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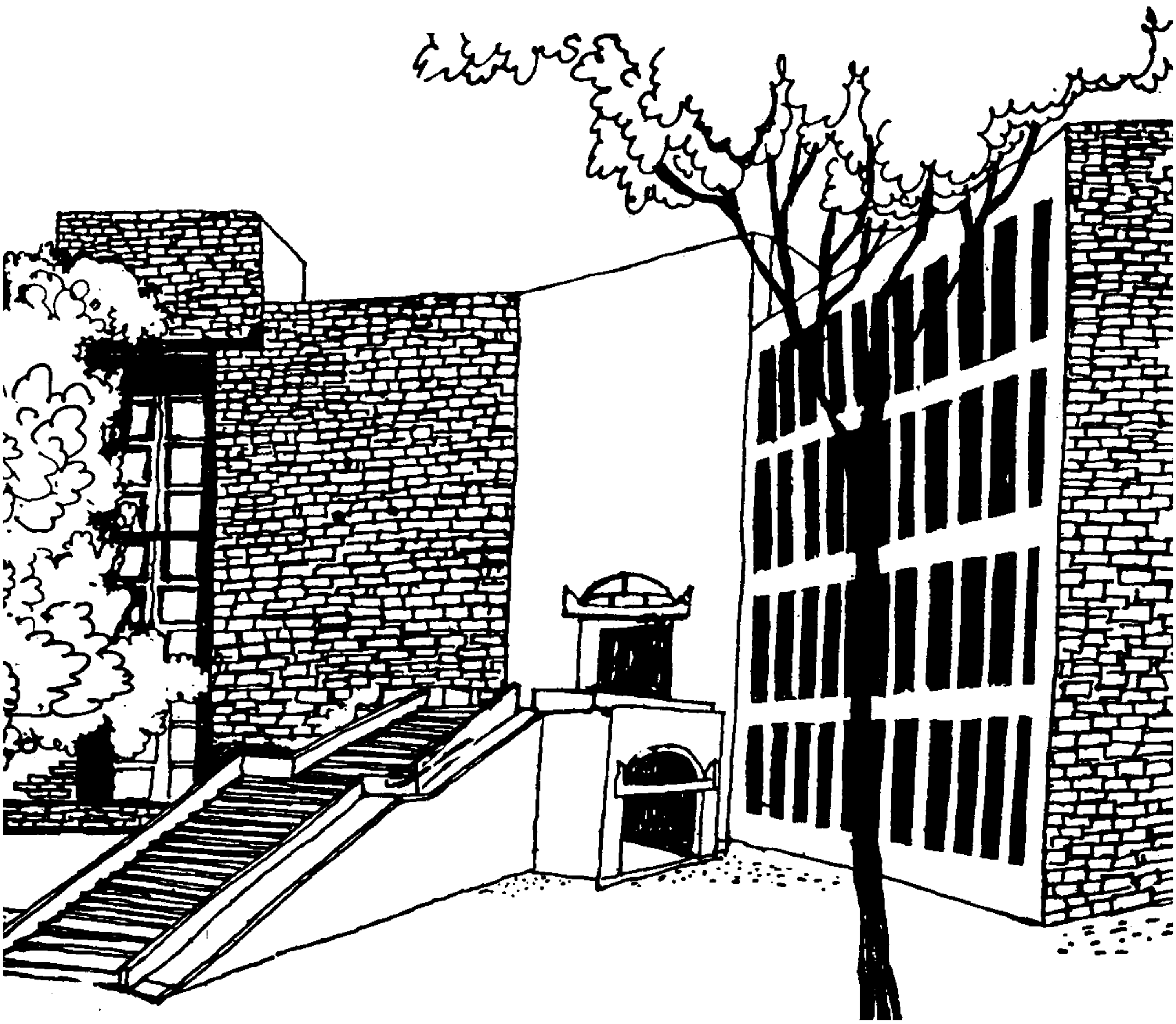
Working Paper



SHIFTS IN MULTIATTRIBUTE CHOICE PROBLEMS

By

Somdeb Lahiri

W P No. 1123
July 1993

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Abstract

In this paper we study the responsiveness of choice functions to shifts in multiattribute choice problems.

