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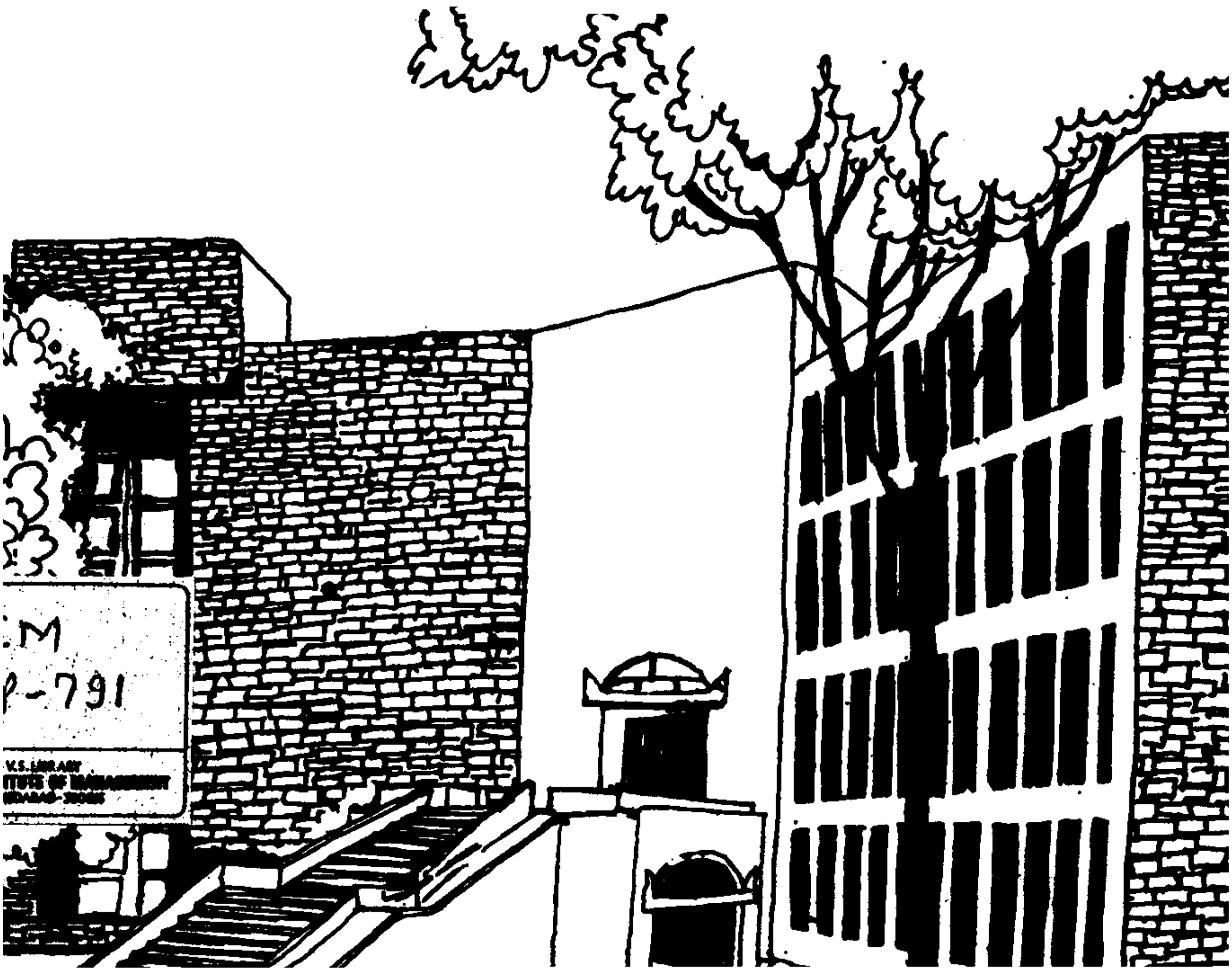

REDUNDANCY OF ADDITIONAL ALTERNATIVES AND SOLUTIONS TO BARGAINING PROBLEMS

By

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ABSTRACT

In this paper we propose a new axiom called the redundancy of additional alternatives axiom, which is satisfied by the proportional solution. A weaker version of the same axiom is satisfied by the Kalai-Smorodinsky [3] solution. The Nash solution satisfies neither. This new axiom seems to be a reasonable compromise between accepting the independence of irrelevant alternatives axiom and rejecting it outright, from the realm of axiomatic models of bargaining.

