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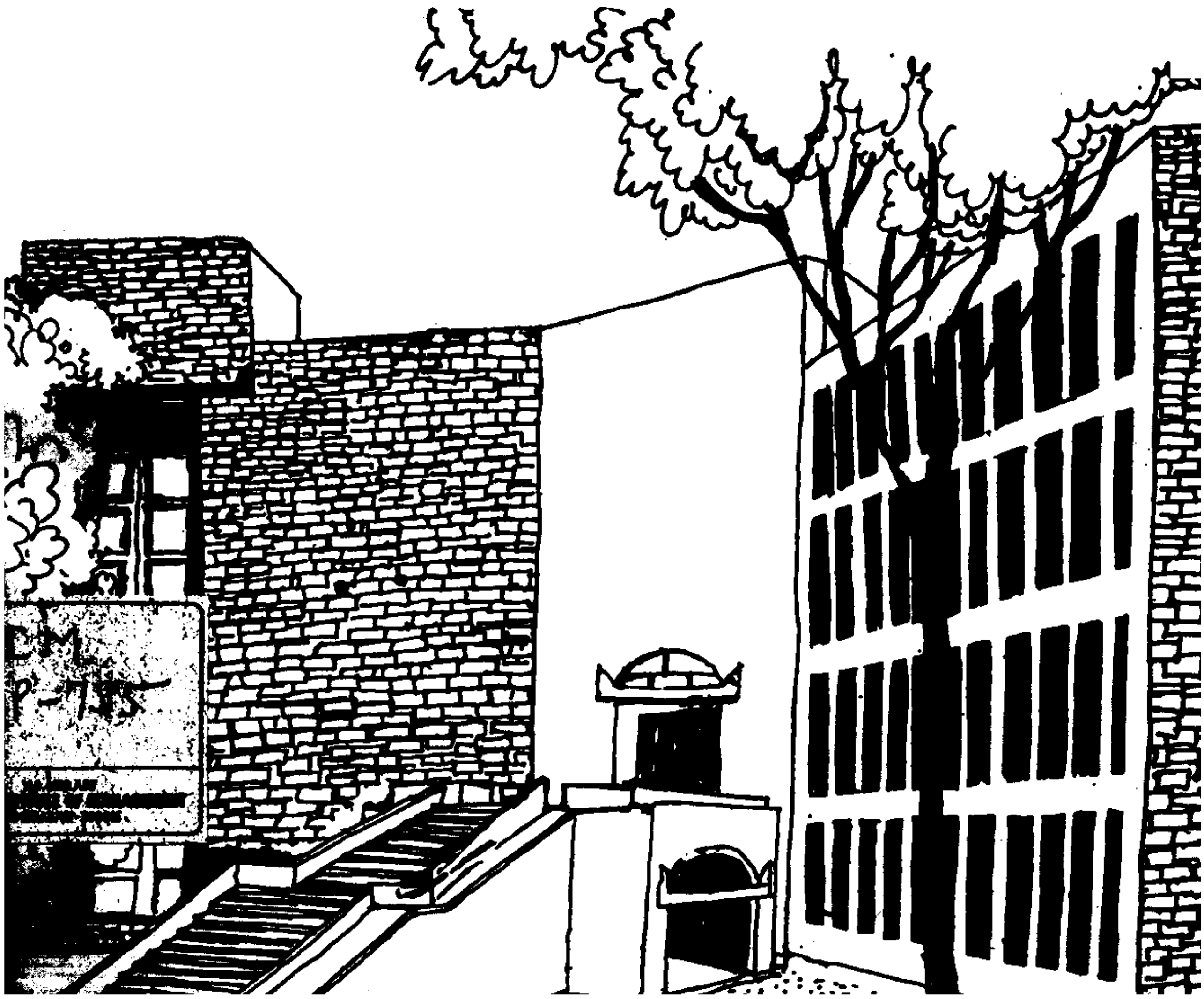

METRIC RATIONALIZATION OF BARGAINING SOLUTIONS

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ABSTRACT

In this paper we represent bargaining solutions by means of a metric which is defined on games, whereby the solutions are precisely those payoffs which are closest to being unanimously highest.