1. Introduction :- In this paper we study the responsiveness of choice functions to shifts in multiattribute choice problems.

A multiattribute choice problem is a feasible set of attribute vectors contained in the nonnegative orthant of a finite dimensional Euclidean space, together with a target point which is also contained in the same orthant. A choice function defined on a set of multiattribute choice problems, assigns to each problem a feasible attribute vector. A shift in a multiattribute choice problem moves the choice problem inwards in a specified direction. The theory of multiattribute choice problems has its origin in a series of papers by Chun (1988), Chun and Thomson (1992), Chun and Peters (1989), Bossert (1992a,b), Lahiri (1993a,b,c.) and Abad and Lahiri (1993).

The first property we study in this paper is monotonicity (of choice functions) with respect to unilateral shifts. This adapts to our chosen domain the concept of monotonicity with respect to the disagreement point due to Thomson (1987). On domains similar to those studied by Thomson (1987), we obtain similar results with slightly modified proofs. On a somewhat extended domain studied in Lahiri (1993a,b,c) we obtain the result that the equal loss choice function and the choice function which selects the unique efficient point on the straight line connecting the origin to the target point both satisfy monotonicity with respect to unilateral shifts.

Subsequently, we proceed to a study of a property called concavity with respect to shift, which adapts to our framework a concept due to Chun and Thomson (1990a,b). We show that concavity with respect to shifts imply a certain stability property of the choice function.

2. Multiattribute Choice Problems :- A multiattribute choice problem is an ordered pair (S,c) where $0 \in S \subseteq \mathbb{R}^n_+$ and $c \in \mathbb{R}^n_+$ for some $n \in \mathbb{N}$ (the set of natural numbers). S is called the feasible set of attribute vectors and c is called the target point. We shall consider the following class $\mathcal{C}$ of admissible choice problems:

$(S,c) \in \mathcal{L}$ if and only if

(i) S is nonempty, compact and convex;

(ii) S satisfies <u>minimal transferability</u> : $\forall x \in S$ $\forall i \in \{1,\ldots,n\}$, if $x_i > 0$, there exists $y \in S$ with $y_i < x_i$ and $y_j > x_j$ $\forall j \neq i$.

(iii) S is comprehensive: $0 \leqq y \leqq x \in S \Rightarrow y \in S$.

A <u>domain</u> is any subset of $\mathcal{L}$.

Let D be a domain. A (multiattribute) choice function is a function $F: D \to \mathbb{R}^n_+$ such that $F(S,c) \in S$ $\forall (S,c) \in D$.

We shall consider two important domains apart from $\mathcal{L}$ itself in the subsequent analysis.

$\mathcal{L}_u = \{(S,c) \in \mathcal{L} / c = u(S)$ where $u_i(S) = \max\{x_i / x \in S\}\}$.

A problem (S,c) in $\mathcal{L}_u$ is denoted simply by S.

$\mathcal{L}^0 = \{(S,c) \in \mathcal{L} / S \neq \{0\} \Rightarrow c >> 0\}$

The domain $\mathcal{L}_u$ is referred to in the literature as the class of bargaining problems. We shall refer to $\mathcal{L}^0$ as the class of <u>proper</u> multiattribute choice problems. It is easy to verify that $\mathcal{L}_u \subseteq \mathcal{L}^0$. (In the above for $x, y \in \mathbb{R}^n$, $x \geqq y \Leftrightarrow x_i \geqq y_i$ $\forall i = 1,\ldots,n$; $x > y \Leftrightarrow x \geqq y$ and $x \neq y$; $x >> y \Leftrightarrow x_i > y_i$ $\forall i = 1,\ldots,n$).

On $\mathcal{L}_u$ we consider the following two choice functions:

(1) $F_N : \mathcal{L}_u \to \mathbb{R}^n_+$ is called the Nash choice function and defined as

$$F_N(S) = \arg\max_{x \in S} \left(\pi^n_{i=1} x_i\right)$$

(2) $F_E : \mathcal{L}_u \to \mathbb{R}^n_+$ is called the egalitarian choice function and defined as

$F_E(S) = \bar{\lambda} e$ where $\bar{\lambda} = \max\{\lambda \geqq 0 / \lambda e \in S\}$.

