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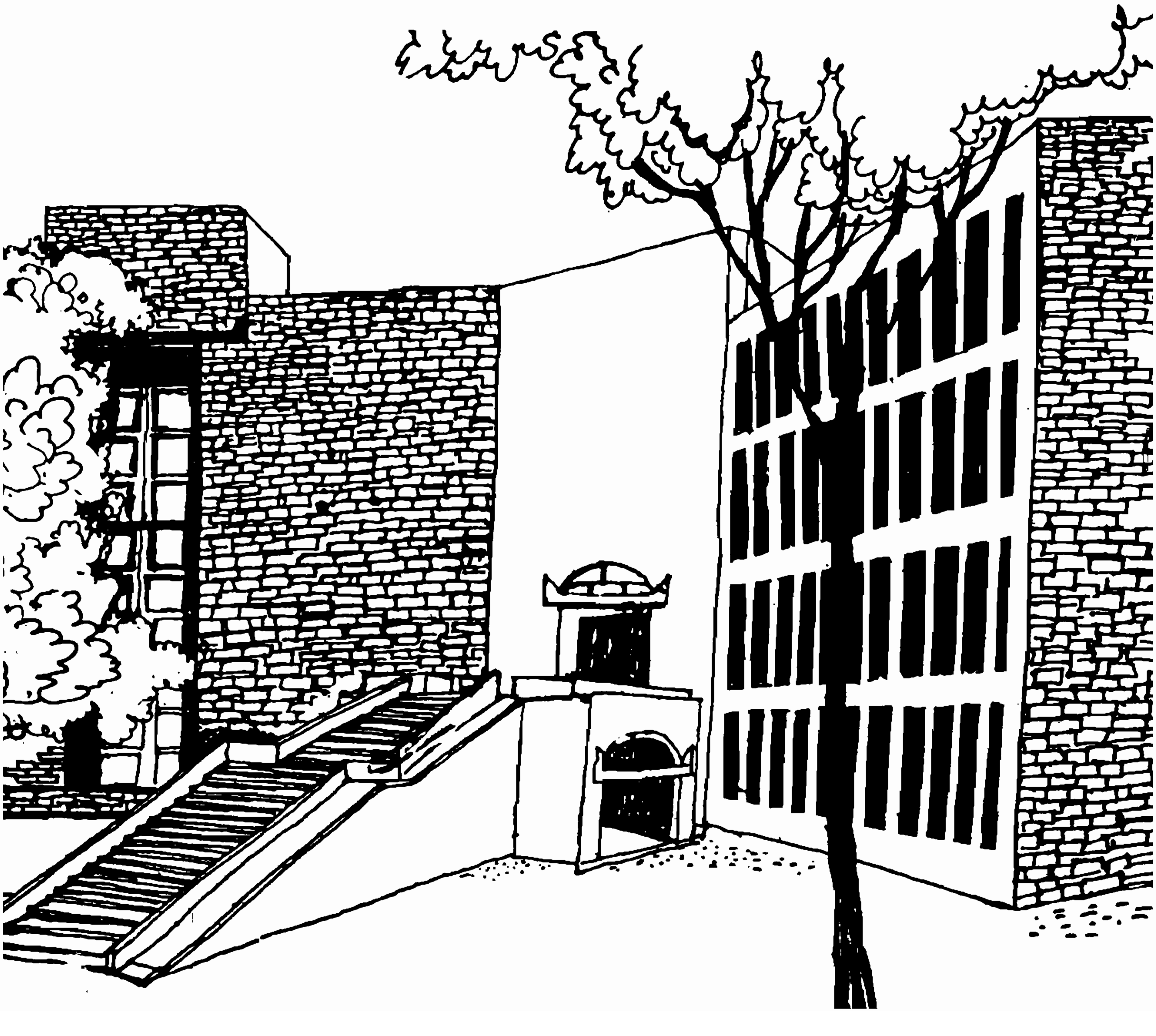

ORDINAL COMPARISONS IN CHOICE PROBLEMS: - A DIAGRAMMATIC EXPOSITION

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ABSTRACT

The basic problem in choice theory is to choose a point from a set of available points. A large literature has grown where the central issue is the choice of a vector from a compact, convex and comprehensive (terms to be defined later) subset of a finite dimensional Euclidean space.

