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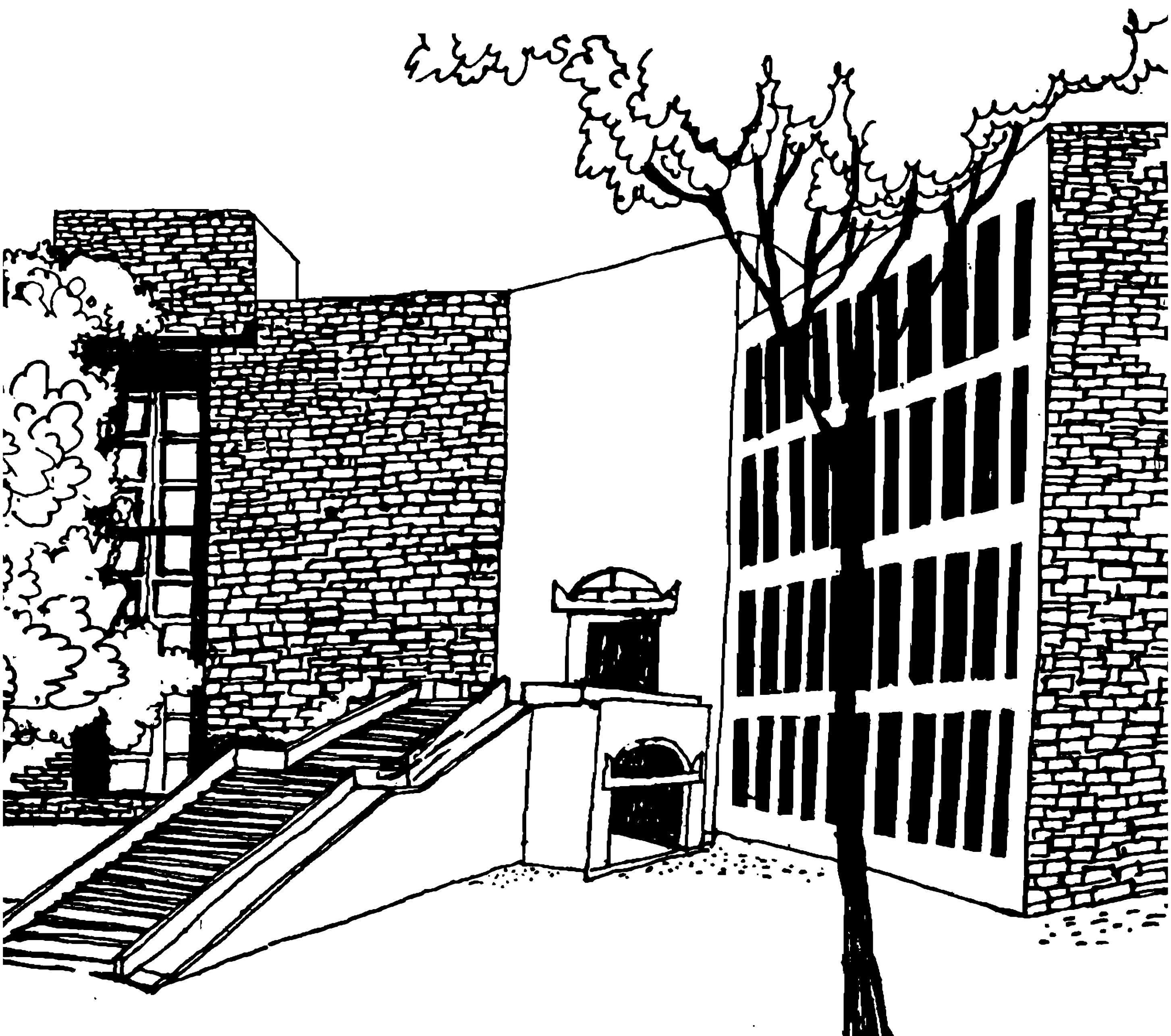

# SOME REMARKS ON PROPERTIES FOR CHOICE FUNCTIONS

By

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## Abstract

In this paper, we present five categories of results by studying the interrelationships between properties for choice functions. The first category of result is about the localization assumption. The second category of result is about replication invariance of the egalitarian solution. The third category of result provides an axiomatic characterization of the equal loss solution. The fourth and fifth categories consist of lexicographic extensions of the equal loss and relative egalitarian solutions respectively.