A two player <u>bargaining problem</u> is a pair (S,d) where S is a compact and convex subset of IR^2 representing the utility vectors, measured in von Neumann-Morgenstern scales, attainable by the two players through some joint action, and d is a point of S, strictly dominated by some other point of S; d is designated as the "status quo" or the "disagreement point" and is interpreted as the outcome that would result if the players failed to reach a compromise. Formally a two player bargaining problem is a pair (S,d) where

(1) The <u>space S of feasible utility payoffs</u> is a compact and convex subset of IR^2.

(2) The <u>disagreement outcome</u> d is an element of S.

(3) There is an $\hat{x} \in S$ with $\hat{x}_i > d_i$ for each $i \in \{1,2\}$.

The third condition merely states that there is a possibility both players to improve their position by bargaining.

Furthermore, for mathematical convenience, we will also assume that

4) $x \geq d$ for all $x \in S$

(5) For all $y \in IR^2$ with $d \leq y \leq x$ for some $x \in S$, we have $y \in S$.

$\sum$ is the class of all such pairs.

For a bargaining problem $(S,d) \in \sum$, the Pareto set $P(S)$ is defined by

$$P(S) := \{x \in S;\ \forall y \in S\ [y \geq x \Rightarrow y = x]\}$$

and the utopia point $h(S) = (h_1(S),\ h_2(S)) \in IR^2$ by

$h_i(S) := \max\left\{x_i;\ (x_{-i}|x_i) \in S\right.$ for all $i \in \{1,2\}$

Here $x_{-i} = x_1$ if $i = 2$

$= x_2$ if $i = 1$

and $(x_{-i}/x_i) = (x_i,\ x_{-i})$ if $i = 1$

$= (x_{-i},\ x_i)$ if $i = 2$

We call a map $\emptyset: \sum \longrightarrow IR^2$ a 2-person bargaining solution. If, additionally, the following two properties hold, we call $\emptyset$ a classical bargaining solution:

(PO): For each $(S,d) \in \sum$, we have $\emptyset\ (S,d) \in P(s)$ (Pareto-optimality)

(IEUR): For each $(S,d) \in \sum$ and each transformation $A: IR^2 \longrightarrow IR^2$ of the form

$$A(x_1,\ x_2) = (a_1x_1 + b_1,\ a_2x_2 + b_2)$$

where b_1, b_2 are real numbers and a_1, a_2 are positive real numbers, we have $\emptyset(A(S),\ A(d) = A(\emptyset(S,d))$ (independence of equivalent utility representations).

Sometimes instead of (PO) the following alternative property is used:

(WPO): For each $(S,d) \in \sum$, we have $\emptyset(s,d) \in W(S)$ where

$$W(S) = \left\{x \in S;\ \forall\ y \in IR^2 \left[y \gg x \Rightarrow y \notin S\right]\right\}.$$

(weak Pareto optimality).

Since here we will consider only classical solutions, we may without loss of generality restrict our attention here to 2-person bargaining games with disagreement outcome 0. From now on, we assume that every $(S,d) \in \sum$, besides (1) through (5), satisfies

(6) $d = 0$,

and we will write S instead of (S,d).

The purpose of this **paper is to provide a reformulation of some solutions in** terms of a redundancy of additional alternatives axiom. This axiom is similar to the independence of irrelevant alternatives axiom. The Nash [4], the Kalai [2] solutions are known to satisfy the independence of irrelevant alternatives axiom. Much in the literature on game theory exists, which provide a strong critique of this axiom. However, as convincingly pointed out in Aumann [1], other than in matters of normative desirability, departures in social decision making from this criterion, is viewed more as an abberation than as a rule. So to begin with we try to formulate a criterion similar to IIA which the Kalai-Smorodinsky Solution satisfies. We call this condition redundancy of additional alternatives.

Let $T: IR^2 \longrightarrow IR^2$ be the symmetry operator i.e. given $z = (x,y) \in IR^2$, $T(z) = (y,x)$.

Given, $S \in \sum$, $T(S) \equiv \{z \in IR^2 / \exists\, z' \in S \text{ with } z = T(z')\}$.

