Approach", Academic Press.

6. Thomson, W. and R.B. Myerson (1980) : "Monotonicity and Independence Axioms", International Journal of Game Theory 9, 37-49.