1. Introduction: In this paper, we consider n-person bargaining games ($n \geq 2$), that is, pairs (S,d) where

(1) The space S of feasible utility payoffs is a compact and convex subset of IR^n.

(2) The disagreement outcome d is an element of S.

Furthermore, for mathematical convenience, we will also assume that

(3) $x \geq d$ for all $x \in S$

(4) There is an $\hat{x} \in S$ with $\hat{x}_i > d_i$ for each $i \in N = \{1,2,\ldots,n\}$

(5) For all $y \in IR^n$ with $d \leq y \leq x$ for some $x \in S$, we have $y \in S$.

Such games (S,d) correspond to situations involving n bargainers (players) 1,2,....,n, who may cooperate and agree upon choosing a point $s \in S$, which has utility s_i for player i, or who may not cooperate. In the latter case, the outcome is the point d, which has utility d_i for player $i \in N$. The family of all such bargaining games, satisfying (1) through (5), is denoted Σ. Following Kaneko [4] we call a multifunction $\emptyset: \Sigma \longrightarrow IR^n$ which assigns to each game $(S,d) \in \Sigma$ a nonempty subset $\emptyset(S,d)$ of S an n-person bargaining solution.

We restrict ourselves to solutions $\emptyset$ which satisfy the following properties:

(IEUR): For each $(S,d) \in \sum$ and each transformation $A: IR^n \to IR^n$ of the form $A(x_1,\ldots,x_n) = (a_1x_1+b_1,\ldots,a_nx_n+b_n)$ for all $x \in IR^n$, where $b_1,b_2,\ldots,b_n$ are real members and $a_1,\ldots,a_n$ are positive real numbers, we have

$\phi(A(S), A(d)) = A(\phi(S,d))$ (independence of equivalent utility representations). Here for $T \subset IR^n$, $A(T) = \{A(x)/x \in IR^n\}$.

(SIR): For each $(S,d) \in \sum$, for each $x \in \phi(S,d)$, $x_i > d_i \; \forall \; i \in N$ (strict individual rationality).

Since we only consider solutions which obey IEUR, we may without loss of generality restrict our attention here to n-person bargaining games with disagreement outcome 0. From now on, we assume that every $(S,d) \in \sum$, besides (1) through (5), satisfies

(6) $d = 0$,

and we will write S instead of (S,d).

Our purpose in this paper is to represent bargaining solutions as defined above by means of a metric which is defined on bargaining problems, whereby the solutions are precisely those payoffs which are closest to being unanimously highest. In general, the purpose of a metric is to define distance and the metric generated by a bargaining solution defines the notion of a solution being close to awarding the highest payoff to all the players.

If some payoff vector awards the highest payoff to all the players then surely it should be declared the consensus solution. This is the unanimity principle which is naturally very appealing and which is satisfied by the Nash [4] solution as well as by the Kalai-Smorodinsky [3] solution to bargaining problems. Of course, for 'most' bargaining problems, a unanimously preferred payoff vector generally does not exist, in as much that 'most' bargaining problems are not representable as the comprehensive, convex hull of a single payoff vector. A significant problem then is to find out in what precise sense different bargaining solutions attempt (if at all) to approximate or respect the ideal of using the unanimity rule.

2. Metrizable Bargaining Solutions: Let $x = (x_1, \ldots\ldots, x_n) \in IR^n_{++}$; let us agree to denote $\{y \in IR^n_+ / y_i \leq x_i, \forall\ i \in N\}$ by $S(x)$. Such games are called unanimity games.

Definition 1 :- A bargaining solution ϕ is Paretian if $\forall\ x \in IR^n_{++}$, $\phi(S(x)) = \{x\}$.

Let $\bar{\delta}$ be a metric on $\sum$ ($\sum$ is metrizable as for instance by the Hausdorff metric (see Goffman and Pedrick [1]). Let $S_+ = S \setminus \{0\}$ and $S_{++} = S \cap IR^n_{++}$.