**Acknowledgement**

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## Some Remarks on Properties for Choice Functions

### 1 Introduction

The main problem in (multi-criteria economic) choice theory is to choose a vector from a compact, convex, comprehensive subset of the non-negative orthant of a finite dimensional Euclidean space. In addition to the above we assume such sets (which are called choice sets or choice problems) admit a vector with strictly positive components. The reason why we may refer to them as multi-criteria economic choice problems is because they arise very naturally in problems of resource allocation in a multi-sector economy, where the returns in each sector are measured by a concave, continuous, non-decreasing, non-constant revenue function with no returns accruing from zero investment. That any finite dimensional choice problem arises from such an investment planning problem has been established in Lahiri [1994].

A solution is a function defined on a class of choice problems, which selects from each admissible choice problem a feasible vector. Axiomatic choice theory (which is more commonly known as Axiomatic Bargaining Game Theory) is a body of mathematics which tries to uniquely characterize solutions with the help of a finite number of axioms or properties which the solutions are required to satisfy. The genesis of axiomatic choice theory is the paper by Nash [1950].

In the Nash [1950] paper a property called Independence of Irrelevant Alternatives (hereafter referred to as NIIA) was used to characterize a solution which is known as the Nash solution, and which is probably the most celebrated of all solutions known till date. However, the NIIA assumption was found seriously wanting in relevance. A property suggested by Lensberg [1987] called localization, has been used by Peters [1992] to characterize the family of all non-symmetric Nash solutions originally proposed by Harsanyi and Selten [1972]. Our first two results in this paper is about the localization assumption. Some consequences of our results are also discussed in this paper. These two results are similar in spirit to the ones proved by Roth [1977] for NIIA.

Our next category of results has been inspired by the work reported in Thomson [1986]. We begin with two dimensional choice problems and then derive another choice problem from it, by replicating each criteria an integer number of times. Thomson [1986] established a simple relationship between the Nash solution for the replicated problem and a certain non-symmetric Nash solution for the original problem. Thomson [1986] also established the same simple relationship between the relative egalitarian solutions (due to Kalai and Smorodinsky [1975]) for the replicated and original problems.

We here establish the same simple relationship between the egalitarian solution (due to Kalai [1977]) for the replicated problem and a certain proportional solution (also due to Kalai [1977]) for the original problem.

Our third category of results concerns the equal loss solution due to Yu [1973] and Friemer and Yu [1976]. Chun [1988] was the first to characterize the equal loss solution axiomatically. A proposed simpler proof appears here on a somewhat different domain.

Our fourth category of results concerns the lexicographic extension of the equal loss solution (originally due to Chun and Peters [1991]. This solution is the lexicographic extension of the equal loss solution cited above. On a somewhat different domain than the one considered by Chun and Peters [1991], we propose a somewhat simpler proof, particularly for three dimensional problems. It is our hunch that the proof carries through for higher dimensional problems.

Our fifth category of result concerns a relatively original solution: the lexicographic relative egalitarian solution. This is the lexicographic extension of the relative egalitarian solution due to Kalai and Smorodinsky [1975]. We are able to obtain an axiomatic characterization which is very similar to the axiomatic characterization for the lexicographic equal loss solution. However, the domains are slightly different. In any event, the domain chosen for the last three categories of results, reflects our innate interest in modelling investment planning problems as choice problems--an endeavor which lacks meaning in the presence of infinite free disposability assumed in earlier characterization results for similar choice functions.

From the point of view of relevance the last three categories of results are the most rewarding as it makes known results accessible to a much wider audience. The second category of results has been presented to show that a certain method works once again. The first category of results can be viewed as a re-statement of an existing characterization theorem for the family of non-symmetric Nash solutions.

## 2 The Framework

A (n-dimensional) choice problem is a non-empty subset S of $\mathbb{R}^n_+$, satisfying the following properties:

i) S is compact, convex

ii) S is comprehensive i.e., $0 \leq y \leq x \in S \rightarrow y \in S$

iii) $\exists\, x \in S$ such that $x \gg 0$.

Let $\Sigma^n$ denote the class of all choice problems. Let B be a nonempty subset of $\Sigma^n$. A solution on B is a function $F: B \to \mathbb{R}^n_+$ such that $F(S) \in S \ \forall\, S \in B$.