We now state some conditions which we would want our bargaining solution to satisfy:

(IR) For each $S \in \sum$, $\phi(s) \geq 0$ (individual rationality)

(SY) For $S \in \sum$, if $T(S) = S$, then $\phi_1(s) = \phi_2(s)$ (symmetry)

Let $g : \sum \rightarrow IR^2$ be a reference function (see Thomson (5)) defined as

$$g_i(S) = \frac{h_i(S)}{h_1(S)+h_2(S)}, \quad g(S) = (g_1(S), g_2(S))$$

(RAg) For all $S, S' \in \sum$, if $S \subset S'$, $g(S) = g(S')$, and $\phi(S) \in P(S')$, then $\phi(S) = \phi(S')$. (redundancy of additional alternatives other than $g(S)$).

We shall now define a solution $K : \sum \longrightarrow IR^2$(originally due to Kalai-Smorodinsky (3) as follows:

$$K(S) = h\ (S)\left\{\max\ t/t.h(S) \in S\right\}.$$

We are now ready to prove:

Proposition 1: (a) $K : \sum \longrightarrow IR^2$ is a solution satisfying (WPO), (IEUR) (IR), (SY) and (RAg).

(b) If $\phi : \sum \rightarrow IR^2$ is a solution satisfying (WPO), (IEUR), (IR), (SY) and (RAg) then $\phi = K$.

Proof : (a) That K satisfies (WPO), (IEUR), (IR) and (SY), has been shown in Kalai-Smorodinsky (3). Let us show that K satisifes (RAg).

Let $S, S' \in \sum$ with $S \subset S'$ and $g(S) = g(S')$.

$$K(S) = h(S).\ \max\left\{t/t.h(S) \in S\right\} = g(S).\ \max\left\{t/t.g(S) \in S\right\}$$

$$K(S') = h(S').\ \max\left\{t/t.h(S') \in S'\right\} = g(S').\ \max\left\{t/t.g(S') \in S'\right\}.$$

$$= g(S).\ \max\left\{t/t.g(S) \in S'\right\}.$$

Clearly $S \subset S' \Rightarrow K(S) \leq K(S')$.

However, $K(S') \in S$ and the definitions of $K(S)$ and $K(S')$ imply,

$K(S) \geq K(S')$.

Hence $K(S) = K(S')$.

b) Let $\phi : \sum \rightarrow IR^2$ be a solution satisfying (WPO), (IEUR), (IR), (SY) and (RAg).

Let $S(\underline{a})$ be the comprehensive convex hull of the points $(a,a) \in IR^2_{++}$ for some $a \in IR_+ \setminus \{0\}$ i.e. $S(\underline{a}) = \left\{(x_1, x_2) \in IR^2_+ / \ 0 \leq x_i \leq a,\ i = 1,2\right\}$.

By (WPO) and (SY), $\phi(S(\underline{a})) = (a,a)$. Observe that $g(S(\underline{a})) = (\frac{1}{2}, \frac{1}{2})$.

Let $A : IR^2 \rightarrow IR^2$ be any positive transformation, of the form

$A(x_1, x_2) = (b_1 x_1,\ b_2 x_2)$ for some $b_1 > 0$, $b_2 > 0$.

$h(A(S(\underline{a}))) = (b_1 a, b_2 a)$ and $g(A(S(\underline{a}))) = \left(\dfrac{b_1}{b_1 + b_2}, \dfrac{b_2}{b_1 + b_2}\right)$

Since ϕ satisfies IEUR

$\phi(A(S(\underline{a}))) = A(\phi(S(\underline{a}))) = (b_1 a,\ b_2 a)$.

Hence $\phi(A(S(\underline{a}))) = h(A(S(\underline{a}))) = h(A(S(\underline{a})))\ \max\left\{t/t.h(A(S(\underline{a}))) \in S(\underline{a})\right\}$

$$= K(A(S(\underline{a}))).$$

Now, let $S \in \sum$ and let $(x_1, x_2) = K(S)$. We want to show that $K(S) = \phi(S)$.

