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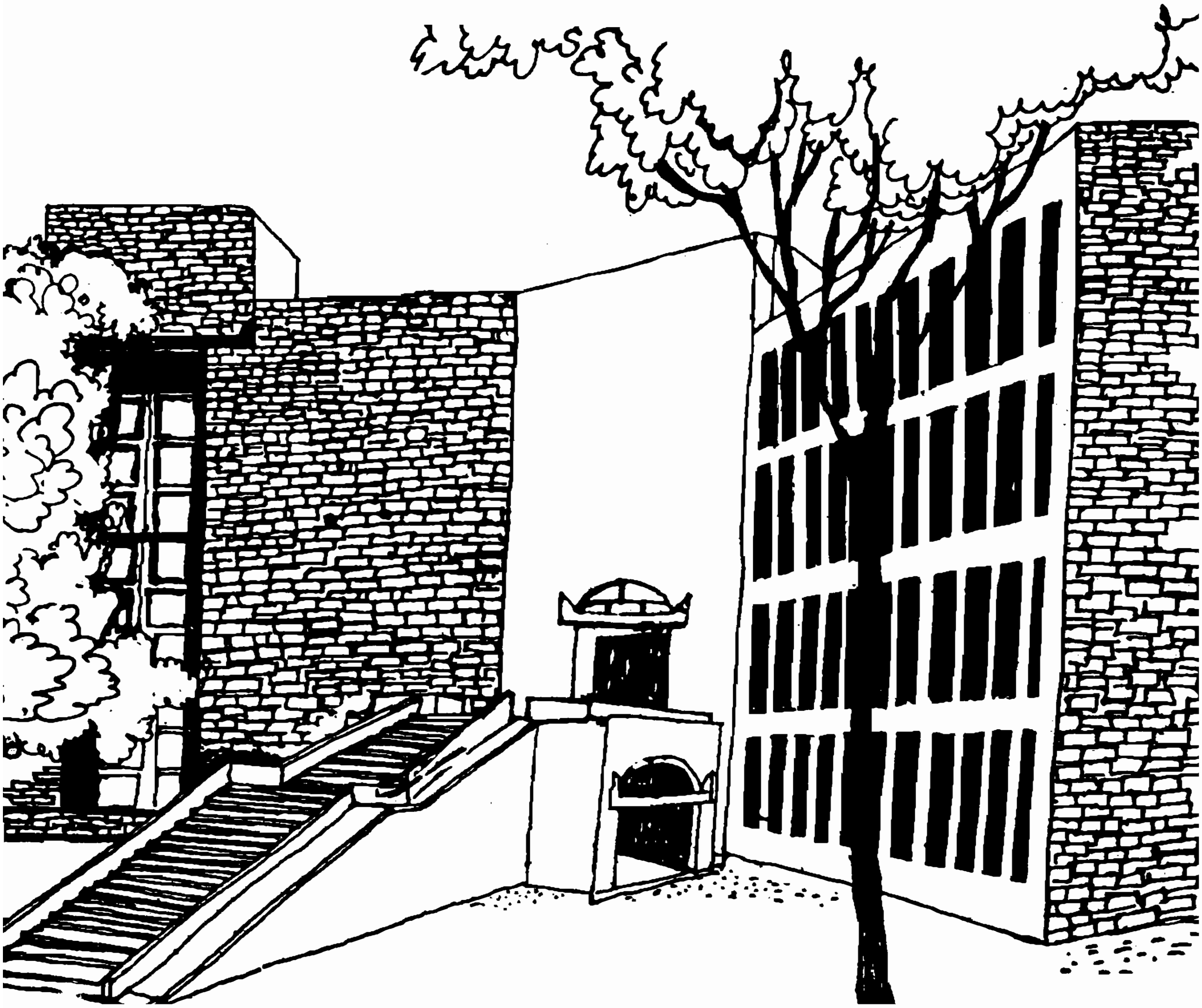

METRIC RATIONALIZATION OF BARGAINING SOLUTIONS AND RESPECT FOR UNANIMITY

By

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ABSTRACT

In this paper we represent bargaining solutions by means of a metric which is defined on games, whereby the solutions are precisely those payoffs which are closest to being unanimously highest. Subsequently we study the properties of "welfare optimal rules" and rationalizability of bargaining solutions by symmetric metrics. This latter condition leads to anonymous bargaining solutions which satisfy "metric respect for unanimity."