## 3 The Localization Axiom

We are interested in the interrelationship between the following axioms on a solution $F: B \to \mathbb{R}^n_+$.

**Axiom 1** **(Strong Individual Rationality):**

$F(S) \gg 0 \ \forall\, S \in B$

**Axiom 2** **(Weak Pareto Optimality):**

$F(S) \in W(S) \ \forall\, S \in B$ where for $S \in B$, $W(S) = \{x \in S / y \gg x,\ y \notin S\}$.

**Axiom 3** **(Pareto Optimality:**

$F(S) \in P(S) \ \forall\, S \in B$, where for

$S \in B$, $P(S) = \{x \in S / y \geq x,\ y \in S \to y = x\}$.

**Axiom 4** **(Scale Translation Covariance):**

Let $A \equiv Diag(a_1, \ldots, a_n)$ be the n x n diagonal matrix with $a_i > 0$ being the $i^{th}$ element on the diagonal. Then $F(AS) = AF(S) \ \forall\, S \in B$ when AS = {Ax/x$\in$S}$\in$B.

**Axioms 5** **(Localization):**

$\forall\, S, T \in B$, if there exists a neighborhood U of F(S) such that $U \cap S = U \cap T$, then F(T) = F(S).

**Lemma 1:**

If F satisfies Axioms 4 and 5, on $\Sigma^n$ then either:

(i) $F(S) = 0 \ \forall\, S \in \Sigma^n$ or

(ii) $F(S) \in W(S) \ \forall\, S \in \Sigma^n$

**Lemma 2:**

If F satisfies Axioms 1, 4 and 5, on $\Sigma^n$ then $F(S) \in P(S) \ \forall\, S \in \Sigma^n$.

**Proof of Lemma 1:**

**Step 1:**

We show that F(S) = 0 for some $S \varepsilon \Sigma^n \rightarrow F(S) = 0 \ \forall \ S \varepsilon \Sigma^n$.

Let F(S) = 0 for some $S \varepsilon \Sigma^n$ and $T \varepsilon \Sigma^n$. Since S and T are comprehensive and contain a strictly positive vector, there exists a neighborhood U of F(S) such that $U \cap S = U \cap T$.

By Axiom 5, F(T) = F(S) = 0.

**Step 2:**

Suppose $F(S) \neq 0$ for some $S \varepsilon \Sigma^n$. Then F(S) εW(S).

Towards a contradiction assume $F(S) \notin W(S)$. Hence $\exists \ a \varepsilon \mathbb{R}^n_{++}, \ a = \{a_1, \ldots, a_n\}, \ 0 < a_i < 1 \ \forall \ i = 1, \ldots, n, A = diag(a_1, \ldots, a_n)$ such that $F(S) \ll y$ for some y ε A S. Hence there exists a neighborhood U of F(S) such that $U \cap S = U \cap T$ where T = AS.

By Axiom 5, F(S) = F(AS). By Axiom 4, F(AS) = AF(S) < F(S), since $F(S) \neq 0$ and $0 < a_i < 1 \ \forall \ i = 1, \ \ldots, \ n$. This contradiction establishes that F(S) ε W(S) and proves the lemma. Q.E.D.

**Proof of Lemma 2:**

Suppose $F(S) \notin P(S)$ for some $S \varepsilon \Sigma^n$. By Axiom 1, $F(S) \gg 0$. Now, $F(S) \notin P(S) \rightarrow \exists \ y \varepsilon S$ such that y > F(S). Suppose $y_i > F_i$ (S). Consider $a_j = 1$ for $j \neq i, \ 0 < a_i < 1$ such that $a_i \ y_i > F_i$ (S) and let A = diag($a_1$, ..., $a_n$). By Axiom 4, $F(AS) \neq F(S)$. However, if T = AS, then there exists a neighborhood U of F(S) such that $U \cap T = U \cap S$. By Axiom 5, F(AS) = F(T) = F(S). But this is impossible since $F_i$ (S) > $a_i$ $F_i$ (S). Thus F(S) ε P(S). Q.E.D

Let $w \varepsilon \mathbb{R}^n_{++}$ with $\sum_{i=1}^{n} w_i = 1$. Define $v^w: \Sigma^n \rightarrow \mathbb{R}^n_{++}$ as follows:

$$v^w(S) = \underset{x \varepsilon S}{argmax} \prod_{i=1}^{n} x_i^{w_i}$$

The family $\left\{ v^w : w \in \mathbb{R}^n_{++}, \sum_{i=1}^{n} w_i = 1 \right\}$ is called the family of non-symmetric Nash choice functions.

