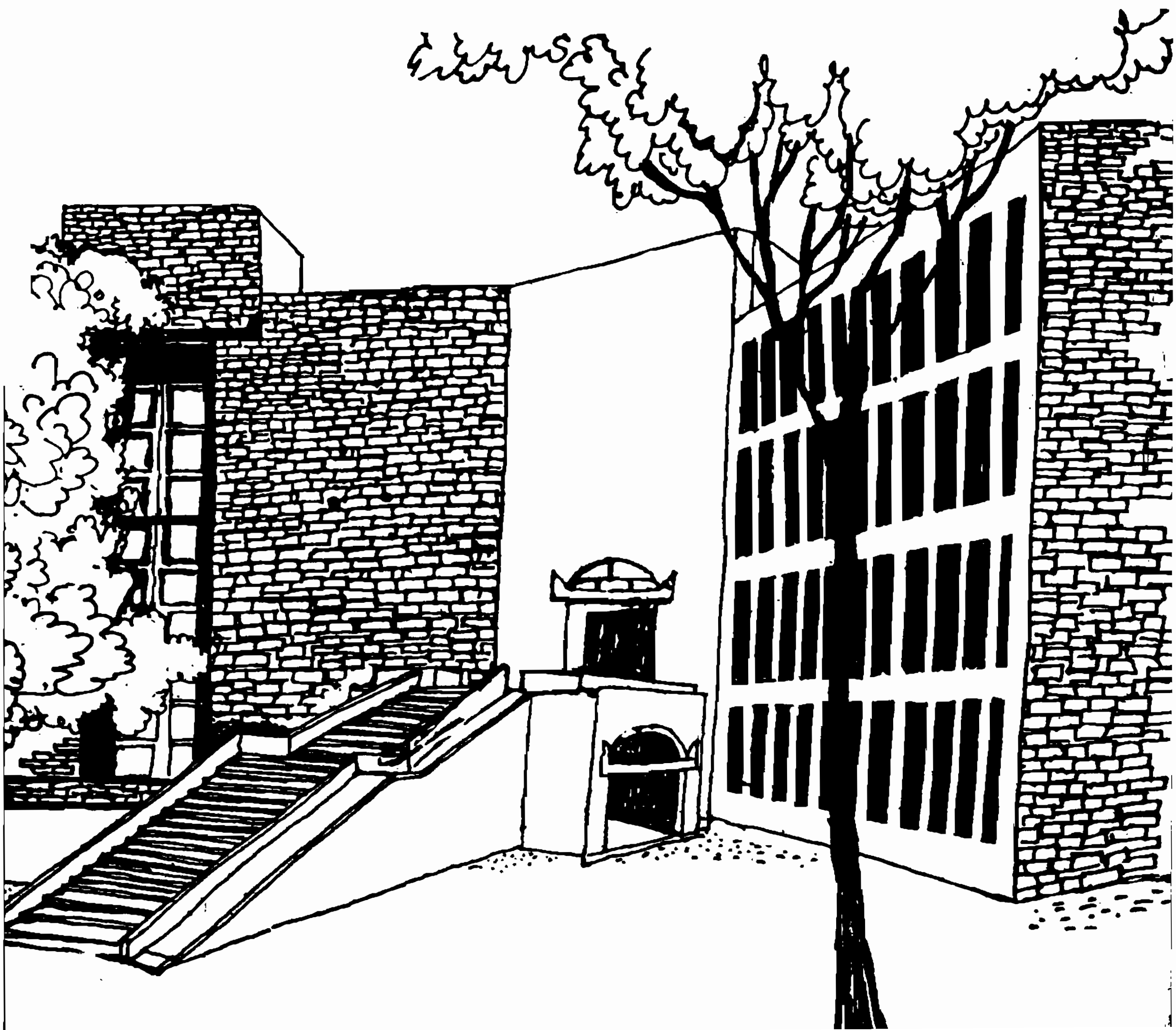
विद्याविनियोगाद्विकासः

IIM AHMEDABAD

Working Paper



VALID INEQUALITIES AND FACETS FOR THE CAPACITATED LOT-SIZING PROBLEM WITH CHANGEOVER COSTS

By

S. Trilochan Sastry

W P No. 987
November 1991

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380 015
INDIA

Valid Inequalities and Facets for the Capacitated Lot-Sizing Problem with Changeover Costs

[redacted]
S. Trilochan Sastry

Subject Category	Descriptive Phrase
1. Programming, integer	Facets for production planning
2. Inventory/production	Production planning with product changeover

Indian Institute of Management, Ahmedabad
India

Abstract

We study a scheduling problem with changeover costs and capacity constraints. The problem is NP-complete and combinatorial algorithms have not been very successful. We identify a general class of facets which subsumes as a special case all facets described earlier. We also develop a cutting plane based procedure for the dynamic version of the problem, and solve problem instances with up to 1200 integer variables to optimality without resorting to branch and bound procedures.

One of the key issues in scheduling is the effective allocation of shared resources to multiple products. A facility processing one product at a time might incur a changeover cost whenever it switches production from one product to another. For example, a facility for producing printed circuit boards might include a machine that places a set of components on to a board. Typically, the facility will produce different types of boards, each with a different set of components. If we switch from one product to another, we need to set up the machine for the new set of components and thus incur a fixed cost. The model also trades off the changeover and set up costs against the production and inventory holding costs.

Although this problem has received considerable attention, most of the earlier researchers have used solution methods that have not performed well in practice. They have focused on dynamic programming or Lagrangean relaxation methods. Since the problem is NP-hard, the running time of all dynamic programming methods increases exponentially with the number of time periods and products. Hence the use of these methods is limited to small problems. Lagrangean relaxation techniques have not been successful either; computational results with this solution approach have not been encouraging (see for example, Karmarkar, Kekre and Kekre (1987)).

On the other hand, recent research has suggested that solution methods based on a cutting plane approach might provide better results. Crowder and Padberg(1980) and Padberg and Hong(1980) were among the first researchers to successfully apply cutting plane approaches to NP-complete problems. They solved large travelling salesman problems using a combination of cutting planes and branch and bound techniques. Van Roy and Wolsey(1984) solved mixed integer 0-1 problems using strong valid inequalities as cutting planes and reported good computational results. Johnson, Kostreva and Suhl(1985) solved a strategic planning problem using a combination of cutting planes and branch and bound techniques. Van Roy and Wolsey(1987) developed another method for mixed 0-1 integer programming problems where they add strong cutting planes when necessary using an algorithm that automatically identifies a violated inequality and reformulates the problem. Groetschel and Holland(1984) implemented a strong cutting plane algorithm for the matching problem and reported that their algorithm performed better than the polynomial time algorithm of Edmonds(1965) for large scale problems. For more information, the reader might consult the survey of computational uses of polyhedral methods by Groetschel (1985) and Hoffman and Padberg(1987), the recent book by Nemhauser and Wolsey(1988), and the collection of articles in two specialized issues of the journal Mathematical Programming edited by Padberg and Rinaldi(1989). Another advantage of studying the polyhedral structure of this model is that it might form a substructure of a larger problem. Facets of the subproblem could provide strong valid inequalities for the larger problem. For example, Crowder, Johnson and Padberg (1983) showed that minimal cover cuts from single constraints of a zero-one program are effective in solving large scale 0-1 programs. Barany, Van Roy and Wolsey(1984a, 1984b) have also reported good computational results for a multi-item capacitated lot-sizing problem using facets of the single item problem.

We study the polyhedral structure of the dynamic, deterministic version of the problem. In the next section we review the research on this problem, and in section 2 we present the model and the integer programming formulation. We then describe valid inequalities and facets for the problem, solve the separation problem and present some computational results.

1. Literature

Research on the product cycling problem has mainly focused o the constant demand, infinite horizon case, and on dynami programming approaches to the general, dynamic, deterministi model. The constant demand, infinite horizon model assumes a cycli pattern of production that repeats over time. It is therefo appropriate when the demand pattern over a long range time horiz is known and is fairly stable. Elmaghraby(1978) calls this mod the Economic Lot Scheduling Problem (ELSP) and surveys meth proposed for solving it. Most of the methods use some variant the EOQ model to obtain a first approximation to the solution, then adjust the solution to conform to the capacity restrictio Although these methods perform well for the ELSP, they do perform well for the general case. See, for exam Bomberger(1966), Madigan(1968), Stankard and Gupta(1969), Hod (1970), Doll and Whybark(1973), Goyal(1973) and Haessler(1979

The general, dynamic, deterministic, case is NP-hard. The running time of all the dynamic programming methods therefore increases exponentially with the number of time periods and products. Glassey(1968) and Tenzer(1969) considered the special case of unit changeover costs. Mitsumori(1972) and Gascon and Leachman(1988) solve the general case with non-sequence dependent costs. Driscoll and Emmons(1977) presented an algorithm that allows sequence dependent changeover costs.

Researchers have also used other approaches to solve the problem. Geoffrion and Graves(1976) proposed a quadratic assignment approach for the multi-machine problem with sequence dependent changeover costs. Schrage(1982) suggested a linear programming based method. Karmarkar, Kekre and Kekre(1987) studied the single item version of the problem. Karmarkar and Schrage(1985) proposed a formulation that is similar to the one we study and solved it using Lagrangean relaxation. However, they report that the computational results are not very encouraging. Eppen and Martin(1987) reformulated the single-item version of the problem and a shortest path problem and report good computational results. Fleischmann(1990) studied the lot-sizing problem using a Lagrangean relaxation based method, and reported good computational results.

Very few researchers have used a polyhedral cutting plane approach for the product cycling problem. Wolsey (1989) used a cutting plane method that performed well for the uncapacitated version of our model. Magnanti and Vachani (1990) developed a solution technique based on cutting planes for the constant capacity case. This approach performed well on problems having up to 20 time periods and 5 products.

Vergin(1978), Graves(1980) and Leachman and Gascon(1988) have studied the stochastic version of the problem.

2.1 Problem Formulation

We now describe a single machine, multi-product, production planning model. Let T denote the finite time horizon over which the facilty is scheduled, P the number of products and d_i the demand in period i. We assume a constant capacity and follow a discrete production policy, i.e, we either do not produce at all or produce to capacity in each time period. This policy is reasonable when it is expensive to run the facility at less than full capacity, or when demand is high and the facility is capacity constrained. As shown in Magnanti and Vachani (1990), we can assume without loss of generality that capacity in each period is 1 unit and that demand is either 0 or 1.