A significant solution for such choice problems is the egalitarian solution which selects the highest possible vector with equal co-ordinates that is available under the given circumstances. There are several axiomatic characterizations of the egalitarian solution available in the literature. Of particular interest is an axiomatic characterization due to Nielsen (1983). There the egalitarian solution is axiomatically characterized using an assumption called 'Independence of Common Monotone Transformations' (ICMT).

Our objective in this paper is to provide a simple diagrammatic (yet completely rigorous) proof of the same result, when the feasible sets of attribute vectors are assumed (in addition to those essentially available in Nielsen (1983)) to be strictly convex.

Introduction: The basic problem in choice theory is to choose a point from a set of available points. A large literature has grown where the central issue is the choice of a vector from a compact, convex and comprehensive (terms to be defined later) subset of a finite dimensional Euclidean space. Excellent surveys can be found in Moulin (1988),Peters (1993) and Thomson (1994).

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2. The Model: As in Nielsen (1983) we consider choice problems embedded in two dimensional Euclidean Space. A choice problem (for our purposes) is thus a non-empty subset S of $\mathbb{R}^2_+$ (the non-negative quadrant of two dimensional Euclidean space) satisfying the following properties:

i) There exists $x \in S$ such that if $x = (x_1, x_2)$, $x_1 > 0$ and $x_2 > 0$.

ii) S is a compact (i.e. closed and bounded) set;

iii) S is comprehensive i.e. $x \in S$, $0 \leq y \leq x \Rightarrow y \in S$

iv) S is strictly convex i.e. $x, y \in S$, $t \in (0,1) \Rightarrow tx + (1-t)y \in$ interior of S.

In the above for $x, y \in \mathbb{R}^2_{++}$, $x \geq y \Leftrightarrow x_i \geq y_i$, $i = 1, 2$.

Let B be the set of all choice problems defined above. Given $S \in B$, let P (S) denote the set of Pareto optimal points in S i.e. $P(S) = \{x \in S / y \in S, y \geq x \Rightarrow y = x\}$.

A choice function is a function $F : B \rightarrow \mathbb{R}^2_+$ such that $F(S) \in S$ for all $S \in B$.

The particular choice function E, which will be characterized here is defined as follows:

$$E(S) = \underset{x \in S}{\operatorname{argmax}} \quad [\min\{x_1, x_2\}] \quad \text{forall } S \in B$$

E is called the egalitarian choice function.

3. The Axioms: Let $F : B \rightarrow \mathbb{R}^2_+$ be a choice function. F is said to satisfy.

Axiom 1: Strict Individual Rationality (SIR), if for all $S \in B$, $F_i(S) > 0$, $i = 1, 2$;

Axiom 2: Pareto Optimality (PO), if for all $S \in B$, $F(S) \in P(S)$;

Axiom 3: Nash's Independence of Irrevalent Alternatives (NIIA), if for all S, T ε B, S $\subseteq$ T, F (T) ε S $\Rightarrow$ F (S) = F (T).

Axiom 4: Independence of Common Monotone Transformations (ICMT) if for all S ε B, for all g: $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ which is strictly increasing and continuous if g (S) = { (g (x_1), g (x_2)) / (x_1, x_2) ε S} belongs to B, then F_i (g(S))=g_i (F_i (S)), i = 1, 2.

In the sequel we shall prove that the only choice function to satisfy Axioms 1 to 4 on B is E.

4. The Main Theorem:

Theorem: The only choice function which satisfies Axioms 1 to 4 on B, is E.

Proof: That E satisfies the four axioms is obvious. So, let us assume F is a choice function which satisfies the four axioms and let S ε B. Let E(S) = (c, c) ε P(S).

Case 1: S is symmetric i.e. (x_2, x_1) ε S for all x = (x_1, x_2) belonging to S.

We will now define a strictly increasing continuous function g: $\mathbb{R}_+ \rightarrow \mathbb{R}_+$ as follows: Let u_i (S) = max { x_i/x ε S }, i = 1,2. Clearly u_1 (S) = u_2 (S), since S is symmetric. We will define g first on [0,u_1 (S)] and then consider any strictly increasing and continuous extension as the required transformation. This is done as shown in figure 1.