For $w_i = \frac{1}{n}\ \forall\ i = 1, \ldots, n$, $v^w$ is denoted N and is called the Nash choice function (see Harsanyi and Selten [1972]).

**Theorem 1:**

The only choice functions to satisfy Axioms 1, 4 and 5 on $\Sigma^n$ are the non-symmetric Nash choice functions.

**Proof:**

Our Lemma 2 and proof of Theorem 2.46 in Peters [1992]. Q.E.D.

Let us now consider another axiom.

**Axiom 6 Symmetry (SYM):**

Given $S \in B$ if $S = \pi S$ for all permutation matrices $\pi$, then $F_i(S) = F_j(S)\ \forall\ i, j \in \{1, \ldots, n\}$.

**Note:**

A permutation matrix is an n x n matrix $\pi$ such that each row and each column of $\pi$ has exactly one entry with value 1 and the remaining entries are zero. Clearly $\pi S = \{\pi x / x \in S\}$.

**Theorem 2:**

There are exactly two choice functions on $\Sigma^n$ which satisfy Axioms 4, 5 and 6. They are:

(i) the dummy solution, $D: \Sigma^n \to \mathbb{R}^n_+$ such that $D(S) = 0\ \forall\ S \in \Sigma^n$; and

(ii) the Nash solution, $N: \Sigma^n \to \mathbb{R}^n_+$.

**Proof:**

Our lemma 1 and proof of Theorem 2.46 of Peters [1992]. Q.E.D.

## 4 Replication Invariance of the Proportional Solution

Let $S \in \Sigma^2$ be given, as well as natural numbers m, l. Let $I_m = \{1, 2, \ldots, m\}$ and

$J_1 = \{m+1, \ldots, m+l\}$. For a pair $(i,j) \in I_m \times J_l$, let

$$S_{i,j} := \left\{x \in \mathbb{R}^{m+l} / \exists (x_1', x_2') \in S \text{ with } x_i = x_1',\ x_j = x_2',\ x_k = 0 \text{ if } k \neq i, j\right\}.$$

The Thomson (m,l) replication of S is defined as $S^{m,l} := Conv\{S_{i,j} / (i,j) \in I_m \times J_l\}$. For, $T \in \Sigma^{m+l}$, define $E(T) = \bar{t}e^{I_m \cup J_l}$ where $\bar{t} = \max\{t \geq 0 / t e^{I_m \cup J_l}\}$ and $e^{I_m \cup J_l}$ is the (m + l) dimensional vector with all coordinates equal to 1. For $T \in \Sigma^2$, define $E^{m,l}(T) = \left(\frac{\bar{s}m}{m+l}, \frac{\bar{s}l}{m+l}\right)$ where

$$\bar{s} = \max\left\{s \geq 0 / \left(\frac{\bar{s}m}{m+l}, \frac{\bar{s}l}{m+l}\right) \in T\right\}.$$

In the above E is called the egalitarian solution and $E^{m,l}$ is called the proportional solution.

**Theorem 3:**

In the above framework $mE_i(S^{m,l}) = E_1^{m,l}(S)\ \forall\ i \in I_m$ and $lE_j(S^{m,l}) = E_2^{m,l}(S)\ \forall\ i \in I_m$.

**Proof:**

Let $(a,b) = E^{m,l}(S)$ and suppose

$a'x_1' + b'x_2' \leq a'a + b'b\ \forall (x_1', x_2') \in S$. This is possible since $E^{m,l}$ satisfies Axiom 2. Thus if $x^{ij}$ denotes a vector in $S^{ij}$, then $a'x_i^{ij} + b'x_j^{ij} \leq a'a + b'b$.