We assume that the relevant costs for each product p in period i are the changeover cost K_{pi}, the fixed cost or the setup cost s_{pi}, and the inventory holding cost h_{pi}. Let z_{pi}, y_{pi} and w_{pi} denote the changeover, setup and production variables respectively. We assume that demands are nonnegative, initial production $w_{p0}=0$, and that there is no starting or ending inventory. Let $d_{ik} = \Sigma_{t=i}^{k} d_t$, denote the total demand in periods i through k. The Changeover Cost Scheduling Problem (CSP) can be formulated as follows:

$$\text{(CSP)}\quad \text{Minimize } U = \Sigma_{p=1}^{P} \Sigma_{i=1}^{T}\{h_{pi}w_{pi} + s_{pi}y_{pi} + K_{pi}z_{pi}\} \qquad (1)$$

subject to

$$\Sigma_{j=1}^{i} w_{pj} = \Sigma_{j=1}^{i} d_{pj} \qquad \text{for all } p,i \qquad (2)$$

$$\Sigma_{j=1}^{T} w_{pj} = n_p \qquad \text{for all } p \qquad (3)$$

$$w_{pi} - y_{pi} \leq 0 \qquad \text{for all } p,i \qquad (4)$$

$$z_{pi} + y_{p,i-1} - y_{pi} \geq 0 \qquad \text{for all } p,i \qquad (5)$$

$$\Sigma_{p=1}^{P} y_{pi} \leq 1 \qquad \text{for all } i \qquad (6)$$

$$w_{pi} \leq 1,\ y_{pi} \leq 1,\ z_{pi} \leq 1 \qquad \text{for all } p,i \qquad (7)$$

$$w_{pi},\ y_{pi},\ z_{pi} \geq 0 \qquad \text{and integer} \qquad (8).$$

Let CSP(L) denote the linear programming relaxation of CSP and let F(CSP) denote the set of feasible integer solutions for CSP. Constraints (2) and (3) are the demand constraints. Constraints (4) ensure that we can produce only if the machine is set up. Constraints (5) ensure that if the machine is set up for product p in period i (i.e., $y_{pi}=1$) but not in period i-1 then the changeover variable z_{pi} equals 1. Constraints (6) ensure that we produce only one product in any period. Magnanti and Vachani (1990) give a detailed formulation with all the underlying assumptions.

To facilitate our discussion we focus on the single product version of the problem. Although a dynamic programming algorithm will solve this problem in polynomial time, we have studied valid inequalities for the problem. There were two motivations for doing so. First, these inequalities can be generalized to the the multi-product problem (which is NP-complete) or for problem settings with arbitrary demands and varying production capacity over time. Second, we can obtain a better understanding of htepolyhedral structure of the problem.

Let SCSP denote the single product version of the problem and SCSP(L) the linear programming relaxation of SCSP. Since this model has only one product, we drop the subscript p. Since there is no need to produce in periods after t_n, we assume that t_n=T. The constraint $\Sigma_{i=1}w_i^{tk} \geq \Sigma_{i=1}d_i^{tk}$ implies the constraints $\Sigma_{i=1}w_i^{t} \geq \Sigma_{i=1}d_i^{t}$ for $t=t_k+1$ through $t_{k+1}-1$ because there is no demand between periods t_k+1 and $t_{k+1}-1$. Hence we can drop the demand constraints for all periods except the periods t_1, $t_2, \ldots, t_n$. If demand equals 1 in periods 1 through j, then $y_i=w_i=1$ for $1 \leq i \leq j$. Hence the problem reduces to finding a schedule for periods j+1 through T. Therefore, to exclude uninteresting cases, we assume that $t_1 \geq 2$.

3. Valid Inequalities

We describe a general class of valid inequalities for SCSP. To motivate the discussion, consider the following example. Assume that the costs K_i, s_i and h_i are constant. Let *K, s, and h* denote these costs respectivley. If *K* is large, and if $nt_k/T \geq k$ for k=1,....,n, then an optimal fractional solution for SCSP(L) is: $z_i=n/T$, $y_i=w_i=n/T$. We incur a fixed cost of nK/T instead of at least *K*. If we keep n constant and increase T, then the gap between the optimal solutions for SCSP(L) and SCSP can be made arbitrarily large. However, since we must produce at least once up to t_1, we must turn on the machine at least once before t_1. Hence $\Sigma_{i=1}^{t1}z_i \geq 1$ is a valid inequality which cuts offthe fractional solution. We replace the variable w_i by z_i in the demand constraint $\Sigma_{i=1}^{t1}w_i \geq 1$.

Consider the inequality $w_1+w_2+....+w_{t1} \geq 1$. Suppose we replace any term w_i by z_i. The following feasible solution violates the inequality: turn the machine on in period i-1, keep it on until period i, and produce in in period i. To satisfy demand beyond t_1, we produce in periods after t_1. However, if we replace w_{i-1} by z_{i-1} or y_{i-1}, the feasible solution satisfies the inequality. Of course if we use z_{i-1} in period i-1, we need to replace w_{i-2} by y_{i-2} or z_{i-2}.

Now consider the inequality $w_1+w_2+....+w_{t2} \geq 2$. Suppose we replace w_i by z_i for some $i \leq t_1$, and impose the condition that period i-1 contains y_{i-1} or z_{i-1}. However, since we need to produce twice to meet the demand up to t_2, the inequality is not valid: we can produce in periods i-1 and i. To obtain a valid inequality, we can replace w_{i-2} by y_{i-2} or z_{i-2} and w_{i-1} by $(y_{i-1}+z_{i-1})$. Then if we produce twice in the interval {i-2,..,i}, the lefthand side of the inequality equals at least two units, and the inequality is valid.

In general, whenever any period i^* contains the term z_{i*}, we need to compensate for it by introducing appropriate terms in periods prior to i. If we produce in i^*, we need to turn on the machine in some period $i' \leq i^*$, and keep it on in the interval $\{i', ..., i^*\}$. We want to ensure that if we produce r times in this interval, then for any feasible solution, the terms in periods i' through i^* add up to at least r units. Before we proceed, let us first define the following terms:

Demand interval j. Recall that t_j denotes the period at which the jth demand occurs. Demand interval j is the interval $\{t_{j-1}+1, t_{j-1}+2, ..., t_j\}$.

Contribution. We say that the sum of the terms on th lefthand side of the inequality associated with som sequence of machine operations (or some set of tim periods) is the *contribution* of that set of operations (o time periods). For example, suppose we turn the machine o in period 2 and keep it on until period 5, producing i periods 3 and 4. Suppose the inequality in the interva from period 2 through 5 has the form:

$$.....+w_2+y_3+z_4+w_5+.....$$

This set of operations contributes 1 unit, since $w_2 = 0$, = 1, $z_4 = 0$ and $w_5 = 0$.

We say that we "turn on" the machine in period if $z_i=1$ and thus incur the changeover cost in thi period.

3.1 Partition Inequalities.

We now describe one class of valid inequalitie which we call the partitioning inequalities (PI). Later, w generalize them to obtain another class of inequalities an show how we can tighten them to obtain facets.

Consider the demand up to time $t = t_q$. Le $L=\{1,....,t_q\}$, and let W, Y, Z, YZ and WZ be disjoin subsets of L that partition L: that is, $W \cup Y \cup Z \cup YZ \cup WZ = L$ In the following development we consider terms up to for $q \leq n$, and partition the interval $\{1,....,t_q\}$. We replac some term w_i by z_i. If $i \geq t_{q-1}+2$, we compensate for thi exchange by introducing the terms $y_t+z_{t+1}+....+z_{i-1}$ in perio

t through $i-1$ for some $t_{q-1}+1 \le t \le i-1$. If period i is in the $(j+1)$st demand interval for $j<q-1$, consider an interval $\{i,i+1,\ldots,i^*\}$. If $i^*+1-i \le q-j$, then we can produce at most i^*+1-i units in the interval $\{i,i+1,\ldots,i^*\}$. We ensure that if we produce r times in this interval, the contribution is at least r units. If $i^*+1-i>q-j$, then the contribution is at least $\min(r, q-j)$. Then each of the first q units of production contribute at least one unit, and the inequality is valid. We ensure this by including terms of the type y_t, $y_t+c_t z_t$ or $w_t+c_t z_t$ for integer $c_t \ge 1$ in periods $t \le i$. If $t_j+1 \le i \le t_{j+1}$, and $i \in Z$, the length of the interval preceding i and containing the terms $(y_t+c_t z_t)$ must be at least $q-j-1$. Moreover, the term w_{t-1} must not precede the term (y_t+z_t). Otherwise the following sequence of machine operations contributes less than $\min(r, q-j)$ units:turn on the machine in period $t-1$ and produce $q-j$ times in periods t through i^*.

Then we can write the following terms from periods i through $i+q-j$:

$$y_i + (y_{i+1}+z_{i+1}) + \ldots + (y_{i+q-j-1}+z_{i+q-j-1}) + z_{i+q-j}.$$

Similarly we can add terms of the type $c_t z_t$ and $y_t+c_t z_t$ for $c_t \ge 1$. For example, we can write

$$\ldots + y_i + (y_{i+1}+z_{i+1}) + (y_{i+2}+2z_{i+2}) + (y_{i+3}+3z_{i+3}) + 4z_{i+4} + 3z_{i+5} + 2z_{i+6} + z_{i+7} + \ldots$$

for the period i through $i+7$ if period i belongs to the $(j+1)$st demand interval and $q-j=4$. Then, if we produce in any four periods from i through $i+7$, the contribution is at least 4 units.

We obtain a further generalization by including terms of the type $w_t+c_t z_t$. We can write

$y_i+(y_{i+1}+z_{i+1})+(y_{i+2}+2z_{i+2})+(w_{i+3}+3z_{i+3})+(y_{i+4}+3z_{i+4})+4z_{i+5}+3z_{i+6}+2z_{i+7}+z_{i+8}$

for the previous example. Here also, any four periods in which we produce contribute at least 4 unit.

Let $Y=\{i(1), i(2), \ldots, i(m)\}$ and for $1 \le r \le m$ let $i''(r)=\min\{i'>i(r): i'\in Z$ and $i'+1 \in W \cup Y\}$. The periods $L=\{1,\ldots,t_q\}$ are partitioned into intervals $B(r)=\{i(r),\ldots,i''(r)\}$ for $1 \le r \le m$, and $W=L\backslash \cup_r B(r)$. All periods in the interval $B(r)$ belong to $Y \cup YZ \cup WZ \cup Z$. We write the inequality as follows.

$$\Sigma_{i\in W} w_i+\Sigma_{i\in Y} y_i+\Sigma_{i\in Z} c_i z_i+\Sigma_{i\in YZ}(y_i+c_i z_i)+\Sigma_{i\in WZ}(w_i+c_i z_i) \ge q$$

$$q = 1, \ldots, n. \quad (\text{PI})$$

where $c_i \ge 1$ is an integer.

Let c_i denote the coefficient of z_i, $m(i,i^*)$ the sum of the coefficients of y_t for $t \in Y \cup YZ$ and $n(i,i^*)$ the number of terms in Z in the interval $\{i,\ldots,i^*\}$. To ensure validity, we impose the following condition:

Compensation Condition. *For any two periods $i^* \in Z$ and $i \le i^*$ in interval $B(r)$ let $i''=\min\{i' \ge i: i' \notin WZ\}$ belong to demand interval $j+1$. Then $m(i,i^*)+c_i \ge \min\{q-j, m(i,i^*)+n(i,i^*)\}$.*

For example suppose we have demands in periods 4, 8, 9 and 10, and $q=4$. Then

$$w_1+y_2+(y_3+z_3)+(w_4+z_4)+(y_5+z_5)+(y_6+2z_6)+2z_7+z_8+w_9+w_{10} \ge 4 \quad (9)$$

is a valid inequality. Then $B(1)=\{2,\ldots,8\}$. Consider $i^*=8\in Z$. For $i=2$ or 3 in demand interval 1, $m(i,i^*)+c_i=q-j=4$. If $i=4$ then $i''=5$

is in demand interval 2, and $m(i,i^*)+c_i=q-j=3$. If $i=5$ or 6 then $m(i,i^*)+c_i=q-j=3$. If $i=7$ then $c_i=n(i,i^*)=2$.