Definition 2 : The metric $\bar{\delta}$ on $\sum$ is a rationalization according to unanimity (henceforth a rationalization) for the bargaining solution ϕ, if $\forall\ S \in \sum$, $\phi(S) = \{x \in S_{++} / \bar{\delta}(S, S(x)) \leq \bar{\delta}(S, S(y))\ \forall\ y \in S_{++}\}$.

That is the metric $\bar{\delta}$ rationalizes ϕ according to the unanimity criterion whenever for any bargaining game S, the solution is the payoff whose convex comprehensive hull is the unanimity game "nearest" to the game and the payoff belongs to S. The characterization of the family of bargaining solutions having such a metric rationalization is provided by the following:

<u>Theorem 1</u>: A bargaining solution ϕ has a metric realization if and only if it is Paretian.

<u>Proof</u> : (i) It is easy to verify that if $\bar{\delta}$ rationalizes ϕ, then ϕ is Paretian.

(ii) Suppose that ϕ is Paretian. Define $\bar{\delta}$ as follows: $\forall$ $S,T \in \Sigma$,

$$\bar{\delta}(S,T) = \begin{cases} 0 \text{ if } S = T \\ 1 \text{ if } S \neq T,\ \phi(S) \cap \phi(T) \neq \emptyset \\ 2 \text{ if } S \neq T,\ \phi(S) \cap \phi(T) = \emptyset \end{cases}$$

$\bar{\delta}$ is obviously symmetric, non-negative, and $\bar{\delta}(S,T) = 0 \Leftrightarrow T = S$. It also satisfies the triangle inequality and so $\bar{\delta}$ is a metric. Let us conclude the proof by showing that the metric $\bar{\delta}$ is a rationalization for ϕ. Define $\bar{\phi}: \Sigma \rightarrow IR^n_+$ as follows: $\forall$ $S \in \Sigma$, $\bar{\phi}(S) = \{x \in S_{++} /$ $\bar{\delta}(S,S(x)) \leq \bar{\delta}(S,S(y))$ $\forall$ $y \in S_{++}$. We now show that $\phi = \bar{\phi}$. If $S = S(x)$ for some $x \in IR^n_{++}$, then $\bar{\delta}(S,S(x)) = 0$. Hence $x \in \bar{\phi}(S)$. Also $\phi(S(x)) = \{x\}$ since ϕ is Paretian. So $\phi(S(x)) \subseteq \bar{\phi}(S(x))$ $\forall$ $x \in IR^n_{++}$. If $y \neq x$, then $y \notin \bar{\phi}(S(x))$ since $S(x) \neq S(y)$ implies $\bar{\delta}(S(x),S(y)) > 0 = \bar{\delta}(S(x),S(x))$. $\bar{\phi}(S(x)) = \{x\} = \phi(S(x))$. Suppose $S \neq S(x)$, $\forall$ $x \in IR^n_{++}$. Then $\forall$ $y \in S_{++}$, $\bar{\delta}(S,S(y)) \neq 0$. If $x \in \bar{\phi}(S)$, then $\bar{\delta}(S,S(x)) = 1$. Hence $\phi(S) \cap \phi(S(x)) \neq \emptyset$. But $\phi(S(x)) = \{x\}$ since ϕ is Paretian. So

$x \in \phi(S)$ i.e. $\bar{\phi}(S) \subseteq \phi(S)$. Now suppose $x \in \phi(S)$. Hence $x \in \phi(S) \cap \phi(S(x))$. Hence $\bar{\delta}(S, S(x)) = 1$. Hence $x \in \bar{\phi}(S)$ i.e. $\phi(S) \subseteq O(S)$.

Q.E.D.

Before we conclude this section, let us note that a reasonable assumption for most bargaining solutions is that it satisfies weak Pareto optimality.

(WPO) For each $S \in \sum$, for each $x \in \phi(S)$, $y \in IR^n_+$, $y_i > x_i$ for all $i \in N$ implies $y \notin S$.

Let $W(S) = \{x \in S / y \in IR^n_+,\ y_i > x_i\ \forall\ i \in N \Rightarrow y \notin S\}$.