Now let $y \in S^{m,l}$. Then, there exists $\mu_{ij} \geq 0\ \forall\ i, j \in I_m \times J_l$ such that $\sum_{(i,j)\in I_m \times J_l} \mu_{ij} x^{ij} \geq y$ for some $x^{ij}, (i,j) \in I_m \times J_l$, and $\sum_{(i,j)\in I_m \times J_l} \mu_{i,j} = 1$.

$$\therefore y_k \leq \sum_{j \in J_l} \mu_{kj} x_k^{kj} \text{ if } k \in I_m$$

$$y_k \leq \sum_{i \in I_m} \mu_{ik} x_k^{ik} \text{ if } k \in J_l$$

$$\therefore a'\sum_{k\in I_m} y_k + b'\sum_{k \in J_l} y_k \leq a'\sum_{k\in I_m}\sum_{j\in J_l} \mu_{kj} x_k^{kj} + b'\sum_{k\in J_l}\sum_{i \in I_m} \mu_{ik} x_k^{ik}$$

$$= a' \sum_{(i,j)\in I_m \times J_l} \mu_{ij} x_i^{ij} + b' \sum_{(i,j)\in I_m \times J_l} \mu_{ij} x_j^{ij} = \sum_{(i,j)\in I_m \times J_l} \mu_{ij}\left(a'x_i^{ij} + b'x_j^{ij}\right)$$

$$\leq a'a + b'b = a'\sum_{i\in I_m} \frac{a}{m} + b'\sum_{j \in J_l} \frac{b}{l}$$

Hence the vector $z \in S^{m,l}$, with $z_i = \frac{a}{m}\ \forall\ i \in I_m$ and $z_j = \frac{b}{l}\ \forall\ j \in J_l$ belongs to the Weak Pareto Optimal frontier of $S^{m,l}$.

Now, $a = \frac{\bar{s}m}{m+l}$, $b = \frac{\bar{s}l}{m+l}$ for some $\bar{s} > 0 \rightarrow \frac{a}{m} = \frac{b}{l}$. Hence z as defined above is equal to $E(S^{m,l})$. Q.E.D.

## 5 The Equal Loss Solution

Let $\Gamma^n$ be the largest subset of $\Sigma^n$ on which the following function $E^*: \Gamma^n \rightarrow \mathbb{R}^n_+$ is defined:

(a) $E^*_i(S) - u_i(S) = E^*_j(S) - u_j(S)\ \forall\ i, j \in N = \{1, \ldots, n\}$

(b) $E^*(S) \in W(S) = \{x \in S / \nexists y \in S \text{ with } y > x\}$

Here $\forall\ i \in N$ and $S \in \Gamma^n$, $u_i(S) = \max\ \{x_i / x \in S\}$. $u(S) = (u_1(S), \ldots, u_n(S))$ is called the ideal point of S. $E^*$ is called the equal loss solution.

<u>Axiom 7:</u> <u>Translation Covariance (TC):</u>

(TC) : $\forall\ S \in B,\ \forall\ a \in \mathbb{R}^n_+$,

$F(T) = F(S) + a$ if

$T = \{z \in \mathbb{R}^n_+ / z \leq x + a \text{ for some } x \in S\} \in B$

<u>Axiom 8:</u> <u>Strong Monotonicity with respect to the Ideal Point (SMON):</u>

$\forall\ S, T \in B$ with $u(S) = u(T)$ and $S \subset T \Rightarrow F(S) \leq F(T)$.

**Theorem 4:** The only solution on $\Gamma^n$ to satisfy Axioms 2, 6, 7 and 8 is $E^*$.

**Proof:** It is easy to check that $E^*$ satisfies the desired properties. Hence let $F: \Gamma^n \rightarrow \mathbb{R}^n$ be a choice function satisfying the above properties and let $S \in \Gamma^n$.

Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be such that $x_i = \max_{j \in N} \{u_j(S) - u_i(S)\}$.

Let $T = \{y \in \mathbb{R}^N_+ / \exists\ z \in S \text{ with } y \leq z + x\}$.

Thus $u(T) = u(S) + x$

Thus $E^*(T) = be$ where $e \in \mathbb{R}^n_+$ with $e_i = 1\ \forall i$ for some $b > 0$

Let V = comprehensive convex hull $\{E^*(T), (u_1(T), 0, \ldots, 0), \ldots, (0, 0, \ldots, u_n(T))\}$

By SYM and WPO, $F(V) = E^*(T)$

Now $u(V) = u(T)$ and $V \subset T$

Thus by (SMON), $F(T) \geq F(V) = E^*(T)$.