We state the following proposition. The proof follows from proposition 2.

Proposition 1. *Inequality (PI) is valid if it satisfies the compensation condition.*

3.2 Skip Inequalities

We can generalize inequalities (PI) to obtain another class of inequalities, which we call the 'skip' inequalities (SI). For an inequality extending up to t_q we say that a time period $i \le t_q$ is skipped if $i \notin W \cup Y \cup Z \cup YZ \cup WZ$. Let S denote the set of all time periods skipped up to t_q. Then for $L=\{1,....,t_q\}$, $W \cup Y \cup Z \cup YZ \cup WZ \cup S=L$. Let $b=|S|$ denote the number of skipped time periods. The inequality is of the form:

$$\Sigma_{i \in W} w_i + \Sigma_{i \in Y} y_i + \Sigma_{i \in Z} c_i z_i + \Sigma_{i \in YZ}(y_i + c_i z_i) + \Sigma_{i \in WZ}(w_i + c_i z_i) \ge q-b \quad \text{(SI)}$$

$$b \le q, \ q=1,....,n.$$

We need to impose some conditions to ensure that these inequalities are valid. Suppose period i is in the (j+1)st demand interval. Let W(j) denote the minimum contribution of the terms 1 through t_j in any feasible solution. Let b_j denote the number of periods skipped up to t_j. We want to ensure that for $k \le j$, the periods 1 through t_k contribute at least $\max(k-b_k, 0)$. Hence the terms 1 through t_j contribute at least $W(j)=\max((k-b_k: k \le j), 0)$. The remaining terms from period t_j+1 through t_q must contribute $q-b-W(j)$. For any period i in demand interval $j+1$ let $u(i)=q-b-\max((k-b_k:k\le j), 0)$. If $Y=\{i(1), i(2),....i(m)\}$, we partition the interval$\{1,....t_q\}$ into intervals $B(r)=\{i(r),....,i''(r)\}$ for $1\le r\le m$. All periods in interval $B(r)$ belong to $Y\cup YZ\cup WZ\cup Z$, and $W\cup S=L\setminus\cup_r B(r)$. We impose a condition similar to the compensation condition.

Skip Condition. *For any two periods $i^*\in Z$ and $i\le i^*$ in interval $B(r)$, let $i''=\min\{i'\ge i: i'\notin WZ\}$. Then $m(i,i^*)+c_i\ge\min\{u(i''), m(i,i^*)+n(i,i^*)\}$.*

For example suppose we have demands in periods 4, 8, 9, 10 and 12 and q=5. We skip periods 8 and 9, and hence b=2. Then

$$w_1+y_2+(y_3+z_3)+(w_4+z_4)+(y_5+z_5)+2z_6+z_7+w_{10}+y_{11}+z_{12} \ge 3 \qquad (10)$$

is a valid inequality. $B(1)=\{2,....,7\}$ and $B(2)=\{11, 12\}$. For $i^*=7\in Z$, and $i=2$ or 3, $m(i,i^*)+c_i=u(i)=4$. For $i=4$, $i''=5$ $m(i,i^*)+c_i=u(i'')=2$.

Proposition 2. *Any inequality (SI) is valid if it satisfies the skip condition.*

Proof.

We show that any feasible solution satisfies the inequality. We use an inductive argument to show that if we produce $r(i)$ times up to period i in periods not in S, then periods 1 through i contribute at least $\min(q-b, r(i))$ units.

If we produce only in periods not in Z, then we are done. Otherwise, let $i^*=\min\{i\le t_q: w_i=1$ and $i\in Z\}$. We must turn on the machine in some period $i\le i^*$ and keep it on from period i through i^*, and hence $z_i=y_i=y_{i+1}=....=y_{i*-1}=y_{i*}=1$. Let $i"=\min\{i'\geq i: i'\notin WZ\}$. If i and i^* are both in the same interval B(r), then from the skip condition $m(i,i^*)+c_i\geq\min\{u(i"), m(i,i^*+n(i,i^*)\}$ for period i" in demand interval j+1. The coefficient c_i and the periods i" through i^* contribute at least $\min\{u(i"), m(i,i^*)+n(i,i^*)\}$ units. Since $i"\geq t_j+1$, we must produce at least j times up to period i"-1. For any $k\leq j$, we produce at least $\max(k-b_k, 0)$ times in periods not in S in the interval $\{1,....,t_k\}$. Hence we produce at least $W(j)=\max((k-b_k: k\leq j), 0)$ times in periods not in S in the interval $\{1,....,t_j\}$. Hence $r(i"-1)\geq W(j)$. Further, the periods 1 through i"-1 contribute at least r(i"-1) units if $z_i=0$, and at least $r(i"-1)+c_i$ units if $z_i=1$ because prior to i" we produce in periods in $W\cup Y\cup WZ\cup YZ\cup S$. If $u(i")\leq m(i,i^*)+n(i,i^*)$ then $m(i,i^*)+c_i\geq u(i")$ and periods 1 through i^* contribute at least $W(j)+u(i")=q-b$ units. If $m(i,i^*)+n(i,i^*)<u(i")$, then for any period t in the interval $\{i",....,i^*\}$, if $t\notin Y\cup YZ\cup Z$, then $t\in W\cup Y\cup S$. Hence if $r(i",i^*)$ denotes the number of times we produce in this interval in periods not in S, then the periods i" through i^* contribute at least $r(i",i^*)$ units. Hence periods 1 through i^* contribute at least $r(i^*)$ units. If $i^*\in B(r)$ and $i\notin B(r)$, then $i<i(r)$ and from the skip condition $m(i(r),i^*)\geq u(i)$ and hence periods i(r) through i^* contribute at least u(i) units. Periods 1 through i(r)-1 contribute at least W(j) units if i is in demand interval j+1. Hence periods 1 through i^* contribute at least $W(j)+u(i)=q-b$ units.

Assume that for some period i and all $i'\leq i$, periods 1 through i' contribute at least $\min(q-b, r(i'))$ units. If we do not produce in period $i+1$, or if $i+1 \in S$, then $r(i)=r(i+1)$ and the assumption holds for period $i+1$ as well. If we produce in period $i+1$ and $i+1 \in W \cup Y \cup WZ \cup YZ$, then $r(i+1)=r(i)+1$ and the periods 1 through $i+1$ contribute at least $\min(q-b, r(i+1))$ units. If $i+1 \in Z$, then we must turn the machine on in some period $i' \leq i+1$, and keep it on from period i' through $i+1$. If $i' \notin WZ$ then let $i''=i'$; otherwise let i'' denote the first period after i' for which $i'' \in WZ$. If $i', i+1 \in B(r)$ then $c_{i'}+m(i',i+1) \geq \min\{u(i''), m(i',i+1)+n(i',i+1)\}$ and as shown earlier, if i' is in demand interval $j'+1$, periods 1 through $i'-1$ contribute at least $\min\{q-b, r(i'-1)\}$ units, $r(i'-1) \geq W(j')$, and periods i' through $i+1$ contribute at least $\min\{u(i''), r(i',i+1)\}$ units. Hence periods 1 through $i+1$ contribute at least $\min\{q-b, r(i+1)\}$ units. If $i' \notin B(r)$, then as shown earlier, periods 1 through $i+1$ contribute at least $q-b$ units.

Since we skip only b periods up to t_q, $r(t_q) \geq q-b$. It follows that periods 1 through t_q contribute at least $q-b$ units and hence the inequality is valid.

□

Corollary 1. In the partition inequalities the number of skipped periods b, up to t_q equals 0. Hence for any period i in demand interval $j+1$, $W(j)=j$ and $u(i)=q-j$. Substituting these values in the preceding proof, it follows that proposition 1 is true.

□

We now show that the partitioning inequalities of Magnanti and Vachani (which we call inequalities (pi)) can be viewed as a special case of our inequalities (PI). Inequalities (pi) are of the form:

$$\Sigma_{i=1}^{j-1} w_i + \Sigma_{i\in W} w_i + \Sigma_{i\in Y} y_i + \Sigma_{i\in Z} z_i \geq q$$

where $t_{q-1}+1 \leq j \leq t_q$, and the sets W, Y and Z do not include periods up to j-1.

The inequalities (pi) confine periods $i\in Z$ to the interval $\{t_{q-1}+2, \ldots, t_q\}$, whereas inequalities (PI) allow $i\in Z$ anywhere in the inequality. The conditions imposed on (pi) were the following:

i) period $j\notin Z$, and

ii) if $i\in W$, then $i+1\notin Z$.

Since we include the periods up to j-1 in W, condition (ii) implies (i), which is the same as condition 1 of (PI): that is, if i lies in the interval $\{t_{q-1}+2, \ldots, t_q\}$ and $i\in Z$, then $i-1\notin W$.

4. Facets

It is possible to tighten the inequalities described if we impose certain conditions in addition to the skip and compensation conditions. We describe these conditions and show that the inequalities are facets.

We show that if $i\in Y$, then for some $i^*\geq i$, $i^*\in Z$, $m(i,i^*)=u(i)$. Otherwise, from the skip condition $m(i,i^*)>u(i)$, and we can replace y_i by w_i and obtain a tighter valid inequality. Similarly, if $i\in(Z\cup YZ\cup WZ)\cap B(r)$ then $c_i=n(i,i^*)$ for some $i^*\in Z\cap B(r)$ such that $m(i,i^*)+c_i\leq u(i)$. Otherwise assume without loss of generality that for any $i'\in Z\cap B(r)$ such that $m(i,i')+c_i\le u(i)$, $c_i-n(i,i^*)\le c_i-n(i,i')$. We can reduce the coefficient c_i until $c_i=n(i,i^*)$ or $c_i=0$. If $c_i=0$ we can replace y_i by w_i. In all cases we obtain a tighter valid inequality satisfying the skip condition. Hence we need the following condition.

Condition 1. *If $i\in(Y\cup YZ\cup WZ\cup Z)\cap B(r)$ then $c_i=n(i,i^*)$ for some $i^*\in Z\cap B(r)$ such that $m(i,i^*)+c_i\le u(i)$.*

So far we have assumed that the coefficient of y_i is either 0 or 1. Suppose for some i the coefficient is $c'_i>1$. Let $m'(i,i^*)$ denote the number of periods in $Y\cup YZ$ in the interval $\{i,....,i^*\}$. We then modify the skip condition and say that for $i,i^*\in B(r)$, $i\le i^*$ and $i^*\in Z$, $m(i,i^*)+c_i\ge\min(u(i),\ m'(i,i^*)+n(i,i^*))$. If $u(i)\ge m'(i,i^*)+n(i,i^*)$, we can produce at most $m'(i,i^*)+n(i,i^*)$ times in periods in $Y\cup YZ$ in the interval $\{i,....,i^*\}$, and if we turn on the machine in period i and keep it on until i^*, then this interval contributes at least $m'(i,i^*)+n(i,i^*)$ units. We can use arguments similar to those in proposition 2 and show that the inequality is valid. Similarly, we modify condition 1 and say that for $i\in B(r)\cap(Y\cup YZ)$, $m(i,i^*)+c_i=m'(i,i^*)+n(i,i^*)$ for some $i^*\in B(r)\cap Z$, and $m(i,i^*)+c_i\le u(i)$. Since $i\notin Z$, $n(i,i^*)=n(i+1,i^*)$. From the modified skip condition

$$\begin{aligned}m(i+1,i^*)+c_{i+1}&\ge m'(i+1,i^*)+n(i+1,i^*)\\&=m'(i,i^*)-1+n(i,i^*)\\&=m(i,i^*)+c_i-1\\&=m(i+1,i^*)+c'_i+c_i-1.\end{aligned}$$

Hence $c_{i+1} \ge c'_i - 1$. We can then use the inequality $y_i + z_{i+1} \ge y_{i+1}$, and replace $c'_i y_i + c_{i+1} z_{i+1}$ by $y_i + (c'_i - 1) y_{i+1} + (c_{i+1} - c'_i + 1) z_{i+1}$ and obtain a tighter valid inequality. Hence the coefficient of y_i for $i \in Y \cup YZ$ is 1. Similarly, if the coefficient of w_i for $i \in W \cup WZ$ is greater than one, we can simply reduce it to 1 because the skip condition is still satisfied.