1. Introduction: In this paper, we consider n-person bargainin games ($n \geq 2$), that is, pairs (S,d) where

(1) The space S of feasible utility payoffs is a compact and convex subset of IR^n.

(2) The disagreement outcome d is an element of S.

Furthermore, for mathematical convenience, we will also assume that

(3) $x \geq d$ for all $x \in S$

(4) There is an $\hat{x} \in S$ with $\hat{x}_i > d_i$ for each $i \in N = \{1,2,\ldots,n\}$

(5) For all $y \in IR^n$ with $d \leq y \leq x$ for some $x \in S$, we have $y \in S$.

Such games (S,d) correspond to situations involving n bargainers (players) 1,2,....,n, who may cooperate and agree upon choosing a point $s \in S$, which has utility s_i for player i, or who may not cooperate. In the latter case, the outcome is the point d, which has utility d_i for player $i \in N$. The family of all such bargaining games, satisfying (1) through (5), is denoted Σ. Following Kaneko [4] we call a multifunction $\phi: \Sigma \longrightarrow IR^n$ which assigns to each game $(S,d) \in \Sigma$ a nonempty subset $\phi(S,d)$ of S an n-person bargaining solution.

We restrict ourselves to solutions ϕ which satisfy the following properties:

(ICO): For each $(S,d) \in \sum$ and each transformation $A: IR^n \to IR^n$ of the form $A(x_1,\ldots,x_n) = (x_1 + b_1,\ldots, x_n + b_n)$ for all $x \in IR^n$, where $b_1,b_2,\ldots,b_n$ are real numbers,

$\phi(A(S), A(d)) = A(\phi(S,d))$ (independence with respect to choice of origin). Here for $T \subset IR^n$, $A(T) = \{A(x)/x \in T\}$.

(IR) : For each $(S,d) \in \sum$, for each $x \in \phi(S,d)$, $x_i \geq d_i \; \forall \; i \in N$ (individual rationality).

Since we only consider solutions which obey ICO, we may without loss of generality restrict our attention here to n-person bargaining games with disagreement outcome 0. From now on, we assume that every $(S,d) \in \sum$, besides (1) through (5), satisfies

(6) $d = 0$,

and we will write S instead of (S,d).

Our purpose in this paper is to represent bargaining solutions as defined above by means of a metric which is defined on bargaining problems, whereby the solutions are precisely those payoffs which are closest to being unanimously highest. In general, the purpose of a metric is to define distance and the metric generated by a bargaining solution captures the notion of a solution being close to awarding the highest payoff to all the players.

If some payoff vector awards the highest payoff to all the players then surely it should be declared the consensus solution. This is the unanimity principle which is naturally very appealing and which is satisfied by the Nash [4] solution as well as by the Kalai-Smorodinsky [3] solution to bargaining problems. Of course, for 'most' bargaining problems, a unanimously preferred payoff vector generally does not exist, in as much as that 'most' bargaining problems are not representable as the comprehensive, convex hull of a single payoff vector. A significant problem then is to find out in what precise sense different bargaining solutions attempt (if at all) to approximate or respect the ideal of using the unanimity rule.

Metrizable Bargaining Solutions: Let $x = (x_1, \ldots\ldots, x_n) \in IR^n_{++}$; let us agree to denote $\{y \in IR^n_+ / y_i \leq x_i, \forall\ i \in N\}$ by $S(x)$. Such games are called unanimity games.

Definition 1 :- A bargaining solution $\emptyset$ is Paretian if $\forall\ x \in IR^n_{++}$, $\emptyset(S(x)) = \{x\}$.

Let $\bar{\delta}$ be a metric on $\sum$ ($\sum$ is metrizable as for instance by the Hausdorff metric (see Goffman and Pedrick [1]). Let $S_+ = S \setminus \{0\}$ and $S_{++} = S \cap IR^n_{++}$.

Definition 2 : The metric $\bar{\delta}$ on $\sum$ is a rationalization according to unanimity (henceforth a rationalization) for the bargaining solution $\emptyset$, if $\forall\ S \in \sum$, $\emptyset(S) = \{x \in S_{++} / \bar{\delta}(S, S(x)) \leq \bar{\delta}\ (S, S(y))\ \forall\ y \in S_{++}\}$.