**Case 1:** $E^*(T) \in \{y \in T / x \in T,\ x \geq y \rightarrow x = y\} = P(T)$; then $F(T) = E^*(T)$

**Case 2:** $E^*(T) \notin P(T)$

Let $T_\varepsilon$ = convex hull $\{\varepsilon e + E^*(T)\}$ for $\varepsilon > 0$.

$E^*(T) \notin P(T) \rightarrow \exists\ y \in T$ such that $y \geq E^*(T),\ y \neq E^*(T) = be$

Let $y_i > b$ for some $i \in N$

Thus $u_i(T) > b$.

Since $u_j(T) = u_i(T)\ \forall\ j \in N,\ u_j(T) > b \forall\ j \in N$.

Hence for $\varepsilon > 0$, small, $u(T_\varepsilon) = u(T)$, $T \subset T_\varepsilon$.

But $E^*(T_\varepsilon) = \varepsilon e + E^*(T) \rightarrow$ (by case 1) $F(T_\varepsilon) = \varepsilon e + E^*(T)$

By SMON, $F(T) \leq F(T_\varepsilon) = \varepsilon e + E^*(T)$

Thus $E^*(T) \leq F(T) \leq \varepsilon e + E^*(T)\ \forall\ \varepsilon > 0$ sufficiently small. Letting $\varepsilon$ go to zero, we get $F(T) = E^*(T)$. Note, $E^*(T) = E^*(S) + x$ and $F(T) = F(S) + x$, both by (TC). Thus $F(S) = E^*(S)$.

## 6 The Lexicographic Equal Loss Solution

Let $\alpha : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be defined as follows:

For $x \in \mathbb{R}^n$, $\alpha(x)$ is any arrangement of x in ascending order. For $x, y \in \mathbb{R}^n$, define $x >^l y$ if and only if either $x_1 > y_1$ or $\exists\ 1 \leq k < n$ such that $x_i = y_i\ \forall\ 1 \leq i \leq k$ and $x_{k+1} > y_{k+1}$. Given $S \in \Gamma^n$, define $L^*(S)$ as follows:

$L^*(S) = u(S) + x^*$ where $x^* = \{z - u(S) : z \in S\}$ and $\alpha(x^*) >^l \alpha(y^*)\ \forall\ y \in \{z - u(S) : z \in S\},\ y \neq x^*$. ($L^* : \Gamma^n \rightarrow \mathbb{R}^n$ is called the lexicographic equal loss solution. Description and discussion in a rather extensive manner of the solution $L^*$ can be found in Chun and Peters [1991].)

**Axiom 9** **Weak Monotonimity (WMON):**

For $S \varepsilon B$, $i \varepsilon \{1, \ldots, n\}$, let $S_{-i} = \left\{x_{-i} / (x_{-i}, x_i) \varepsilon S\right\}$. (Here for i $\varepsilon$ {1, ..., n} and $x \varepsilon \mathbb{R}^n$, $x_{-i} = (x_j)_{j \neq i}$. ) $\forall$ $T \varepsilon B$, if $S \subset T$ and $S_{-i} = T_{-i}$ $\forall$ $i = 1, \ldots, n$, then F(T) $\geq$ F(S).

**Axiom 10:** **Independence of Irrelevant Alternatives other than Ideal Point (IIA*):**

$\forall$ $S, T \varepsilon B$, if $S \subset T$, $u(S) = u(T)$ and F(T)$\varepsilon$S, then F(S) = F(T).

The issue here is the following theorem due to Chun and Peters [1991]:

**Theorem 5:**

The only solution on $\Gamma^n$ to satisfy Axioms 3, 6, 7, 9 and 10 is $L^*$.

In this paper we do not dispute the sequence of lemmas leading to the statement that $L^*$ satisfies Axioms 3, 6, 7, 9 and 10. What we submit is an alternative simpler proof of the statement that any solution which satisfies Axioms 3, 6, 7, 9 and 10 must agree with $L^*$. That too we suggest for the case n = 3, with a conjecture that the proof may be adapted to higher dimensions.