Condition 2. *The coefficient of y_i for $i \in Y \cup YZ$ and of w_i for $i \in W \cup WZ$ is 1.*

Suppose $i \in Z$ for some $i \le q-b$. If $i=1$, we can replace $c_i z_i$ by $c_i y_i$ and obtain a tighter valid inequality. Hence period $1 \notin Z$. If $i \ge 2$, then from the skip condition, the sum of the coefficient of z_t and the coefficients of $y_{i'}$ for $t \le i' \le i$ is at least $i+1-t$ for all $t \le i$. The coefficients of $y_{i'}$ are at most 1. Hence the first i terms are:

$$(c_1 z_1 + y_1) + (y_2 + c_2 z_2) \ldots . + (y_{i-1} + c_{i-1} z_{i-1}) + c_i z_i,$$

where c_t is the coefficient of z_t for $t \le i$, and $c_2 \ge 1$. We can use the inequality $y_1 + z_2 \ge y_2$ to replace $y_1 + z_2$ by y_2 and obtain a tighter valid inequality. However the coefficient of y_2 is then 2, which contradicts condition 2. Using similar arguments we can establish the following condition.

Condition 3. *If $i \in Z$, then $i' \in Y$ for some $i' < i$. In particular, period $i \notin Z$ for $i \le q-b$.*

We now consider period $i \in Y$. If period $i+1 \in W \cup Y$ or is skipped, we can replace y_i by w_i and still have a valid inequality. Since $y_i \ge w_i$, the inequality is dominated. Similarly, if $i \in YZ \cup WZ$ and $i+1 \in W \cup Y$ or is skipped, we can replace the term $(y_i+c_iz_i)$ or $(w_i+c_iz_i)$ in period i by y_i or w_i. Hence we need the following condition:

Condition 4. *If $i \in Y \cup YZ \cup WZ$ then $i+1 \in YZ \cup WZ \cup Z$. In particular, $t_q \notin Y \cup YZ \cup WZ$.*

We next consider the location of the skipped periods. For example, suppose all the skipped periods are in the last demand interval q and the number of skipped periods is $b \geq 1$. The sum of the terms in the inequality up to t_{q-1} for any feasible solution is $W(q-1)=\max((k-b_k: k \leq q-1), 0)=q-1 \geq q-b$. We can therefore drop all variables with indices greater than t_{q-1} from the inequality. It is therefore clear that we must impose some condition on the location of the skipped periods. If the number of skipped periods after period t_j is q-j or more, it means that the right hand side is at most q-(q-j). Moreover the number of skipped periods up to t_j is at most b-(q-j), and the sum of the terms in the inequality up to t_j is at least j-(b-(q-j)) = q-b. Therefore, we can simply drop all variables with indices more than or equal to t_j+1 and still have a valid inequality.

Therefore we impose the following condition:

Condition 5. *The number of periods skipped from t_j+1 through t_q is strictly less than $q-j$, for $j = 0, \ldots, q-1$.*

Suppose we skip b_j periods up to t_j, that $b_j \leq j$, and that $i \in W$ for $i \leq t_j$, or i is skipped. Let us denote the terms in the inequality (SI) after t_j as (SI'). Then the inequality (SI) can be written as a linear combination of the following inequalities:

$$\sum_{i=1}^{t_j} w_i + (SI') \geq q-b+b_j \text{ for } b_j \leq j$$

and $1 \geq w_s$ for each skipped period up to t_j. We show that the first inequality is valid. The terms up to t_j contribute $r(t_j)-b_j$ units in the original skip inequality where $r(t_j)$ is the number of times we produce up to t_j. Therefore the terms after t_j contribute at least $q-b-r(t_j)+b_j$ units. In the first inequality obtained by replacing all skipped periods i by w_i, the terms up to t_j then contribute $r(t_j)$ units. Hence the inequality is valid. We therefore impose the following condition:

Condition 6. *If $0 < b_j \leq j$ periods are skipped up to period t_j, then $i \in Y \cup Z \cup YZ \cup WZ$ for some $i \leq t_j$.*

Consider the situation when $q=n$. If $Y \cup Z \cup YZ \cup WZ = \emptyset$, then by the previous condition no periods are skipped. Hence the inequality reduces to $\Sigma_{i=1}^{T} w_i \geq n$. But this is implied by $\Sigma_{i=1}^{T} w_i = n$. Hence we impose the condition that if $q=n$, then $Y \cup Z \cup YZ \cup WZ \neq \phi$.

For q=n, show that $|Y|=1$. We first argue that $|Y|\geq 1$. If $Z=\phi$, then from condition 1 $Y \cup YZ \cup WZ = \phi$. Hence $Y \cup Z \cup YZ \cup WZ = \phi$, which contradicts the previous condition. Therefore $Z \neq \phi$. Further, from condition 3, period $i \notin Z$ for $i \leq q-b$. Hence from condition 1, period $1 \notin WZ \cup YZ$. Therefore, the first $i \in Z$ (i.e., $i \in Z$: $i \leq i'$ for all $i' \in Z$) is preceded by a sequence in $YZ \cup WZ$, which in turn is preceded by $i \in Y$. Therefore $|Y| \geq 1$.

Suppose that $|Y|>1$ and that $S=\emptyset$. By condition 4, each $t \in Y$ is followed by a sequence of periods in $YZ \cup WZ \cup Z$. Let t^* denote the last period in Z in this sequence such that $t^*+1 \notin YZ \cup WZ$, $SI(t,t^*)$ denote the terms in periods t through t^*, and $(SI)\backslash SI(t,t^*)$ denote the rest of the terms in the inequality. We can write this inequality (SI) as the sum of the following inequalities (and hence it cannot be a facet):

$n = \Sigma_{i=1}^{T} w_i$ written $|Y|-1$ times.

$\Sigma^{T}_{i=1,\ i \notin \{t, \ldots t^*\}} w_i + (SI(t,t^*)) \geq n$ for each $t \in Y$.

Hence we impose the following condition:

Condition 7. *If $q=n$, then $Y \cup Z \cup YZ \cup WZ \neq \phi$, and if $S=\emptyset$, then $|Y| = 1$.*

Finally, we consider special cases. Suppose $|Y|\geq 1$ and $B(1)=\{1,\ldots,t_j\}$ for some $1\leq j\leq q$. For example, if $t_1=3$, $t_2=5$, $t_3=8$ and $q=3<n$, then

$$y_1+(y_2+z_2)+(y_3+2z_3)+2z_4+z_5+w_6+y_7+z_8 \geq 3$$

is a valid inequality. However we can obtain it as the sum of the following inequalities

$$y_1+z_2+z_3 \geq 1$$

and $$y_2+(y_3+z_3)+2z_4+z_5+w_6+y_7+z_8 \geq 2,$$

and hence it cannot be a facet. Notice that for all $i\in B(1)$, $i<i''(1)$ and corresponding $i^*\in Z$ for which $c_i=n(i,i^*)$, we have $m(i,i^*)+c_i=u(i)$. Using similar arguments we can establish the following condition.

Condition 8. *If $B(1)=\{1,....,i''(1)\}$ for $i''(1)$ in demand interval $j+1$, $0\leq j\leq q-1$ then $c_i=n(i,i^*)$ and $m(i,i^*)+c_i<u(i)$ for some $i(1)<i<i''(1)$ and $i^*\in Z\cap B(1)$.*

We impose one more condition. Suppose we have demands in two consecutive periods, t_q and $t_{q+1}=t_q+1$. If $Y\cup Z\cup YZ\cup WZ=\phi$ for an inequality extending up to t_q, then the inequality reduces to $\sum_{i=1}^{t_q} w\geq q$. But this is implied by the constraints $\sum_{i=1}^{t_q+1} w\geq q+1$ and $1\geq w_{t_q+1}$.

Hence we impose the following condition:

Condition 9. *For any inequality extending up to t_q if $t_{q+1}=t_q+1$, then $i\in Y\cup Z\cup YZ\cup WZ\neq\emptyset$ for some $i<t_q$.*

Using conditions 1 through 9 we establish three lemmas. Let c_i denote the coefficient of z_i.

Lemma 1. If $u(i+1)=u(i)-1$ then $i\notin Y$ and if $i\in WZ$ then $c_i=n(i,i^*)$ for some $i^*\in Z$ such that $m(i,i^*)+c_i\leq u(i)-1$.

Proof. Suppose $i\in Y$ and $i^*=\min\{i'\geq i:\ i'\in Z\}$. Then since $c_i=0$, from the skip condition, $m(i,i^*)\geq u(i)$ and from condition 1 $m(i,i^*)=u(i)$.

Since $i \in Y$, $m(i+1,i^*)=m(i,i^*)-1$, and since $u(i+1)=u(i)-1$, $m(i+1,i^*)=u(i+1)$. However from condition 4, since $i \in Y$, $i+1 \in WZ \cup YZ \cup Z$ and hence $c_{i+1} \geq 1$. Therefore $c_{i+1} > n(i+1,i')=0$ for all i' such that $m(i+1,i')+c_{i+1} \leq u(i+1)$. This contradicts condition 1 for period $i+1$, and hence $i \notin Y$. We can tighten the inequality by replacing y_i by w_i, and replacing the term $y_{i+1}+c_{i+1}z_{i+1}$ (or $w_{i+1}+c_{i+1}z_{i+1}$) by y_{i+1} (or w_{i+1}).

Suppose $i \in WZ$. From condition 1, $c_i=n(i,i^*)$ for some $i^* \in Z$ such that $m(i,i^*)+c_i \leq u(i)$. If $c_i > n(i,i')$ for all $i' \in Z$ such that $m(i,i')+c_i \leq u(i)-1$, then $c_i=n(i,i^*)$ for $i^* \in Z$ such that $m(i,i^*)+c_i=u(i)$. Without loss of generality assume that $i^*=\min\{i': i' \geq i, i' \in Z$ and $m(i,i')+c_i=u(i)\}$. Since $i \in WZ$, $m(i+1,i^*)=m(i,i^*)$ and since $u(i+1)=u(i)-1$, $m(i+1,i^*)=u(i+1)$. From condition 4, since $i \in WZ$, $i+1 \in WZ \cup YZ \cup Z$ and hence $c_{i+1} \geq 1$. Therefore $c_{i+1} > n(i+1,i')=0$ for all i' such that $m(i+1,i')+c_{i+1} \leq u(i+1)$. This contradicts condition 1 and hence $i \notin WZ$. We can tighten the inequality by replacing $w_i+c_iz_i$ by w_i, and replacing the term $y_{i+1}+c_{i+1}z_{i+1}$ (or $w_{i+1}+c_{i+1}z_{i+1}$) by y_{i+1} (or w_{i+1}).