That is the metric $\bar{\delta}$ rationalizes ϕ according to the unanimity criterion whenever for any bargaining game S, the solution consists of payoff vectors whose convex comprehensive hull for each such vector is the unanimity game "nearest" to the given game and each such payoff vector belongs to S. The characterization of the family of bargaining solutions having such a metric rationalization is provided by the following:

<u>Theorem 1</u>: A bargaining solution ϕ has a metric rationalization if and only if it is Paretian.

<u>Proof</u> : (i) It is easy to verify that if $\bar{\delta}$ rationalizes ϕ, then ϕ is Paretian.

(ii) Suppose ~~the~~ that ϕ is Paretian. Define $\bar{\delta}$ as follows: $\forall\ S, T \in \Sigma$,

$$\bar{\delta}(S,T) = \begin{cases} 0 \text{ if } S = T \\ 1 \text{ if } S \neq T,\ \phi(S) \cap \phi(T) \neq \emptyset \\ 2 \text{ if } S \neq T,\ \phi(S) \cap \phi(T) = \emptyset \end{cases}$$

$\bar{\delta}$ is obviously symmetric, non-negative, and $\bar{\delta}(S,T) = 0 \Leftrightarrow T=S$. It also satisfies the triangle inequality and so $\bar{\delta}$ is a metric. Let us conclude the proof by showing that the metric $\bar{\delta}$ is a rationalization for ϕ. Define $\bar{\phi} : \Sigma \to IR^n_+$ as follows: $\forall\ S \in \Sigma$, $\bar{\Phi}(S) = \{x \in S_{++} / \bar{\delta}(S,S(x)) \leq \bar{\delta}(S,S,(y))\ \ \forall\ y \in S_{++}\}$. We now show that $\phi = \bar{\phi}$. If $S = S(x)$ for some $x \in IR^n_{++}$, then $\bar{\delta}(S,S(x)) = 0$. Hence $x \in \bar{\phi}(S)$. Also $\phi(S(x)) = \{x\}$ since ϕ is Paretian. So $\phi(S(x)) \subseteq \bar{\phi}(S(x))\ \forall\ x \in IR^n_{++}$. If $y \neq x$, then $y \notin \bar{\phi}(S(x))$ since $S(x) \neq S(y)$ implies $\bar{\delta}(S(x), S(y)) > 0 = \bar{\delta}(S(x),S(x))$. $\bar{\phi}(S(x)) = \{x\} = \phi(S(x))$. Suppose $S \neq S(x)$, $\forall\ x \in IR^n_{++}$. Then $\forall\ y \in S_{++}$, $\bar{\delta}(S,S(y)) \neq 0$. If $x \in \bar{\phi}(s)$, then $\bar{\delta}(S,S(x)) = 1$. Hence $\phi(S) \cap \phi(S(x)) \neq \emptyset$. But $\phi(S(x)) = \{x\}$ since ϕ is Paretian. So

$x \in \phi(S)$ i.e. $\bar{\phi}(S) \subseteq \phi(S)$. Now suppose $x \in \phi(S)$. Hence $x \in \phi(S) \cap \phi(S(x))$. Hence $\bar{\delta}(S, S(x)) = 1$. Hence $x \in \bar{\phi}(S)$ i.e. $\phi(S) \subseteq O(S)$.

Q.E.D.

Before we conclude this section, let us note that a reasonable assumption for most bargaining solutions is that they satisfy weak Pareto optimality.

(WPO) For each $S \in \sum$, for each $x \in \phi(S)$, $y \in IR^n_+$, $y_i > x_i$ for all $i \in N$ implies $y \notin S$.

Let $W(S) = \{x \in S / y \in IR^n_+,\ y_i > x_i \ \forall\ i \in N \Rightarrow y \notin S\}$.

A solution $\phi : \sum \longrightarrow IR^n_+$ satisfying (WPO), (I.C.) and (IR) is called a <u>classical bargaining solution</u>.

For classical bargaining solutions we have the following metric characterizations:

Claim 1 :- A classical bargaining solution ϕ has a metric rationalization if and only if it is Paretian.

<u>Proof</u> : The proof is the same as that of Theorem 1, except that

$$\bar{\phi}(S) = \left\{x \in W(S) \cap IR^n_{++} / \bar{\delta}(S, S(x)) \leq \bar{\delta}(S, S(y)) \ \forall\ y \in W(S) \cap IR^n_{++}\right\}$$ is the modified definition of ϕ .

Q.E.D.

The Nash [4] and Kalai-Smorodinsky [3] solutions are Paretian and hence are metric rationalizable. Note, however, that the Kalai [2] solution is not Paretian and hence not metric rationalizable.