**Lemma 3:**

If $F: \Gamma^3 \rightarrow \mathbb{R}^3$ satisfies Axioms 3, 6, 7, 9 and 10 then F(S) $\geq$ $E^*$(S) where for $S \varepsilon \Gamma^3$, $E^*(S) = u(S) - \bar{t}e$, $\bar{t} = \max\{t \geq 0 / u(S) - te \; \varepsilon S\}$; e is the vector in $\mathbb{R}^3$ with all coordinates equal to 1. $E^*$ is called the equal loss solution.

**Proof:**

Let F satisfy Axioms 3, 6, 7, 9, and 10 and let $S \varepsilon \Gamma^3$. By Axiom 7, assume $u(S) = \lambda e$ for some $\lambda > 0$. Let T be the comprehensive convex hull of the smallest symmetric set containing $S_{-1}$, $S_{-2}$, $S_{-3}$ and $E^*$(S). By Axioms 3 and 6, F(T) = $E^*$(S). Let $T' =$ comprehensive convex hull $\{S_{-1}, S_{-2}, S_{-3}, \{E^*(S)\}\}$

$u(T') = u(T) = u(S) = \lambda e$

and $T' \subset T$

further $F(T) = E^*(S) \varepsilon T'$

By Axiom 10, $F(T') = F(T) = E^*(S)$

By Axiom 9, $F(S) \geq F(T') = E^*(S)$ Q.E.D.

**Lemma 4:**

If $F: \Sigma^3 \to \mathbb{R}^3_+$ satisfies Axioms 3, 6, 7, 9, and 10 then $F(S) = L^*(S)$.

**Proof:**

Let $S \varepsilon \Sigma^3$ and let T = comprehensive convex hull $\{S_{-1}, S_{-2}, S_{-3}, \{L^*(S)\}, \{E^*(S)\}\}$. Clearly u(T) = u(S), $E^*(S) \varepsilon T$, $T \subset S \to E^*(T) = E^*(S)$. Similarly

$u(T) = u(S)$, $T \subset S$, $E^*(T) = E^*(S)$ and $L^*(S) \varepsilon T \to L^*(T) = L^*(S)$.

By Lemma 3, $F(T) \geq E^*(T) = E^*(S)$.

By Axiom 3, $F(T) = L^*(T) = L^*(S)$. Now, $T \subset S$, $T_{-i} = S_{-i}\ \forall i$

implies by Axiom 9, $F(S) \geq F(T) = L^*(S)$. But since $L^*$ satisfies Axiom 3, $F(S) = L^*(S)$.

Q.E.D.

## 7 The Lexicographic Relative Egalitarian Solution

The relative egalitarian solution due to Kalai and Smorodinsky [1975] is defined as follows:

$K(S) = \bar{t}u(S)\ \forall S \varepsilon \Sigma^n$, where $u(S) = (u_i(S))_{i \varepsilon N}$

and $u_i(S) = \max\{x_i / x \varepsilon S\}$ whenever $S \varepsilon \Sigma^n$. As before U(S) is called the ideal point of S.

It is well known (see for instance Roth [1979]) that $K(S)\ \varepsilon\ W(S)\ \forall\ S \varepsilon \Sigma^n$ and that there exists $S \varepsilon \Sigma^n$ for which $K(S) \varepsilon W(S) \setminus P(S)$. We therefore consider the lexicographic completion of K, denoted $K_L$ which is defined as follows:

$\forall\ S \varepsilon \Sigma^n$, such that u(S) = e, set $K_L(S) = L^*(S)$. (Clearly such an S would belong to $\Gamma^n$ and hence there is no problem with the above definition.) For all other $S \varepsilon \Sigma^n$, set $K_L(S) = u(S)\ K(S')$ where

$$S' = \left(\frac{1}{u_1(S)}, \ \ldots, \ \frac{1}{u_n(S)}\right) S.$$

We now have the following theorem whose proof is analogous to the proof of Theorem 5.

**Theorem 6:**

The only solution on $\Sigma^n$ to satisfy Axioms 3, 4, 6, 9 and 10 is $K_L$.

## 8 Conclusion

In this paper we have established interrelationships between properties for choice functions by using a group of these properties at a time to uniquely characterize a particular solution. With the sole exception of section 4 (where the egalitarian solution was shown to satisfy replication invariance) the other sections dealt with axiomatic characterizations. Some well known and some new solutions were discussed. However, the entire discussion was restricted to a grand domain which makes the study of investment planning problems meaningful.

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