□

Lemma 2. If $i \in Y$ then $c_{i+1}=1$, if $i \in YZ$ and $u(i+1)=u(i)-1$ or if $i \in WZ$ then $c_{i+1}=c_i$, and if $i \in YZ$ and $u(i+1)=u(i)$ then $0 \leq c_{i+1}-c_i \leq 1$.

Proof. Suppose $i \in Y$ and let $i^*=\min\{i'>i: i' \in Z\}$. As shown earlier, $m(i,i^*)=u(i)$. Moreover since $i \in Y$, $m(i+1,i^*)=m(i,i^*)-1$. From condition 4, $i+1 \in YZ \cup WZ \cup Z$ and hence $c_{i+1} \geq 1$. From condition 1, $c_{i+1}=n(i+1,i')$ for some $i' \in Z$ and $m(i+1,i')+c_i \leq u(i+1)$. From lemma 1, since $i \in Y$, $u(i+1)=u(i)$. It follows that $i'=i^*$ and $c_{i+1}=n(i,i^*)=1$.

If $i \in YZ \cup WZ$ then from condition 1 $c_i=n(i,i^*)$ for some $i^* \in Z$ such that $m(i,i^*)+c_i \leq u(i)$. From the skip condition either

$m(i+1,i^*)+c_{i+1}\geq u(i+1)$ or $c_{i+1}\geq n(i+1,i^*)$. If $c_{i+1}\geq n(i+1,i^*)$, then since $i\notin Z$, $n(i,i^*)=n(i+1,i^*)$ and hence $c_{i+1}\geq c_i$. Suppose $m(i+1,i^*)+c_{i+1}\geq u(i+1)$. If $i\in YZ$, then $m(i,i^*)=m(i+1,i^*)+1$. Therefore $m(i+1,i^*)+c_{i+1}\geq u(i+1)\geq u(i)-1\geq m(i+1,i^*)+c_i$. Hence $c_{i+1}\geq c_i$. If $i\in WZ$ then $m(i,i^*)=m(i+1,i^*)$, and from lemma 1, $m(i,i^*)\leq u(i)-1$. Hence $m(i+1,i^*)+c_{i+1}\geq u(i+1)\geq u(i)-1\geq m(i+1,i^*)+c_i$ and therefore $c_{i+1}\geq c_i$.

Similarly, for period i+1 $c_{i+1}=n(i+1,i^0)$ for some $i^0\in Z$ and $m(i+1,i^0)+c_{i+1}\leq u(i+1)$. Since $c_{i+1}\geq 1$, $m(i+1,i^0)<u(i+1)$. If $i\in YZ$ and $u(i)=u(i+1)+1$ or if $i\in WZ$ then $m(i,i^0)<u(i)$. Hence from the skip condition $m(i,i^0)+c_i\geq \min\{u(i),\ m(i,i^0)+n(i,i^0)\}$. It is easy to see that therefore

$$\begin{aligned} m(i+1,i^0)+c_i &\geq \min\{u(i+1),m(i+1,i^0)+n(i+1,i^0)\} \\ &= m(i+1,i^0)+n(i+1,i^0)\} \\ &= m(i+1,i^0)+c_{i+1}. \end{aligned}$$

Hence $c_{i+1}=c_i$.

Suppose $i\in YZ$, $u(i)=u(i+1)$ and that $c_i=n(i,i^*)$ for some $i^*\in Z$ and $m(i,i^*)+c_i<u(i)$. Then $i'\notin Z$ for any $i'>i^*$ such that $m(i,i')+c_i\leq u(i)$. Otherwise the skip condition is not satisfied. Moreover, for period i+1 we need that $c_{i+1}=n(i+1,i')$ for some $i'\in Z$ and $m(i+1,i')+c_{i+1}\leq u(i+1)=u(i)$. The only period satisfying this is $i'=i^*$ and hence $c_{i+1}=c_i$. If on the other hand $m(i,i^*)+c_i=u(i)$, then since $c_{i+1}\geq c_i$, $m(i+1,i^*)+c_{i+1}\geq u(i+1)-1$. From condition 1, $c_{i+1}=n(i+1,i')$ for some $i'\in Z$ such that $m(i+1,i')+c_{i+1}\leq u(i+1)$. Hence either $i'=i^*$ or $i'=i^*+1$. If $i'=i^*$ then $c_{i+1}=c_i$ and if $i'=i^*+1$ then $c_{i+1}=c_i+1$.

□

Using similar arguments we can establish that

Lemma 3. If $i \in Z$, then $c_i \le u(i)$ and $0 \le c_i - c_{i+1} \le 1$.

□

The following theorem shows that the conditions are not only necessary, but also sufficient to define facets.

Theorem 1. *The inequalities (SI) and (PI) are facets of* $\mathbf{C}$ *if and only if W, Y, Z, S, WZ and YZ satisfy the conditions 1 through 9.*

Proof.

We have already established the necessity of the conditions.

Sufficiency of the conditions .

Let $C^* = \{(w, y, z) \in \mathbf{C} : (w, y, z)$ satisfies (SI) at equality$\}$. To show that (SI) is a facet, we let $\alpha w + \beta y + \gamma z = \delta$ represent an arbitrary equation that is satisfied by all $(w, y, z) \in C^*$. We show that $\alpha w + \beta y + \gamma z = \delta$ must be a linear combination of

$$\Sigma_{i \in W} w_i + \Sigma_{i \in Y} y_i + \Sigma_{i \in Z} c_i z_i + \Sigma_{i \in YZ} (y_i + c_i z_i) + \Sigma_{i \in WZ} (w_i + c_i z_i) = q - b.$$

and the only equality in SCSP which is $\Sigma_{i=1}^{T} w_i = n$.

We first define two solutions in C^* which we call the $k(i^*)$ and the k^0 solutions. We shall use them as the basis for generating alternate solutions in C^*. From the skip condition and from condition 1, if $i^* \in Z$ then $m(i, i^*) + c_i = u(i)$ for some $i \le i^*$. Suppose i is in demand interval j+1 for $0 \le j \le q-1$. Then we define the period $k(i^*)$ as follows:

$$k(i^*) - c = \max(j' - b_{j'} : j' \le j)$$

where c is the number of periods skipped up to perio $k(i^*)$ and b_j is the number of periods skipped up to t_j

If $k(i^*)$ satisfies the equation, and $k(i^*)+1$ is also skipped, then $k(i^*)+1$ also satisfies the equation. We choose the minimum of all such values. We produce in periods 1 through $k(i^*)$ and in all the skipped periods. This contributes $W(j)=\max(j'-b_{j'}:j'\leq j)$ units. We then turn on the machine in period i, keep it on until period i^* and produce in periods i through i^* in periods belonging to $Y \cup YZ$. From conditions 1 and 2, this contributes $u(i)=q-b-W(j)$ units. For demand beyond t_q we produce in periods after t_q. Using conditions 5 and 6 we can establish that the $k(i^*)$ solution is feasible. It is easy to verify that the solution is in C^*. We refer to the production in periods 1 through $k(i^*)$ as the *first sequence*, and the production in i through i^* in periods belonging to $Y \cup YZ$ as the *second sequence.*

If we let $k^0=q-b+c$, where c is the number of periods skipped up to i^0 and produce in periods 1 through i^0 and in all skipped periods, we obtain the k^0 solution.

1) $\gamma_i=0$ for all $i \notin (YZ \cup WZ \cup Z)$.

Consider the k^0 solution. For any $i \notin (YZ \cup WZ \cup Z)$, $2 \leq i \leq t_q$, we can set $z_i = 0$ or 1 and still be in C^*. Hence $\gamma_i = 0$. For any skipped period i, we can shift production from i to i^* for some $k(i^*)$ solution. This is always possible since from condition 6 $Y \cup YZ \cup WZ \cup Z \neq \emptyset$ if we skip any period, and from condition 4, if $Y \cup YZ \cup WZ \neq \emptyset$, then $Z \neq \emptyset$. Further, since $t_1 \geq 2$, the modified solution is still feasible. We can then set $z_i=0$ or 1, and hence $\gamma_i=0$. For $i=1$, $i \in W \cup Y$, we can modify the k^0

solution by shifting production by one period and produce from period 2 onwards. Whether $z_1=0$ or 1 the solution is in C^*, and hence $\gamma_1=0$.

If $i>t_q$, then if $(T-t_q)>(n-q)$, we can choose a solution in C^* with $z_i=y_i=w_i=0$. Otherwise, $t_{q+1}=t_q+1$ and from condition 9, $Y\cup YZ\cup WZ\cup Z\neq\emptyset$, and from condition 4, $Z\neq\emptyset$. Hence we can choose a $k(i^*)$ solution for some $i^*\in Z$. We can modify it by shifting production from period i to i^*, and hence $z_i=y_i=w_i=0$. We can set $z_i=0$ or 1, and still be in C^*. Hence $\gamma_i=0$.

Note that for any skipped period, we can modify the $k(i^*)$ solution by shifting production from i to i^*, and then set $z_i=y_i=0$ or 1. Hence $\beta_i=0$ as well.

2) $\beta_i=0$ for all $i\notin(YZ\cup Y)$.

Suppose $i>t_q$. We have shown that we can always choose a feasible solution in C^* with $z_i=y_i=w_i=0$. Another solution is to set $z_i=y_i=1$. We have shown that $\gamma_i=0$ for $i>t_q$. Therefore $\beta_i=0$ for $i>t_q$.

Suppose $i\le t_q$. By assumption $i\notin Y\cup YZ$. If i is a skipped period, we showed earlier that $\beta_i=0$. If $i\in Z$ then $Y\cup Z\cup YZ\cup WZ\neq\phi$, and there is a $k(i^*)$ solution for $i^*=i$. We can set $y_i=0$ or 1, and hence $\beta_i=0$. If $i\in W$ and $w_i=0$, then whether $z_i=y_i=0$ or 1, period i does not contribute to the left hand side. Hence $\beta_i=0$. If $i\in W$ and $w_i=1$, then i is in the first sequence of productions from 1 through $k(i^*)$, and therefore $i\le k(i^*)$. Hence we can shift production from i to the first available period i' with $w_{i'}=0$, and still have a feasible solution in C^*. We can then set $z_i=y_i=0$ or 1, and therefore $\beta_i=0$. If $i\in WZ$, then from condition 4 $i^*\in Z$ and $i^*-1\notin Z$ for some $i^*>i$. We consider the $k(i^*)$ solution, which produces in periods 1 through $k(i^*)$, and in a sequence of periods preceding i^* except in periods in WZ. If we produce m times in the interval $\{i+1,\ldots,i^*\}$ in the second sequence, let $W(j)=q-b-m$ for some $j\geq j^*$. We can modify the $k(i^*)$ solution by shifting the production from $i+1$ through i_*-1 to unskipped periods starting from t_j+1. From condition 1, this contributes m units. The second sequence of productions now ends in period $i-1$. We can set $y_i=0$ or 1. Thus $\beta_i=0$.