On $\mathcal{L}^0$ we consider the following choice function

(3) $F_{RE} : \mathcal{L}^0 \to \mathbb{R}^n_+$ is called the relative egalitarian choice function and defined as

$F_{RE}(S,c) = \bar{\lambda} c$ where $\bar{\lambda} = \max\{\lambda \geqq 0 / \lambda c \in S\}$.

To define our final choice function we consider a subdomain $\bar{\mathcal{L}}$ of $\mathcal{L}^0$:

$\bar{\mathcal{L}} \equiv (S,c) \in \mathcal{L}^0 / c - (\min_i c_i) e \in S\}$ where e is the vector in $\mathbb{R}^n$ with all coordinates equal to 1.

(4) $F_{EL}: \bar{\ell} \to \mathbb{R}^n_+$ is called the equal loss choice function and defined as

$$F_{EL}(S,c) = [c-(\min_i c_i)e]+\bar{\lambda}(\min_i c_i)e$$

where $\bar{\lambda} = \max\{\lambda \geq 0 / [c-(\min_i c_i)e+\lambda(\min_i c_i)e \in S\}$.

Let D be a domain and $(S,c) \in D$. If given $a \in \mathbb{R}^n_+, (S(a),c-a) \in D$ where $S(a)=\{x-a/x \in S\} \cap \mathbb{R}^n_+$, then we say that $(S(a),(c-a))$ is a shift of $(S,c) \in D$.

Let $F:D \to \mathbb{R}^n_+$ be a choice function. We say that F satisfies monotonicity with respect to unilateral shifts if $\forall$ $i \in \{1,\ldots,n\}$ $\forall$ $\alpha_i \geq 0, a=\alpha_i e, (S,c),(S(a),c-a) \in D$ implies $F^i(S(a),c-a)+\alpha_i \geq F^i(S,c)$. **We say that F satisfies** strict monotonicity with respect to unilateral shifts if $\forall$ $i \in \{1,\ldots,n\}$ $\forall \alpha_i > 0, a=\alpha e_j (S,c),(S(a),c-a) \in D$ implies $F^i(S(a),c-a)+\alpha_i > F^i(S,c)$.

We say that $F:D \to \mathbb{R}^n_+$ is concave with respect to shifts if $\forall (S,c) \in D, (S(a),c-a),(S(a'),c-a'),S(ta+(1-t)a'),c-ta-(1-t)a') \in D \forall t \in [0,1]$ implies $F(S(ta+(1-t)a'),c-ta-(1-t)a') \geq tF(S(a),c-a)+(1-t)F(S(a'),c-a')$

3. Monotonicity with respect to unilateral shifts on ℓ_u :-

A choice function $F:D \to \mathbb{R}^n_+$ is said to be efficient if $\forall (S,c) \in D, x \geq F(S,c), x \in S \Rightarrow x=F(S,c)$. It is said to be scale independent if $(S,c) \in D,(a.S,a.c) \in D$ for $a \in \mathbb{R}^n_{++}$ implies $F(a.S,a.c)=a.F(S,c)$. (Here for $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, $x.y \equiv (x_1 y_1,\ldots,x_n y_n)$ and for $S \subseteq \mathbb{R}^n_+$ $x.S=\{x.y/y \in S\}$. $\mathbb{R}^n_{++} \equiv \{x \in \mathbb{R}^n / x_i > 0 \forall i=1,\ldots,n\}$.) A choice function $F:D \to \mathbb{R}^n_+$ is said to satisfy strict individual rationality if $\forall (S,c) \in D$, $F(S,c) \gg 0$.

Theorem 1 :- Both $F_N : \ell_u \to \mathbb{R}^n_+$ and $F_E: \ell_u \to \mathbb{R}^n_+$ satisfy monotonicity with respect to unilateral shifts. In fact they do so strictly.

Proof :- F_N satisfies efficiency, scale independence and strict individual rationality. Let $S \in \ell_u$, $S \neq \{0\}$, $a=\alpha_i e, \alpha_i > 0 S(a) \in \ell_u$. By scale independence we may assume, $F_N(S)=e$.