[Insert Figure 1 here]

In the above figure connect L to E(S) and M to E (S) by two straight lines. The function g is defined on [o,c] as follows: take any arbitrary point H in P(S) which lies above or on the 45° line. We proceed vertically down from H onto LE(S) to intersect at I and then horizontally from I to intersect P(S). The point of incidence on LE(S) is I and the intersection of the horizontal line from I to P(S) is at J. This procedure maps the horizontal coordinate of H to the horizontal co-ordinate of J. This procedure defines the function g on [o,c]. To define g on [c, u_i (S)] one treats U, V, and W as one treated H, I and J respectively and repeats the same procedure. Once again this procedure maps the horizontal co-ordinate of U to the horizontal co-ordinate of W. Now extend g as defined above to a strictly increasing and continuous function on $\mathbb{R}_+$.

Observation 1: Since S is symmetric, g(S) = S. In fact if H = (x_1, x_2) and J = (y_1, y_2) in figure 1 above, then $y_i = g(x_i)$, i = 1, 2. This follows from the symmetry of S.

Observation 2: g (o) = o, g (u_i (S)) = u_i (S), g (c) = c. This follows from the method of construction of g. Hence the only strictly individually rational fixed point under the above construction is E(S). Thus by ICMT and SIR F(S) = F(g (S)) = (c, c).

Case 2: S is not necessarily symmetric.

In this case let T be the smallest symmetric set containing S. It is easy to verify that $T \in B$ and $(c, c) \in P(T)$. Thus by Case 1, $F(T) = (c, c)$.

By NIIA, Since $S \subseteq T$ and $(c, c) \in S$ we have $F(S) = (c, c) = E(S)$. Q.E.D

Note 1: In establishing that a choice function which satisfies Axioms 1 to 4 must be E, we did not appeal to Axiom 2 (i.e. PO) at all. Thus we have essentially proved a stronger result.

Theorem 2: The only choice function on B to satisfy Axioms 1, 3 and 4 is E.

Note 2: Our proof differs from that of Nielsen [1993] in a major respect. In case 1, which is the essential step in our proof we do not appeal to NIIA at all, whereas in the original proof NIIA is used in Case 3 (which is our Case 1). Case 1 and 2 in Nielsen's proof do not arise in our context, owing to our choice of domain.

Note 3: Conventional choice theory is defined on a domain which is slightly larger than what we have chosen. Let $\overline{B} = \{ S \subseteq \mathbb{R}^2_+ \,/\, S$ is non-empty, convex and satisfies (i), (ii) and (iii)$\}$.

The question that naturally arises is whether after making appropriate changes in the definition of a choice function, similar results continue to hold on $\overline{B}$. For that purpose we need the following property.

Axiom 5 (Hausdorff Continuity): Let $F : \overline{B} \rightarrow \mathbb{R}^2_+$ be a choice function. F is said to be continuous if given a sequence $\{S^k\}_{k \in N} \subseteq \overline{B}$,
$S^k \rightarrow S \in \overline{B}$ (in the Hausdorff topology) implies $F(S^k) \rightarrow F(S)$. We now have the following theorem.

Theorem 3: The only choice function on $\overline{B}$ to satisfy Axioms 1,3,4 and 5 is E.

Proof: The proof appeals to the fact that any $S \in \overline{B}$, can be approximated arbitrarily closely by a choice problem in B.
Q.E.D
However, E no longer satisfies Axiom 2. It satisfies instead,

Axiom 2' (Weak Pareto Optimality): Given $S \in B$, there does not exist $x \in S$ with $x_i > F_i(S)$, $i = 1, 2$.
It is instructive to note how seemingly complicated proofs can be simplified by appealing to Axiom 5, whenever it is valid.

References:

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Appendix

In this appendix we provide a mathematical expression for the function g constructed in the proof of Theorem 1.

Let $S \varepsilon B$, then there exists a function $\phi : [o, u_1(S)] \rightarrow [o, u_2(S)]$ such that ϕ is continuous, strictly decreasing and strictly concave and $P(S) = \{ (x_1, \phi(x_1)) : x_1 \varepsilon [o, u_1(S)] \}$.
If S is symmetric, $u_1(S) = u_2(S) > 0$.
Define $g: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ as follows:

$$g(x_1) = \phi^{-1}\left[c + \frac{(c-x_1)}{c}(u_2(S) - c)\right], 0 \leq x_1 \leq c$$

$$= \phi^{-1}\left[c + \frac{c\,(c - x_1)}{u_1(S) - c}\right], \quad c \leq x_1 \leq u_1(S)$$

$$= x_1, \quad u_1(S) \leq x_1$$

The above function defines the process described in the proof of theorem 1.