Consider the problem $S(K(S)) \in \sum$. Since S is comprehensive, $S(K(S)) \subseteq S$. Further, $h(S(K(S))) = (K_1(S), K_2(S))$ implies,

$$g(S(K(S))) = \left(\frac{K_1(S)}{K_1(S)+K_2(S)}, \frac{K_2(S)}{K_1(S)+K_2(S)}\right) = g(S).$$

Also, $\phi(S(K(S))) = K(S)$.

Since $K(S) \in P(S)$, by (RAg) we get $\phi(S) = K(S)$

Q.E.D.

The following corollary shows that we can substitute (PO) for (WPO), (SY) and (IEUR) in the statement of the above proposition.

Corollary :- (a) $K: \sum \to IR^2$ is a solution satisfying (PO)

(b) If $\phi: \sum \to IR^2$ is a solution satisfying (PO) and (RAg), then $\phi = K$.

Proof : (b) Let $a \in IR_+ \sim \{0\}$. Then $\phi(A(S(\underline{a}))) = K(A(S(\underline{a})))$ by (P.O). The rest of the proof follows as in Proportion 1.

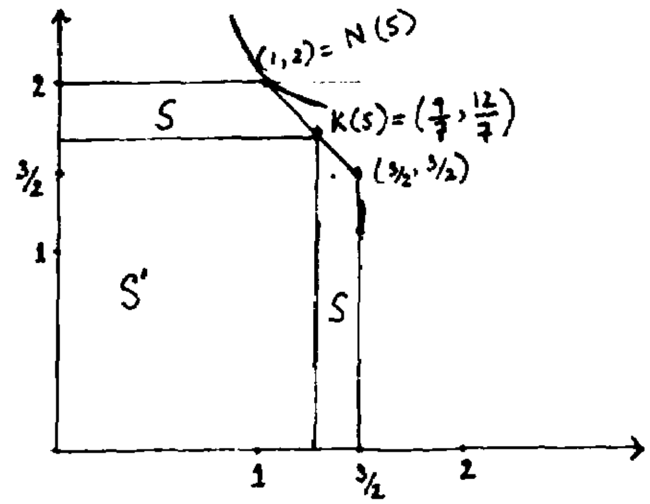
(a) That K satisfies (PO) has been established in Kalai-Smorodinsky (1975).

Q.E.D.

Remark: (i) The Nash Solution $N: \sum \to IR^2$ defined by

$$N(S) = \arg\max_{(x_1,x_2) \in S} x_1 x_2$$

does not satisfy RAg as the following example shows.



Let $S = cch\left\{(1,2),\ (3/2,\ 3/2)\right\}$ where cch $\equiv$ comprehensive convex hull. $N(S) = (1,2)$; $K(S) = (\frac{9}{7}, \frac{12}{7})$. Let $S' = cch\left\{(\frac{9}{7}, \frac{12}{7})\right\}$.

$K(S') = N(S') = (\frac{9}{7}, \frac{12}{7}) \in P(S)$. But $N(S) \neq N(S')$.

(ii) In the statement of (RAg), $P(S')$ could be replaced by $W(S')$ without either affecting the statements or proofs of Proposition 1 and Corollary 1.

We now introduce a stronger axiom:

(RAA) : For all $S, S' \in \sum$, if $S \subset S'$ and $\phi(S) \in W(S')$, then $\phi(S) = \phi(S')$ (redundancy of additional alternatives).

We call this axiom redundancy of additional alternatives as the reference function plays no role here. (RAA) is stronger than (RAg).

We now define a solution $E:\Sigma \rightarrow IR^2$ (originally due to Kalai [2]) which goes as follows:

$E(S) = (1,1) \max\{t/(t,t) \in S\}$.

This solution is known in the literature as the egalitarian solution.

A condition which will be found necessary in the sequel is the following:

(HOM): For each $S \in \Sigma$ and for each $A: IR^2 \rightarrow IR^2$ satisfying

$$A(x_1, x_2) = (ax_1, ax_2),\ a > 0,$$

$$\phi(A(S)) = A(\phi(S)).$$

(HOM) is weaker than (IEUR).

Proposition 2:- (a) The solution $E:\Sigma \rightarrow IR^2$ satisfies (WPO), (SY), (HOM), (IR) and (RAA).

(b) If $\phi:\Sigma \rightarrow IR^2$ satisfies (WPO), (SY), (IR) and (RAA), then $\phi = E$.