A solution $\phi : \sum \rightarrow IR^n_+$ satisfying (WPO), (IEUR) and (SIR) is called a <u>classical bargaining solution</u>.

For classical bargaining solutions we have the following metric characterizations:

Theorem 2 :- A classical bargaining solution ϕ has a metric rationalization if and only if it is Paretian.

<u>Proof</u> : The proof is the same as that of Theorem 1, except that

$\bar{\phi}(S) = \{x \in W(S) \cap IR^n_{++} / \bar{\delta}(S, S(x)) \leq \bar{\delta}(S, S(y))\ \forall\ y \in W(S) \cap IR^n_{++}\}$ is the modified definition of O .

Q.E.D.

The Nash [4] and Kalai-Smorodinsky [3] solutions are Paretian and hence are metric rationalizable. Note, however, that the Kalai [2] solution is not Paretian and hence not metric rationalizable.

• <u>Monotonic Metric Rationalizations</u>: The metric used in the proofs of Theorem 1, $\bar{\delta}(S,T)$, is induced by the specific bargaining solution under consideration. That is, loosely speaking, the metric is based on what the bargaining games do, rather than what the games actually are. The lack of dependence on the internal structure of the bargaining games suggests that some bargaining solutions may be rationalized by metrics which do not satisfy some intuitively desirable criteria.

<u>Definition 3</u>: A metric δ on $\sum$ is monotonic (strongly monotonic) if $\forall\ S \in \sum$ and $\forall\ x, y \in S_{++}$, $x_i > y_i\ \forall\ i \in N \Rightarrow \delta(S,S(x)) \leq \delta(S,S(y))$, $\big(\forall\ x, y \in S_{++}$, $x \geq y$, $x \neq y \Rightarrow \delta(S,S(x)) < \delta(S,S(y))\big)$.

<u>Definition 4</u>: A bargaining solution $\emptyset$ is a <u>welfare optimal rule</u> if there exists a function $F: IR^n_{++} \longrightarrow IR_{++}$ such that

(i) F is integrable on compact subsets of IR^n_{++}

(ii) $\forall\ x,y \in IR^n_{++}$, $x \geq y \Rightarrow \displaystyle\int_{S(x)} F\,d\lambda \geq \int_{S(y)} F d\lambda$

(iii) $\emptyset(S) = \left\{x \in S / \displaystyle\int_{S(x)} F d\lambda \geq \int_{S(y)} F d\lambda \quad \forall\ y \in S\right\}$.

where λ is the lebesgue measure on IR^n.

A bargaining solution $\emptyset$ is a <u>strongly welfare optimal rule</u> if there exists a function $F: IR^n_{++} \rightarrow IR_+$ such that (i) and (iii) above hold and in addition we have

(ii) $\forall\ x, y \in IR^n_{++}$, $x \geq y$, $x \neq y \Rightarrow \displaystyle\int_{S(x)} F d\lambda > \int_{S(y)} F d\lambda$.

In the above definition F is assumed to be integrable on compact subsets of IR^n_{++}. Formally, this means that $F \in L^1(K; IR_{++})$ where K is a compact subset of IR^n_{++}.

Theorem 3 : If a bargaining solution is a welfare optimal rule, then it is rationalizable by a monotonic metric on $\sum$.

Proof: Let F be as in definition 4. Define a metric $\bar{\delta}$ on $\sum$ as follows:

$$\bar{\delta}(S,T) = \int_{S_{++}\setminus T_{++}} F d\lambda + \int_{T_{++}\setminus S_{++}} F d\lambda .$$

Observe that $\bar{\delta}(S,T) = 0$ if and only if $S = T$ (if $x \in S \setminus T$ then there exists a neighbourhood of x belonging to $S \setminus T$). Also since for every $S,T,U \in \sum$, $(S\setminus T)\cup(T\setminus S) \subseteq (S\setminus U)\cup(U\setminus S)\cup(T\setminus U)\cup(U\setminus T)$,

$$\bar{\delta}(S,T) \leq \bar{\delta}(S,U) + \bar{\delta}(U,T).$$

That the metric $\bar{\delta}$ is symmetric is also easily seen.