3) $\alpha_i=\alpha_*$ *for* $i\in W\cup WZ$ *and* $\alpha_i=\alpha$ *for* $i\notin W\cup WZ$.

To show that $\alpha_i=\alpha_j$ we construct two solutions in C^*, one with $w_i=1$, $w_j=0$ and the other with $w_i=0$, $w_j=1$. All other variables have the same value in both solutions. First consider $i>t_q$. We can always pick a solution in C^* with $w_i=0$, $w_j=1$ for $i,j>t_q$. Alternately, we could produce in period i, and set $w_j=0$. Hence $\alpha_i=\alpha_j$ for all $i>t_q$.

If $i\leq t_q$ and $i\notin W\cup WZ$, then for any $i^*\in Z$ consider the associated $k(i^*)$ solution. We can modify it by shifting production from any period in the second sequence to period i^*. Hence α_i is the same for all periods in $Y\cup YZ$ in the second sequence of productions, and for i^*. Notice that for any $k(i^*)$ solution we can shift production from any period $i>t_q$ to period i^*. Hence α_i is the same for all periods in $Y\cup YZ\cup Z$. If period i is skipped, then for some $k(i^*)$ solution we can shift production from i to i^*. Hence $\alpha_i=\alpha$ for all $i \notin W \cup WZ$.

Next we consider a period $i \in W \cup WZ$. If $Z = \phi$, then from condition 1, $Y \cup YZ \cup WZ = \phi$, and from condition 6, there are no skipped periods. Hence all periods are in W. For any period $i \geq q$, we can produce from periods 1 through q-1, and in period i. This shows that α_i is the same for all periods $i \geq q$. For $i \leq q-1$, we can produce in all periods 1 through q except i, and in period t_q. Hence $\alpha_i = \alpha^*$ for all periods $i \in W$ if $WZ = \phi$.

If $Z \neq \emptyset$ then for some $i^* \in Z$ we have a $k(i^*)$ solution. If $i \in W$ and $w_i=0$, we can shift production from $i''=\max\{i': i'<i^*, w_{i'}=1\}$ in the second sequence to period i. Suppose $i \in W$ and $w_i=1$. Let i^0 denote the first period of production in the second sequence. We can show using the fact that $t_1 \geq 2$ and the definition of $k(i^*)$ that some period is not skipped in the interval $\{k(i^*)+1, \ldots, i^0-1\}$ for i^0 in demand interval j+1. We can shift production from period i to the first period after $k(i^*)$ that is not skipped. Notice that we can modify the $k(i^*)$ solution by shifting production from i'' to the first period not skipped after $k(i^*)$. Hence α_i is the same for all $i \in W$. If $i \in WZ$, then let $i^*=\max\{i': i'>i$, and $i' \in Z\}$. We can shift production from i'' to i or to the first period after $k(i^*)$ that is not skipped. Hence $\alpha_i=\alpha^*$ for all $i \in W \cup WZ$.

Therefore the inequality has the form:

$$\alpha^* \Sigma_{i \in W \cup WZ} w_i + \alpha \Sigma_{i \notin W \cup WZ} w_i + \Sigma_{i \in Y \cup YZ} \beta_i y_i + \Sigma_{i \in Z} \gamma_i z_i \geq \delta.$$

4) $\beta_i=\beta$ for all $i\epsilon Y\cup YZ$, and $\gamma_i=c_i\beta$, where c_i denotes the coefficient of z_i.

If $Y=\emptyset$ then from condition 3 $Z=\emptyset$ and hence from condition 1, $YZ\cup WZ=\emptyset$. Otherwise for $i\in(Y\cup YZ\cup Z)\cap B(r)$, we construct the following solution in C^*. If $m(i(r),i)<u(i(r))$, then for some $i'\in B(r)$, and $i^*\in Z\cap B(r)$, $m(i',i^*)+c_{i'}=u(i(r))-m(i(r),i)$. If $i(r)$ is in demand interval $j+1$, then let $k(i(r))=W(j)+c$ where c is the number of periods skipped up to $k(i(r))$. We produce in periods 1 through $k(i(r))$, all the skipped periods, and in all periods belonging to $Y\cup YZ\cup Z$ in the intervals $\{i(r),....i\}$ and $\{i',....,i^*\}$. If $m(i(r),i)\geq u(i(r))$, then for some $i'\in B(r)$ and $i(r)\leq i'\leq i$, $m(i',i)+c_{i'}=u(i')$. We then produce in periods 1 through $k(i')$, all the skipped periods, and in all periods in $Y\cup YZ\cup Z$ in the interval $\{i',....,i\}$. Using the fact that $t_1\geq 2$ and condition 8 we can establish that for some period $t\notin S$, $t\in\{k(i^*)+1,....,i(r)\}$ if $B(1)=\{1,....,u(i''(1))\}$ or if $r>1$. If $i(1)>1$, then using condition 8 we can establish that for some $t\notin S$, $t\in\{i''(1)+1,....,t_q\}$. We can therefore modify the solution by shifting one unit of production from i to t. Hence $\beta_i+\alpha=\alpha_t+\beta_t+\gamma_t=\beta_{i(r)}$ for all $i\in(Y\cup YZ)\cap B(r)$.

Consider the k^0 solution. We can modify it by shifting one unit of production from k^0 to any $i\in Y$ and $i>k^0$. Hence $\beta_{i(r)}=\beta$ for all $i\in Y$, $i>k^0$. If $i\in Y$ and $i\leq k^0$, then for any $i(r)\in Y$ and $i(r)>k^0$ in demand interval $j+1$, we can produce in periods 1 through $k(i(r))$, all the skipped periods, and in all periods in the interval $\{i(r),....,i(r)+u(i(r))-1\}$. This gives us a solution in C^*. We can then shift production from period $i(r)+u(i(r))-1$ to period i. Hence

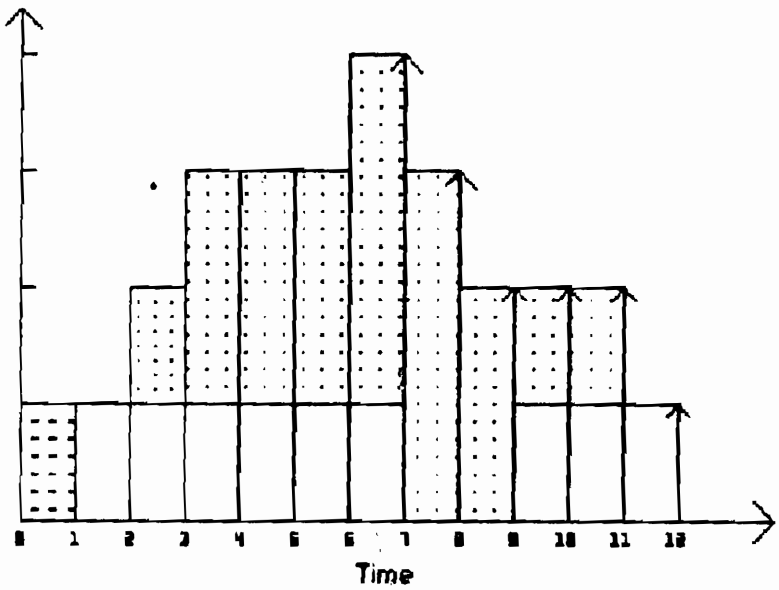
$\beta_i=\beta$ for all $i \in Y \cup YZ$.

For $i \in (YZ \cup Z) \cap B(r)$, $c_i=n(i,i^*)$ for some $i^* \in Z \cap B(r)$ such that $m(i,i^*)+c_i \le u(i) \le u(i(r))$. We can produce in periods 1 through $k(i(r))$, all the skipped periods, and in periods $t \in (Y \cup YZ \cup Z) \cap \{i,\ldots,i^*\}$. If $m(i,i^*)+c_i<u(i(r))$, we produce in the periods $i(r)$ through $t=i(r)+\{u(i(r))-(m(i,i^*)+c_i)-1\}$. Using condition 1 and lemma 1 we can establish that $t<i$, and hence the solution is feasible. It is easy to verify that it is in C^*. From condition 1, $m(i(r),i^{**})=u(i(r))$ for some $i^{**} \in Z \cap B(r)$. Another solution is to produce in periods 1 through $k(i(r))$, all skipped periods, and in all periods $t \in (Y \cup YZ) \cap \{i(r),\ldots,i^{**}\}$. We call this the $k(i(r))$ solution. By comparing the two solutions we obtain $\gamma_i+\beta m(i,i^*)=\beta m(t+1,i^{**})$. Since $u(i(r))=m(i(r),i^{**})= m(i,i^*)+c_i+m(i(r),t)$, $m(i,i^*)+c_i=m(t+1,i^{**})$. Hence $\gamma_i=c_i\beta$.

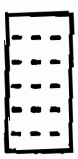
For any r, consider the $k(i(r))$ solution desrcibed earlier and let $i(m)=\max\{i \le t_q:\ i \notin S \text{ and } w_i=1\}$. Clearly $i(m) \in Y \cup YZ$. We can shift production from $i(m)$ to any period $t \in W$ for which $w_t=0$ or to any $t \in WZ \cap B(r)$. Hence $\alpha^*=\alpha+\beta$.

Consider the k^0 solution. Let m_1 and m_2 denote the number of periods in $W \cup WZ$ and $Y \cup YZ$ respectively in which we produce. Let $m_3=n-(m_1+m_2)$ denote the number of other periods in which we produce. From the definition of the k^0 solution, $m_1+m_2=q-b$. Hence the lefthand side of the inequality is $m_1\alpha^*+m_2(\beta+\alpha)+m_3\alpha=n\alpha+(q-b)\beta$. Hence the righthand side $\delta=n\alpha+(q-b)\beta$. If (SI) denotes the skip inequality, then the equation $\alpha w+\beta y+\gamma z=\delta$ can be expressed as a linear combination of $\Sigma^T_{i=1} w_i=n$ and (SI) and hence is a facet.