3. Monotonic Metric Rationalizations: The metric used in the proofs of Theorem 1, $\bar{\delta}(S,T)$, is induced by the specific bargaining solution under consideration. That is, loosely speaking, the metric is based on what the bargaining games do, rather than what the games actually are. The lack of dependence on the internal structure of the bargaining games suggests that some bargaining solutions may be rationalized by metrics which do not satisfy some intuitively desirable criteria.

Definition 3: A metric δ on Σ is monotonic (strongly monotonic) if $\forall\ S \in \Sigma$ and $\forall\ x, y \in S_{++}$, $x_i > y_i\ \forall\ i \in N \Rightarrow \delta(S,S(x)) \leq \delta(S,S(y))$, $(\forall\ x, y \in S_{++}, x \geq y, x \neq y \Rightarrow \delta(S,S(x)) < \delta(S,S(y)))$.

Definition 4: A bargaining solution $\emptyset$ is a welfare optimal rule if there exists a function $F: IR^n_+ \longrightarrow IR_+$ such that

(i) F is integrable on compact subsets of IR^n_+.

(ii) $\forall\ x,y \in IR^n_+$, $x \geq y \Rightarrow \int_{S(x)} F\, d\lambda \geq \int_{S(y)} F d\lambda$

(iii) $\emptyset(S) = \left\{ x \in S / \int_{S(x)} F d\lambda \geq \int_{S(y)} F d\lambda \quad \forall\ y \in S \right\}$.

where λ is the lebesgue measure on IR^n.

A bargaining solution $\emptyset$ is a strongly welfare optimal rule if there exists a function $F: IR^n_+ \rightarrow IR_+$ such that (i) and (ii) above hold and in addition we have

(ii) $\forall\ x, y \in IR^n_+$, $x \geq y$, $x \neq y \Rightarrow \int_{S(x)} F d\lambda > \int_{S(y)} F d\lambda$.

In the above definition F is assumed to be integrable on compact subsets of IR^n_+. Formally, this means that $F \in L^1(K; IR_+)$ where K is a compact subset of IR^n_+.

<u>Theorem 2</u>: If a bargaining solution is a welfare optimal rule, then it is rationalizable by a monotonic metric on Σ.

<u>Proof</u>: Let F be as in definition 4. Define a metric $\bar{\delta}$ on Σ as follows:

$$\bar{\delta}(S,T) = \int_{S_{++}\setminus T_{++}} F d\lambda + \int_{T_{++}\setminus S_{++}} F d\lambda .$$

Observe that $\bar{\delta}(S,T) = 0$ if and only if $S = T$ (if $x \in S \setminus T$ then there exists a neighbourhood of x (in the topology induced on S) belonging to $S \setminus T$). Also since for every $S,T,U \in \Sigma$, $(S\setminus T) \cup (T\setminus S) \subseteq (S\setminus U) \cup (U\setminus S) \cup (T\setminus U) \cup (U\setminus T)$,

$$\bar{\delta}(S,T) \leq \bar{\delta}(S,U) + \bar{\delta}(U,T).$$

That the metric $\bar{\delta}$ is symmetric is also easily seen.

Let $\bar{\phi}(S) = \left\{ x \in S / \bar{\delta}(S,S(x)) \leq \bar{\delta}(S,S(y)) \ \forall\ y \in S \right\}$.

Let $x \in \bar{\phi}(S)$.

$$\therefore \int_{S_{++}\setminus S_{++}(x)} F d\lambda \leq \int_{S_{++}\setminus S_{++}(y)} F d\lambda \quad \forall\ y\ S.$$

$$\Rightarrow \int_{S_{++}} F d\lambda - \int_{S_{++}(x)} F d\lambda \leq \int_{S_{++}} F d\lambda - \int_{S_{++}(y)} F d\lambda \qquad \forall\ y \in S_{++}, \text{since } S(y) \subseteq S \ \forall\ y \in S_{++}$$

$$\Rightarrow \int_{S_{++}(x)} F d\lambda \geq \int_{S_{++}(y)} F d \quad \forall\ y \in S_{++}.$$

Since ϕ is welfare optimal with respect to F, $x \in \phi(S)$

Hence $\bar{\phi}(S) \subseteq \phi(S)$.