Since $e-a \in S(a)$, $1-\alpha_i < \pi^n_{j=1} x_j$. Since $x+a \in S$, $(x_i + \alpha_i) \pi_{j \neq i} x_j < 1$. By strict individual rationality, $x_i > 0$, if $S(a) \neq \{0\}$.

$\therefore (x_i + \alpha_i) \pi^n_{j=1} x_j < x_i$. Similarly $(1-\alpha_i)(x_i + \alpha_i) < (x_i + \alpha_i) \pi^n_{j=1} x_j$. Thus $x_i > (1-\alpha_i)(x_i + \alpha_i)$. Thus $x_i > x_i - \alpha_i x_i + \alpha_i - \alpha_i^2$. This implies since $\alpha_i > 0$, $x_i + \alpha_i > 1$. If $S(a) = \{0\}$, then $1-\alpha_i < 0$ i.e. $\alpha_i + 0 > 1$ and hence $F^i_N(S(a)) + \alpha_i > F^i_N(S)$.

Now let us consider $F_E : \mathcal{l}_u \to \mathbb{R}^n_+$. F_E is efficient. Let $S \in \mathcal{l}_u$, $\alpha_i > 0$, $a = \alpha_i e$, $S(a) \in \mathcal{l}_u$. Suppose $F^i_E(S(a)) + \alpha_i \leqq F^i_E(S)$. Thus $F^i_E(S(a)) \leqq F^i_E(S) - \alpha_i < F^i_E(S)$. Since $F^j_E(S(a)) = F^i_E(S(a))$ $\forall j = 1, \ldots, n$,

$F^j_E(S) = F^i_E(S)$ $\forall$ $j = 1, \ldots, n$ and $F_E(S) - a \in S(a)$, we get $F_E(S) - a > F_E(S(a))$, contradicting the efficiency of F_E.

Q.E.D.

4. Monotonicity with respect to unilateral shifts on other domains :-

Theorem 2 :- (i) On $\mathcal{l}^0$, F_{RE} satisfies monotonicity with respect to unilateral shifts. In fact, F_{RE} satisfies strict monotonicity with respect to unilateral shifts

(ii) On $\mathcal{l}$, F_{EL} satisfies monotonicity with respect to unilateral shifts.

Proof :- (i) F_{RE} satisfies scale independence and efficiency. Let $\{0\} \neq (S, c) \in \mathcal{l}^0$ be such that $c = e$. We can do this by scale invariance. Let $F_{RE}(S, c) = \bar{\lambda} e$, $\bar{\lambda} > 0$. Let $\alpha_i > 0$, $a = \alpha_i e$. $(S(a), c-a) \in \mathcal{l}^0$ and $F_{RE}(S(a), c-a) = \bar{\mu}(c-a)$. Assume towards a contradiction that $\alpha_i + \bar{\mu}(1-\alpha_i) \leq \bar{\lambda}$. Since the point x with $x_i = \bar{\lambda} - \alpha_i$, $x_j = \bar{\lambda}$ $\forall$ $j \neq i$ belongs to $S(a)$, we must have that $\bar{\mu} \geq \bar{\lambda}$ otherwise we would be contradicting the efficiency of $F_{RE}(S(a), c-a)$. But then $\bar{\mu}(1-\alpha_i) \leq 0$. Since $\bar{\mu} \geq \bar{\lambda} > 0$, we must have $\alpha_i \geq 1$. But $\alpha_i \geq 1$ contradicts $c-a >> 0$ i.e. $(S(a), c-a) \in \mathcal{l}^0$. Hence $\alpha_i + \bar{\mu}(1-\alpha_i) > \bar{\lambda}$ and F_{RE} satisfies strict monotonicity with respect to unilateral shifts.

(ii) Let $(S,c)\in\vec{\mathcal{Q}}$, $\alpha_i > 0$, $a=\alpha_i e$. It is easy to see that F_{EL} satisfies the following property discussed in Lahiri (1993c):

<u>c-Shift Invariance</u> :- $\forall (S,c)\in\vec{\mathcal{Q}}\ \forall b\in\mathbb{R}^n_+$ such that $b\leqq c-[\min_i(c_i)]e$,

$F(S(b),c-b)=F(S,c)-b$.