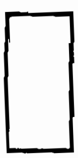
Examples



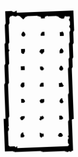
$$w_1+y_2+(y_3+z_3)+(y_4+2z_4)+(y_5+2z_5)+(y_6+2z_6)+(y_7+3z_7)+3z_8+2z_9$$
$$+(y_{10}+z_{10})+(y_{11}+z_{11})+z_{12}\geq 6 \qquad (11)$$



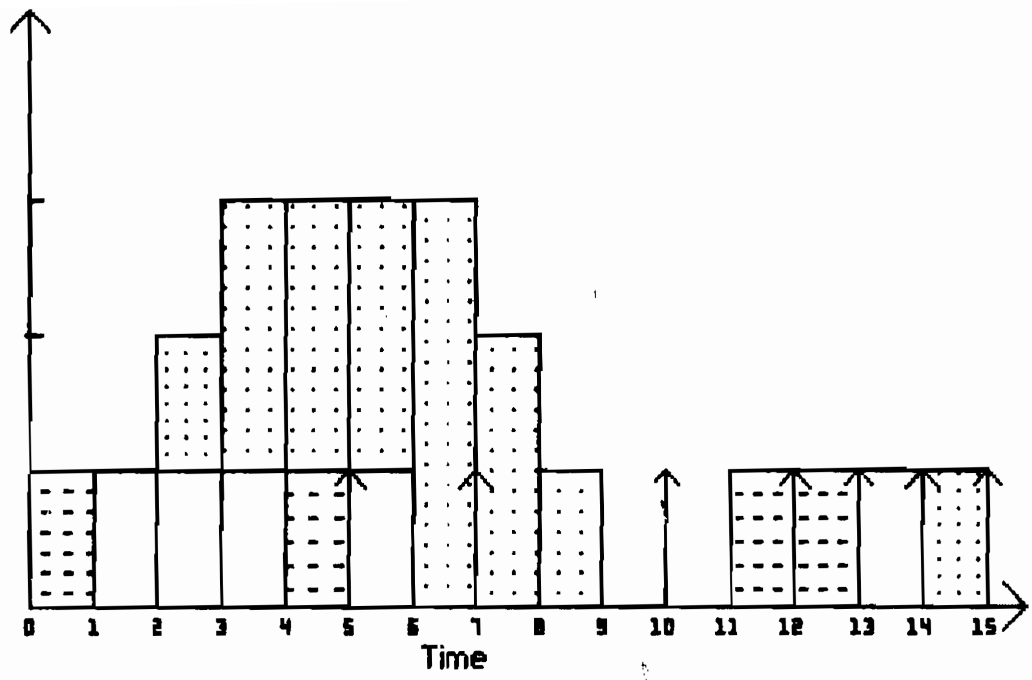
periods in W U WZ



periods in Y U YZ



periods in Z U WZ U YZ



$$w_1+y_2+(y_3+z_3)+(y_4+2z_4)+(w_5+2z_5)+(y_6+2z_6)+3z_7+2z_8+z_9$$

$$+w_{12}+w_{13}+y_{14}+z_{15}\geq 4 \qquad (12)$$

$q = 6 \qquad b = 2$

For $i \leq 5$, $\quad u(i) = q - b = 4$

For $i = 6$ or 7, $\quad u(i) = 3$

For $8 \leq i \leq 13$, $\quad u(i) = 2$

For $i = 14$ or 15 $\quad u(i) = 1$

□

5. Separation Problem

We solve the separation problem for the inequalities (PI). Eppen and Martin (1987) have suggested a simlar approach. The basic idea is to formulate the separation problem as an integer program, and show that the constraints describe a network flow problem. Therefore the linear programming relaxation of the separation problem gives us the same optimal solution as the integer program.

There are two advantages to using a linear programming based approach as opposed to a dynamic programming approach. First, as we show later, it enables us to reformulate the original problem more compactly. Second, this reformulation enables us to prove that our inequalities guarantee integer solutions for certain types of objective functions.

Given a solution (w^*, y^*, z^*) to the linear programming relaxation of the problem SCSP, we wish to identify any violated inequality (PI). For each inequality extending up to t_q, and $1 \le q \le n$, we associate a variable with each of the terms w_i^*, y_i^*, $y_i^*+cz_i^*$, $w_i^*+cz_i^*$ and cz_i^*. The following table gives the terms and their associated variables.

Term	Variable	
w_i^*	a_{iq}	$1 \le i \le t_q$
y_i^*	b_{iq}	$1 \le i \le t_q$
$(y_i^*+cz_i^*)$	$e^k_{iq}(c)$	$t_j+1 \le i \le t_{j+1}$, $0 \le j \le q-1$, $1 \le c \le q-j-1$ and $1 \le k \le q-j-c$;

	$e_{iq}(c)$	$1 \le i \le t_q$
$(w_i^*+cz_i^*)$	$g^k_{iq}(c)$ for $t_j+1 \le i \le t_{j+1}$, $0 \le j \le q-1$, $1 \le c \le q-j-1$ and $1 \le k \le q-j-c$;	
	$g_{iq}(c)$	$1 \le i \le t_q$
cz_i^*	$m_{iq}(c)$ for $t_j+1 \le i \le t_{j+1}$, $0 \le j \le q-1$, $1 \le c \le q-j$;	

For the Partition Inequalities (PI), we can write the following separation problem for inequalities extending up to t_q.

$$\text{Min } v = \Sigma^n_{q=1} [\Sigma^{q-1}_{j=0} \Sigma^{tj+1}_{i=tj+1} \Sigma^{q-j-1}_{c=1} [\Sigma^{q-j-c}_{k=1} \{(y_i^*+cz_i^*)e^k_{iq}(c)+(w_i+cz_i^*)g^k_{iq}(c)\}$$
$$+(y_i^*+cz_i^*)e_{iq}(c)+(w_i+cz_i^*)g_{iq}(c)+cz_i^*m_{iq}(c)+(q-j)z_i^*m_{iq}(q-j)]$$
$$+\Sigma^{tq}_{i=1} \{w_i^*a_{iq}+y_i^*b_{iq}\} - q\delta_q] \qquad (13)$$

subject to:

$$a_{iq}+b_{iq}+\Sigma_{c=1}^{q-j-1} \{\Sigma_{k=1}^{q-j-1} (e^k_{iq}(c)+g^k_{iq}(c)\}+m_{iq}(c) - \delta_q = 0$$
$$t_j+1 \le i \le t_{j+1},\ 0 \le j \le q-1,\ 1 \le q \le n \qquad (14)$$

$$a_{i+1,q}+b_{i+1,q} - a_{iq} \ge 0 \qquad 1 \le i \le t_q-1,\ 1 \le q \le n \qquad (15)$$

$$e^{q-j-1}_{i+1,q}(1)+g^{q-j-1}_{i+1,q}(1) - (b_{iq}+m_{iq}(1)) \ge 0$$
$$0 \le j \le q-2,\ t_j+1 \le i \le t_{j+1}-1,\ 1 \le q \le n \qquad (16)$$

$$e^k_{i+1,q}(c+1)+e^k_{i+1,q}(c)+g^k_{i+1,q}(c+1) - (e^{k+1}_{iq}(c)+g^k_{iq}(c+1)) \ge 0$$
$$t_j+1 \le i \le t_{j+1},\ 0 \le j \le q-3,\ 1 \le c \le q-j-1,\ 1 \le k \le q-j-2,\ 1 \le q \le n \qquad (17)$$

$$g^1_{i+1,q}(c)+m_{i+1,q}(c) - (e^1_{iq}(c)+g^1_{iq}(c)) \ge 0$$
$$t_j+1 \le i \le t_{j+1},\ 0 \le j \le q-2,\ 1 \le c \le q-j-1,\ 1 \le q \le n \qquad (18)$$

$$m_{i+1,q}(q-j) - e^1_{iq}(q-j-1) \ge 0$$
$$t_j+1 \le i \le t_{j+1},\ 0 \le j \le q-3,\ 1 \le q \le n \qquad (19)$$

$$m_{i+1,q}(c)+e_{i+1,q}(c)+e_{i+1,q}(c+1)+g_{i+1,q}(c) - (m_{iq}(c+1)+e_{iq}(c)+g_{iq}(c)) \ge 0$$
$$t_j+1 \le i \le t_{j+1},\ 0 \le j \le q-3,\ 1 \le c \le q-j-1,\ 1 \le q \le n \qquad (20)$$

$$\delta_q \le 1 \qquad 1 \le q \le n \qquad (21)$$

$$a,\ b,\ e,\ g,\ m \ge 0 \text{ and integer.} \qquad (22)$$

The constraints ensure that the compensation condition is satisifed.

Notice that each variable appears at most twice in constraints (15) through (22). It is straightforward though tedious to show that the constraints define a network. We omit the details. Therefore, the separation problem always has an integer optimal solution. Hence we can solve the problem as a linear program.

Separation Lemma. *The optimum objective function value of the separation problem is zero if and only if the solution* (w^*, y^*, z^*) *of the linear programming relaxation of SCSP satisfies all the partitioning inequalities (PI).*

Proof. If some inequality is violated, then we can set δ_q and the variables associated with the inequality in the separation problem to 1. This is a feasible solution to the separation problem with an objective function value strictly less than zero.

Suppose all the inequalities are satisfied. Since there always exists an optimum integer solution and $\delta_q \leq 1$, all the variables are either 0 or 1 in the integer solution. Hence the set of positive variables in an integer solution to the separation problem correspond to a partitioning inequality. Since all the inequalities are satisfied, the objective function is at least equal to zero. Notice that we can set all the variables in the separation problem to zero. Hence the optimum objective function value is zero.

Corollary. *If we drop the constraint $\delta \leq 1$ from the separation problem, then the dual has a feasible solution if and only if all the partitioning inequalities are satisfied.*

If any inequality is violated, then we can arbitrarily increase δ_0 and the objective function is unbounded from below. Hence the dual is infeasible. If all the inequalities are satisfied, then the optimal objective function is zero, and hence there exists a feasible dual solution.

6. Computational Results

The literature contains very little test data for the product cycling problem. Karmarkar and Schrage (1985) report computational experience for the continuous production policy version of the product cycling problem. In our model, we follow a discrete production policy in which we produce either zero or one unit in each period. In the continuous policy, we can produce any amount between zero and the production capacity. Karmarkar and Schrage use Lagrangean relaxation to solve problem instances of up to 4 products and 8 time periods. Magnanti and Vachani (1987) report computational results for the discrete version of the problem. They solve problem instances of up to 5 products and 15 time periods.

We use the same approach as Magnanti and Vachani to generate problem instances. For all problem instances, we assume that the initial inventory is zero, and that the machine is in the off state at the start of the time horizon.

The cost parameters k_i and s_i are the same for all the products and for all machines, and are constant over the time horizon. The inventory holding cost h_i = 20(T-i). It can be shown that for this cost structure, the subset of the partitioning inequalities (PI) which partition only the last demand interval are sufficient to guarantee optimal integer solutions if $s_i \le h_i$. We therefore do not test problem instances with $s_i \le h_i$. We conjecture that for the case $s_i > h_i$, the partitioning inequalities (PI) are sufficient to guarantee optimal solutions. Our computational results confirm this result. We tested two categories of problems:

1) The single item, single machine problem. We tested problems of up to 100 time periods and 30 demands. The largest problem instance had 300 variable (100 each of the w_i production variables, y_i setup variables and z_i changeover variables).

2) The two item, two machine problem. We tested problems of up to 100 time periods and 15 demands for each item. The largest problem instance had 1200 variables (300 variables for each item and machine).

For both the problem categories we used only a subset of the partitioning inequalities (PI). For any inequality extending up to t_q, we partitioned only the last 5 demand intervals t_{q-5} through t_q. Further, we used only those inequalities whose coefficients for the variables z_i are 0, 1 and 2. We did not use any of the skip inequalities (SI). For all the problem instances tested, we obtained optimal integer solutions.

We use a machine utilization of up to 30%. For example, a 10 period problem has three demands, and a 100 period problem has 30 demands. The computations were performed on a IBM 4341 computer using the GAMS package. Tables 8.1, 8.2 and 8.3 summarize the computational results. Let v(IP) and v(LP) denote the optimal objective function values of the original integer program SCSP and the linear programming relaxation SCSP(LP) respectively. Let v(p) denote the optimal objective function value of SCSP (LP) after including the inequalities discovered by Magnanti and Vachani (1987) and let v(n) denote the optimal objective function value of SCSP (LP) after including the partitioning inequalities (PI).