Conversely suppose $x \in \phi(S)$.

$$\therefore \int_{S_+(x)} F d\lambda \geq \int_{S_+(y)} F d\lambda \quad \forall\ y \in S$$

$$\therefore \int_{S_+ \setminus S_+(x)} F d\lambda \geq \int_{S_+ \setminus S_+(y)} F d\lambda \quad \forall\ y \in S$$

$$\therefore \quad x \in \bar{\phi}(S).$$

Hence $\phi(S) = \bar{\phi}(S)$.

Q.E.D.

__Claim 2__ : If a bargaining solution is a strongly welfare optimal rule, then it is rationalizable by a strongly monotonic metric on Σ.

__Proof:__ Similar to the above, except that now we have $\delta(S,S(x)) > \delta(S,S(y))$ if $x \geq y$ and $x \neq y$.

Q.E.D.

The converse of the above results are valid when a certain additional condition is satisfied.

__Proposition 1__: If a bargaining solution is rationalizable by a monotonic metric δ on Σ which obeys the additional condition

$$\begin{cases} \text{there exists a continuous function } F: \mathbb{R}^n_+ \longrightarrow \mathbb{R}_{++} \text{ such that} \\ \text{for all } S \in \Sigma, \ \delta(S, S(x)) = \int_{S_{++} \setminus S_{++}(x)} F d\lambda \quad \text{whenever } x \in S_{++} \end{cases}$$

then, ϕ is a welfare optimal rule.

Proof: Since F is continuous, F satisfies (i)

Since F is monotonic, F satisfies (ii)

Since ϕ is rationalizable by δ, F satisfies (iii)

Q.E.D.

Remark: (i) Once again if δ is a strongly monotonic metric rationalization of ϕ, then ϕ is strongly welfare optimal.

(ii) The Nash solution is welfare optimal, in fact, strongly welfare optimal. The Kalai-Smorodinsky solution is not. Hence the latter solution is not rationalizable by a monotone metric whereas the former is.

(iii) In case a solution $\phi : \Sigma \to R^n$ satisfies the stronger invariance condition:

(IEUR) : For each $(S,d) \in \Sigma$ and each transformation $A: R^n \to R^n$ of the form $A(x_1,\dots,x_n) = (a_1x_1+b_1,\dots,a_nx_n+b_n)$ for all $x \in R^n$, where $b_1,\dots,b_n$ are real numbers and $a_1,\dots,a_n$ are positive real numbers, $\phi(A(S), A(d)) = A(\phi(S,d))$ (independence with respect to equivalent utility representations);

then the class of strongly welfare optimal rules coincides with the family of generalized Nash bargaining solutions.

(iv) The essential features of a metric used in our analysis is that it is non-negative, symmetric and $\delta(S,T) = 0$ if and only if $S = T$. Hence we are really interested in quasi-metric rationalizations where the triangle inequality plays no role.

4. Symmetric Metric Rationalizations and Respect for Unanimity:-

An interesting property satisfied by many bargaining solutions is anonymity

(A) Let $S \in \Sigma$ and $\pi : \{1,\ldots,n\} \to \{1,\ldots n\}$ be any permutation. For $x \in R^n$, $x = (x_1 \ldots, x_n)$ let $x_\pi \in R^n$ be defined as $x_\pi = (x_{\pi(1)}, \ldots, x_{\pi(n)})$ and $S_\pi = \{x_\pi \in R^n / x \in S$. Then $\phi(S_\pi) = (\phi(S))_\pi$ (i.e. nothing is affected by renaming the players).

In this section we find conditions that a metrizable bargaining solution needs to satisfy in order to be anonymous.

Let δ be a metric on Σ. We say that δ is a symmetric metric if for all permutation $\pi : N \to N$ and for all games S, $T \in \Sigma$, $\delta(S,T) = \delta(S_\pi, T_\pi)$.

We now define the concept of rationalization by a symmetric metric.

Definition 5 : A bargaining solution $\phi : \Sigma \to\to R^n$ is rationalizable by a symmetric metric if there exists a symmetric metric δ on Σ, such that for any

$S \in \Sigma$, $\phi(S) = \{x \in S_{++} / \delta(S(x),S) \leq \delta(S(y),S)$ for any $y \in S_{++}\}$

i.e. the symmetric metric on games rationalizes ϕ according to the unanimity criterion whenever, for any game S, the solution set of S is given by the payoffs x such that the unanimity game S(x) is closest to S with respect to the given metric.

We also introduce the notion of metric respect for unanimity.