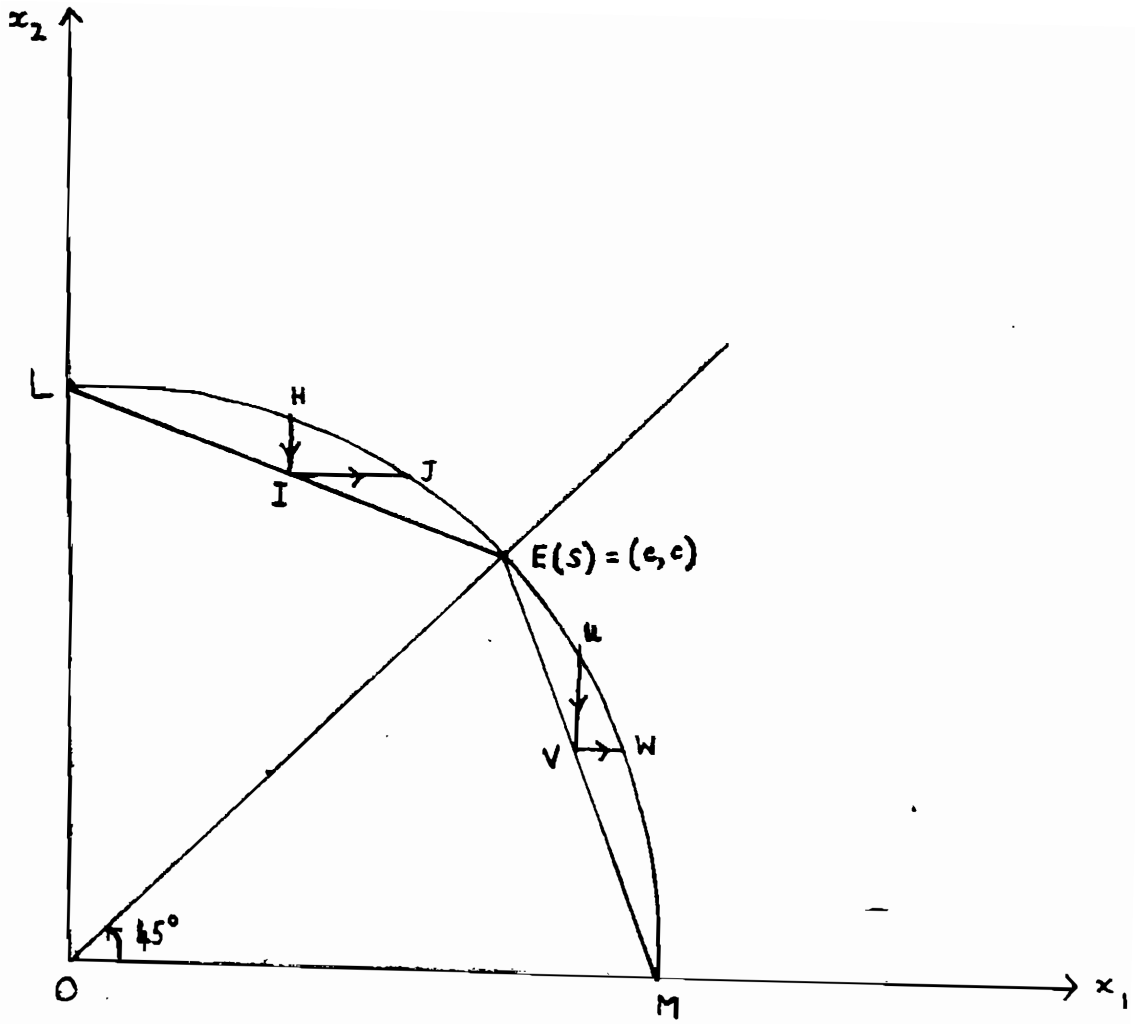

Figure - 1