Let $\bar{\phi}(S) = \left\{x \in S / \bar{\delta}(S,S(x)) \leq \bar{\delta}(S,S(y)) \;\forall\, y \in S\right\}$.

Let $x \in \bar{\phi}(S)$.

$$\therefore \int_{S_{++}\setminus S_{++}(x)} F d\lambda \leq \int_{S_{++}\setminus S_{++}(y)} F d\lambda \quad \forall\, y\, S.$$

$$\Rightarrow \int_{S_{++}} F d\lambda - \int_{S_{++}(x)} F d\lambda \leq \int_{S_{++}} F d\lambda - \int_{S_{++}(y)} F d\lambda \qquad \forall\, y \in S_{++}, \text{since } S(y) \subseteq S \;\forall\, y \in S_{++}$$

$$\Rightarrow \int_{S_{++}(x)} F d\lambda \geq \int_{S_{++}(y)} F d \quad \forall\, y \in S_{++}.$$

Since ϕ is welfare optimal with respect to F, $x \in \phi(S)$

Hence $\bar{\phi}(S) \subseteq \phi(S)$.

Conversely suppose $x \in \phi(S)$.

$$\therefore \int_{S_+(x)} F d\lambda \geq \int_{S_+(y)} F d\lambda \quad \forall\ y \in S$$

$$\therefore \int_{S_+ - S_+(x)} F d\lambda \geq \int_{S_+ - S_+(y)} F d\lambda \quad \forall\ y \in S$$

$$\therefore \quad x \in \bar{\phi}(S).$$

Hence $\phi(S) = \bar{\phi}(S)$.

Q.E.D.

Theorem 4: If a bargaining solution is a strongly welfare optimal rule, then it is rationalizable by a strongly monotonic metric on Σ.

Proof: Similar to the above, except that now we have $\delta(S,S(x)) > \delta(S,S(y))$ if $x \geq y$ and $x \neq y$.

Q.E.D.

The converse of the above results are valid when a certain additional condition is satisfied.

Theorem 5: If a bargaining solution is rationalizable by a monotonic metric δ on Σ which obeys the additional condition

$$* \begin{cases} \text{there exists a continuous function } F: \mathbb{R}^n_{++} \longrightarrow \mathbb{R}_{++} \text{ such that} \\ \text{for all } S \in \Sigma, \ \delta(S, S(x)) = \int_{S_{++} \setminus S_{++}(x)} F d\lambda \quad \text{whenever } x \in S_{++} \end{cases}$$

then, ∅ is a welfare optimal rule.

Proof: Since F is continuous, F satisfies (i)

Since is monotonic, F satisfies (ii)

Since ∅ is rationalizable by δ, F satisifes (iii)

Q.E.D.

Remark: (i) Once again if δ is a strongly monotonic metric rationalization of ∅, then ∅ is strongly welfare optimal.

(ii) The Nash solution is welfare optimal, in fact, strongly welfare optimal. The Kalai-Smorodinsky solution is not. Hence the latter solution is not rationalizable by a monotone metric whereas the former is.

(iii) The essential features of a metric used in our analysis is that it is non-negative, symmetric and $\delta(S,T) = 0$ if and only if $S = T$. Hence we are really interested in quasi-metric rationalizations where the triangle inequality plays no role.

REFERENCES

1. C. Goffman and G. Pedrick: "First Course in Functional Analysis", Prentice Hall of India, 1987.

2. E. Kalai: "Proportional Solutions to Bargaining Situations", Econometrica, 45 (1977), 1623–1630.

3. E. Kalai and M. Smorodinsky: "Other Solutions to Nash's Bargaining Problem", Econometrica 43 (1975), 513–518.

4. M. Kaneko: "An Extension of the Nash Bargaining Problem And the Nash Social Welfare Function", Theory and Decision 12 (1980) 135–148.

5. J.F. Nash: "The Bargaining Problem", Econometrica 18 (1950), 155–162.