Hence F_{EL} satisfies monotonicity with respect to unilateral shifts.

Q.E.D.

5. Concavity With Respect to Shifts :- Let D be a domain such that if $(S,c)\in D$, $(S(a),c-a),(S(a'),c-a')\in D$ for some $a,a'\in\mathbb{R}^n_+$, then $(S(ta+(1-t)a'),c-ta-(1-t)a')\in D\ \forall\ t\in[0,1]$.

<u>Theorem 3</u>:- Let $F:D\to\mathbb{R}^n_+$ be an efficient choice function, which satisfies $F(S,c)\leqq c$. Then if F satisfies concavity with respect to shifts, then given $(S,c)\in D\ \forall\ a=tF(S,c)$, $t\in[0,1]$, $F(S(a),c-a)=F(S,c)-a$

<u>Proof</u> :- Let $a'=F(S,c)$ and $b=0$.

Then $(S(a'),c-a')=(\{0\},c-a')$, so that $F(S(a'),c-a')=0$.

$(S(b),c-b)=(S,c)$.

Thus

$F(S(ta'),c-ta')\geqq(1-t)F(S,c)\ \forall\ t\in[0,1]$ (which follows from concavity). The efficiency of $(1-t)F(S,c)$ in $S(ta',c-ta')$ implies $F(S(ta'),c-ta')=(1-t)F(S,c)\ \forall\ t\in[0,1]$

i.e. $F(S(a),c-a)=F(S,c)-a\ \forall\ a=tF(S,c)$, $t\in[0,1]$.

Q.E.D.

<u>Note</u> :- The assumption that $F:D\to\mathbb{R}^n_+$ satisfy $F(S,c)\leqq c$ implies a domain restriction. Thus for instance an $(S,c)\in\mathcal{Q}$ with $x\in S$ such that $x>c$ would automatically be excluded. However the domain is large enough to include bargaining problems as a strict subset.

Conclusion :- As discussed in the introduction this paper adapts and extends existing concepts in axiomatic bargaining to multiattribute choice problems. The proof of Theorem 1 is almost identical to the corresponding theorem in Thomson (1987). It has been provided primarily for completeness and for whatever ingenuity there exists in the first half of the proof.

References :-

1. P. Abad and S. Lahiri (1993) : "Group Decision Theory and Production Planning Problems", I.I.M.A. W.P. No. 1106.

2. W. Bossert (1992a) : "Monotonic Solutions for Bargaining Problems With Claims", Economics Letters 39, 359-399.

3. W. Bossert (1992b) : "An alternative solution to bargaining problems with claims", Mathematical Social Sciences (forthcoming).

4. Y. Chun (1988) : "The Equal-Loss Principle for Bargaining Problems", Economic Letters 26, 103-106.

H. Peters (1991) : "The lexicographic equal-loss solution", Mathematical Social Sciences 22, 151-161.

6. Y. Chun and W. Thomson (1990a) : "Bargaining with Uncertain Disagreement Points", Econometrica 58, 951-959.

7. Y. Chun and W. Thomson (1990b) : "Egalitarian Solutions and Uncertain Disagreement Points", Economics Letters 33, 39-33.

8. Y. Chun and W. Thomson (1990c) : "Nash Solution and Uncertain Disagreement Points", Games and Economic Behavior 2, 213-223.

and W. Thomson (1992) : "Bargaining Problems with Claims", Mathematical Social Sciences (forthcoming).

10. S. Lahiri (1993a) : "A Compromise Solution for Claims Problems", European Journal of Operations Research (forthcoming).

11. S. Lahiri (1993b) : "Monotonicity of Solutions With Respect to the Claims Point", I.I.M.A. (mimeo).

12. S. Lahiri (1993c) : "An Axiomatic Characterization of the Equal Loss Choice Function for Multiattribute Choice Problems", I.I.M.A. (mimeo).

13. W. Thomson (1987) : "Monotonicity of Bargaining Solutions With Respect to the Disagreement Point", Journal of Economic Theory 42, 50-58.