<u>Proof</u>: (a) That E satisfies (WPO), (SY), (HOM) and (IR) has been established by Kalai [2] . That E satisfies (RAA) is obvious.

(b) Let $\phi:\Sigma \rightarrow IR^2$ satisfy (WPO), (SY), (IR), and (RAA).

Let $a > 0$ be given and let $S(a) = \text{cch}\{(a,a)\}$

By (WPO) and (SY), $\phi(S(a)) = E(S(a)) = (a,a)$

Let $S \in \Sigma$ and let $(a,a) \in W(S)$ for some $a > 0$. Such an (a,a) exists and is unique since S is compact, convex, comprehensive and contains a point $x \in IR^2_+$ such that $x_1 > 0$, $x_2 > 0$.

$\therefore$ $S(\underline{a}) \subset S$ since S is comprehensive and $\phi(S(a)) = (a,a)$ from above. Further $(a,a) \in W(S)$ by construction. Hence by RAA, $\phi(S) = (a,a) = E(S)$.

Q.E.D.

A class of solutions studied in Kalai [2] is obtained as follows: Let $\Delta = \left\{ (p_1, p_2) \in IR^2_+ / p_1 + p_2 = 1 \right\}$. Then for $p = (p_1, p_2) \in \Delta$, we define $E^p : \sum \rightarrow IR^2$ thus:

$$E^p(S) = p. \max \left\{ t / (t\, p_1,\ tp_2) \in S \right\}.$$

Clearly $E = E^{(\frac{1}{2}, \frac{1}{2})}$. The first thing we need to notice is that E^p does not satisfy (SY) if $p \neq (\frac{1}{2}, \frac{1}{2})$. However, the following is true:

Proposition 3 : (a) $\forall\ p \in \Delta$, $E^p : \sum \rightarrow IR^2$ satisfies (IR), (WPO), (HOM) and (RAA).
(b) If $\phi : \sum \rightarrow IR^2$ satisfies (IR), (WPO), (HOM) and (RAA) then there exists $p \in \Delta$ such that $\phi = E^p$.

Proof : (a) That E^p satisfies (IR), (WPO), (HOM) and (RAA) is an easy exercise.
(b) Suppose $\phi : \sum \rightarrow IR^2$ satisifes (IR), (WPO), (HOM), and (RAA). Consider the game $S(\underline{1}) \equiv cch \left\{ (1,1) \right\}$. Clearly by (IR) and (WPO) there exists a $p \in \Delta$ and a unique $t > 0$ such that $\phi(S(\underline{1})) = tp$. Fix 'p'. Let $a > 0$ and consider $S(\underline{a}) \in \sum$.
By (HOM), $\phi(S(\underline{a})) = a. \phi(S(\underline{1})) = a. E^p(S(\underline{1})) = E^p(S(\underline{a}))$.
Now let $S \in \sum$. By the convexity, compactness, comprehensiveness and the fact that there exists $(x_1, x_2) \in IR^2$ such that $x_i > 0$, $i = 1,2$,

there exists an $a > 0$, such that $(a,a) \in W(S)$. Hence $S(\underline{a}) \subset S$ by the comprehensiveness of S and $\phi(S(\underline{a})) = (a,a) \in W(S)$. By (RAA), $\phi(S) = \phi(S(\underline{a}))$.

But $E^P(S(\underline{a})) = E^P(S)$ (by definition of E^P).

Hence $\phi(S) = E^P(S)$.

Q.E.D.

Earlier it was noted that in the game theory literature IIA has often come under fire in spite of its intuitive appeal. What IIA says is that a contraction of the feasible set would not affect the solution to a bargaining problem if the solution to the original problem was a member of the revised problem. The Kalai-Smorodinsky [3] solution does not satisfy this property. However, what it does satisfy is a modified redundancy of additional alternatives condition. What RAA says is that if we add alternatives to an existing game, without affecting the Pareto-optimality (or weak Pareto optimality)property of the existing solution, then the solution to the expanded game remains the same as before. Were we to view the IIA property as undesirable, the RAA axiom could be viewed as a substitute in defining the proportional solution, as has been shown above. Hence RAA seems to be a reasonable compromise between accepting the IIA axiom and rejecting it outright.

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