Definition 6 : A bargaining solution $\phi : \Sigma \rightarrow\rightarrow R^n$ has a metric respect for unanimity if there exists a metric δ on Σ and a metric m on R^n such that for any games $S \in \Sigma$ and vectors $x, y \in S_{++}$, $\delta(S, S(x)) \leq \delta(S, S(y))$ implies that $m(\phi(S), x) \equiv \min_{z \in \phi(S)} m(z, x) \leq \min_{z \in \phi(S)} m(z, y) \equiv m\{\phi(S), y\}$

Notice that if ϕ has a metric respect for unanimity, then it is Paretian. The appeal of this property of proximity preservation lies in it offering a natural and consistent conception of a bargaining solution as a mechanism attempting to approximate or respect the social ideal of using the unanimity rule.

Theorem 3: If a bargaining solution $\phi : \Sigma \rightarrow\rightarrow R^n$ is rationalizable by a symmetric metric, then it satisfies the anonymity property, and it has a metric respect for unanimity.

Proof : Let ϕ be rationalizable by a symmetric metric. Then there exists a metric δ on Σ such that for all $S, T \in \Sigma$, $\delta(S, T) = \delta(S_\pi, T_\pi)$ and for all $S \in \Sigma$

$$\phi(S) = \{x \in S_{++} / \delta(S(x), S) \leq \delta(S(y), S) \quad \forall y \in S_{++}\}$$

Hence $(\phi(S))_\pi = \{x_\pi \in (S_{++})_\pi \; / \delta\{S_\pi(x_\pi), S_\pi\} \leq (S_\pi(y_\pi), S_\pi) \; \forall y_\pi \in (S_{++})_\pi\}$.

$$= \{x \in (S_{\pi++}) / \delta(S_\pi(x), S_\pi) \leq \delta(S_\pi(y), S_\pi) \; \forall y \in (S_\pi)_{++}\}$$
$$= \phi(S_\pi)$$

Hence ϕ is anonymous.

To show that ϕ has a metric respect for unanimity, let us define the metric on R^n as follows: for any $x, y \in R^n$

$$m(x,y) = \begin{cases} 0 \text{ if } x = y \\ 1 \text{ if } x \neq y \end{cases}$$

Now consider a game S in Σ and two alternatives x,z in S_{++} such that $\delta\{S,S(x)\} < \delta\{S,S(z)\}$. Here there are two possibilities:

(i) $x \in \phi(S)$, $z \notin \phi(S)$ in which case

$\delta(S,S(x)) < \delta(S,S(z))$, since δ rationalizes ϕ

In turn

$$m(\phi(S), x) = m(x,x) = 0 < m(\phi(S),z) = 1$$

(ii) $x \notin \phi(S)$, $z \notin \phi(S)$

Here $m(\phi(S),x) = m(\phi(S),z) = 1$

In both cases, we obtain that $m(\phi(S),x) \leqq m(\phi(S),z)$

<u>Remark</u> :

We know for instance that the family of non-symmetric Nash bargaining solutions as defined by Harsanyi and Selten (1972) or Kalai (1977 b) are not anonymous. For more on this see Peters (1988). Hence they are not rationalizable by a symmetric metric. However, non-symmetric Nash solutions are Paretian and are therefore amenable to metric rationalizability.

5. Conclusion:

Let us briefly recaputulate what we have achieved in this paper. Here we deal with solutions to bargaining games. Our point of departure in this analysis has been that unanimity wherever possible is of fundamental importance in social decision making. Based on this premise we have attempted to characterize solutions to bargaining games. The approach we adopt is to first define a "Paretian" solution and then define the concept of rationalizability of a bargaining solution by a metric. The idea behind the second concept is to locate feasible outcomes whose comprehensive convex hull is closest to the given game and declare these outcomes to be the solution set. Our first major result establishes the equivalence of these two concepts. Thus, a wide class of solutions is "rationalizable" in terms of a metric.

Subsequently, we restrict our solution to be a welfare optimal rule and observe that after this restriction rationalizability in terms of a "monotonic" metric is possible. Under our present assumptions the family of non-symmetric Nash bargaining solutions all belong to the set of welfare optimal rules.

To extend our analysis we then invoke the notion of rationalizability by a symmetric metric and show that solutions satisfying this property also satisfy anonymity and metric respect for unanimity. This last concept is stronger than the concept of a "Paretian" solution and implies the existence of a metric on the space of outcomes (not merely a metric on the space of games), which once again respects the social ideal of implementing the unanimity rule if situations permit.

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