For the single item, single machine problem we computed optimal solutions for three different problems: (1) the linear programming relaxation, which we call the LP solution, (2) the linear programming relaxation along with the cuts discovered by Magnanti and Vachani (1987), which we call the previous cuts solution, and (3) the linear programming solution with the partitioning inequalities (PI), which we call the new cuts solution.

Table 1

Single machine, single item problems

# of demands	v(LP)	v(p)	v(n)	(v(IP)-v(LP))*100 / v(IP)	(v(IP)-v(P))*100 / v(IP)	(v(IP)-v(n))*100 / v(IP)
3	26.7	146.7	160	83.3	8.3	0.0
5	56.7	267	300	81.1	11.1	0.0
10	117.8	497	540	78.2	8.0	0.0
15	195.6	746.7	840	76.7	11.1	0.0
20	282.2	1047	1160	75.7	9.8	0.0
25	340	1277	1440	76.4	11.3	0.0
30	371	1496	1680	77.9	10.9	0.0

Notes: Constant turn on and setup cost

Production cost = H*(T-I)

Table 2

Two Machine, Two Item Problems

Turn on cost k = 100

# of demands	v(LP)	$\frac{v(IP)-v(LP)*100}{v(IP)}$	v(n)	v(IP)	$\frac{v(IP)-v(n)*100}{v(IP)}$
3	1453.3	15.51	1720	1720	0
5	4700	16.37	5620	5620	0
10	13569	5.77	14400	14400	0
15	32278	4.56	33820	33820	0

Turn on cost k = 200.

3	1520	20.83	1920	1920	0
5	4820	20.72	6080	6080	0
10	13787	9.06	15160	15160	0
15	32633	6.07	34740	34740	0

Notes: Identical machines

Constant turn on and setup costs

Production cost = H*(T-I)

We obtain optimal integer solutions for all the problem instances tested. This result is not surprising in view of our earlier conjecture that the partitioning inequalities (PI) guarantee integer solutions for the cost structures that we are testing. In fact, we included only a subset of the partitioning inequalities. The gaps between the optimal objective function value of the linear programming relaxation and the optimal integer program objective function value are large for the single item, single machine problem, and vary between 75% and 83%. A small subset of the partitioning inequalities that partition only the last demand interval reduces the gap considerably to between 8% and 11%. However, we still obtain fractional solutions, and need to introduce more general partitioning inequalities to reduce the gaps to zero. We were able to optimally solve problem instances with 1200 variables.

7. Conclusions

In this study we described a general class of valid inequalities that subsumes inequalities described in earlier work. We identified conditions under which these inequalities describe facets. We also solve the separation problem for one class of valid inequalities efficiently. Using this, we used a cutting plane based procedure to solve various problem instances. Our computational results show that these facets are extremely useful in reducing integrality gaps. In all the problem instances tested, we obtained optimal integer solutions without resorting to a branch and bound procedure.

A useful line of research is to determine whether the skip inequalities describe the convex hull of feasible integer solutions. However, the convex hull for the capacitated lot-sizing problem without additional changeover variables has not yet been described. It might therefore be more useful to identify cost structures for which the skip inequalities guarantee integer solutions. Since the problem can be cast as a fixed charge capacitated network design problem, the inequalities could perhaps be extended to the network design problem.

References

Afentakis, P., B. Gavish and U. S. Karmarkar (1984) Exact Solutions to the Lot Sizing Problem in Multistage Assembly Systems, *Management Science*, 30, pp. 222-239.

Baker, K. R. (1974) *Introduction to Sequencing and Scheduling*, John Wiley & Sons, New York.

Balas, E. (1985) On the Facial Structure of Scheduling Polyhedra, *Mathematical Programming Study*, 24, pp. 179-218.

Balas, E. and M. W. Padberg (1976) Set Partitioning: A Survey, *SIAM Review*, 18, pp. 710-760.

Barany, I., T. J. Van Roy and L. A. Wolsey (1984a) Uncapacitated Lot-Sizing: The Convex Hull of Solutions, *Mathematical Programming Study*, 22, 32-43.

Barany, I., T. J. Van Roy and L. A. Wolsey (1984b) Strong Formulations for Multi-Item Capacitated Lot-Sizing, *Management Science*, 30, pp. 1255-1261.

Bitran, G. R., T. L. Magnanti and H. H. Yanasse (1984) Approximation Methods for the Uncapacitated Dynamic Lot-Size Problem, *Management Science*, 30, pp. 1121-1140.

Bitran, G. R. and H. Matsuo (1986a) Approximation Formulations for the Single-Product Capacitated Lot-Size Problem, *Operations Research*, 34, pp. 63-74.

Bitran, G. R. and H. Matsuo (1986b) The Multi-Item Capacitated Lot-Size Problem: Error Bounds of Manne's Formulations, *Management Science*, 32, pp. 350-359.

Bitran, G. R. and H. H. Yanasse (1982) Computational Complexity of the Capacitated Lot Size Problem, *Management Science*, 28, pp. 1174-1186.

Bomberger, E. E. (1966) A Dynamic Programming Approach to a Lot Size Scheduling Problem, *Management Science*, 12, pp. 778-784.

Cho, D. C., E. L. Johnson, M. W. Padberg and M. R. Rao (1983a) On the Uncapacitated Plant Location Problem I: Valid Inequalities and Facets, *Mathematics of Operations Research*, 8, pp. 579-589.

Cho, D. C., M. W. Padberg and M. R. Rao (1983b) On the Uncapacitated Plant Location Problem II: Facets and Lifting Theorems, Mathematics of *Operations Research*, 8, pp. 590-612.

Crowder, H., E. L. Johnson and M. W. Padberg (1983) Solving Large-Scale Zero-One Linear Programming Problems, *Operations Research*, 31, pp.803-834.

Crowder, H., and M. W. Padberg (1980) Solving Large-Scale Symmetric Traveling Salesman Problems to Optimality, *Management Science*, 26, pp.495-509.

Doll, C. L. and D. C. Whybark (1973) An Iterative Procedure for the Single-Machine, Multi-Product Lot Scheduling Problem, *Management Science*, 20, pp. 50-55.

Driscoll, W. C. and H. Emmons (1977) Scheduling Production on One Machine with Changeover Costs, *AIIE Transactions*, 9, pp. 388-395.

Edmonds, J. (1965) Maximum Matching and a Polyhedron with 0,1 Vertices, *Journal of Research National Bureau of Standards*, 69B, pp.125-130.

Elmaghraby, S. E. (1978) The Economic Lot Scheduling Problem (ELSP): Review and Extensions, *Management Science*, 24, pp. 587-598.

Eppen, G. D. and R. K. Martin (1987) Solving Multi-Item Capacitated Lot Sizing Problems Using Variable Redefinition, *Operations Research*, 35, 832-848.

Florian, M., J. K. Lenstra and A.H.G.Rinnooy Kan (1980) Deterministic Production Planning: Algorithms and Complexity, *Management Science*, 26, pp. 669-679.

Gascon, A. and R. C. Leachman (1988) A Dynamic Programming Solution to the Dynamic Multi-Item Single Machine Scheduling

Problem, *Operations Research*, 36, pp. 50-56.
Geffrion, A. M. (1974) Lagrangean Relaxation for Integer Programming, *Mathematical Programming Study*, 2, pp. 82-114.

Geffrion, A. M. and G. W. Graves (1976) Scheduling Parallel Production Lines with Changeover Costs: A Practical Application of a Quadratic Assignment / LP Approach, *Operations Research*, 24, pp. 595-610.

Glassey, C. R. (1968) Minimum Change-over Scheduling of Several Products on One Machine, *Operations Research*, 16, pp. 342-352.

Gorenstein,S. (1970) Planning Tire Production, *Management Science*, 17, pp. B72-B82.

Graves, S. C. (1980) The Multi-Product Production Scheduling Problem, *AIIE Transactions*, 12, pp. 233-240.

Graves, S. C. (1982) Using Lagrangean Techniques to Solve Hierarchical Production Planning Problems, *Management Science*, 28, pp. 260-275.

Groetschel, M. (1985) Polyhedral Combinatorics, in Combinatorial Optimization: Annotated Bibliographies, O'Eigeartaigh,M., J.K.Lenstra and A.H.G.Rinnooy Kan, eds., Wiley Interscience, New York.

Groetschel, M. and O. Holland (1984) Solving Matching Problems with Linear Programming, Preprint No. 37, Mathematisches Institut, Universitat Augsburg, FRG.

Groetschel, M., M. Juenger and G. Reinelt (1984) A Cutting Plane Algorithm for the Linear Ordering Problem, *Operations Research*, 32, pp.1195-1220.

Groetschel, M., M. Juenger and G. Reinelt (1985) Facets for the Linear Ordering Problem, *Mathematical Programming*, 33, pp.43-60.

Groetschel, M., and M. W. Padberg (1979a) On the Symmetric Traveling Salesman Problem I: Inequalities, *Mathematical Programming*, 16, pp.281-302.

Groetschel, M., and M. W. Padberg (1979b) On the Symmetric Traveling Salesman Problem II: Lifting Theorems and Facets, *Mathematical Programming*, 16, pp.265-280.

Groetschel, M., L. Lovasz and A. Schrijver (1981) The Consequences of the Ellipsoid Method for Combinatorial Optimization, *Combinatorica*, 1, 169-198.

Haessler, R. W. (1979) An Improved Extended Basic Procedure for Solving the Economic Lot Scheduling Problem, *AIIE Transactions*, 11. pp. 336-340.

Hax, A. C. and C. Candea (1984) *Production and Inventory Management*, Prentice-Hall, Englewood Cliffs, New Jersey.

Hodgson, T. J. (1970) Addendum to Stankard and Gupta's Note on Lot Size Scheduling, *Management Science*, 16, pp. 514-517.

Inman, R.R. and P.C. Jones (1989) When is the Economic Lot Sizing Problem Easy? *IIE Transactions*, vol.21, No. 1, pp. 11-20.

Jagannathan, R. and M. R. Rao (1973) A Class of Deterministic Production Planning Problems, *Management Science*, 19, pp. 1295-1300.

Johnson, E. L., M. M. Kostreva and H. H. Suhl (1985) Solving 0-1 Integer Programming Problems Arising from Large-Scale Planning Models, *Operations Research*, 33, pp. 803-820.

Karmarkar, U. S., S. Kekre and S. Kekre (1987) The Dynamic Lot Sizing Problem with Startup and Reservation Costs, *Operations Research*, 35, pp. 389-398.

Karmarkar, U. S. and L. Schrage (1985) The Deterministic Dynamic Product Cycling Problem, *Operations Research*, 33, pp. 326-345.

Lasdon, L. S. and R. C. Terjung (1971) An Efficient Algorithm for Multi-Item Scheduling, *Operations Research*, 19, pp.946-969.

Lawler, E. L., J. K. Lenstra and A. H. G. Rinnooy Kan (1982) Recent Developments in Sequencing and Scheduling: A Survey. In *Deterministic and Stochastic Scheduling*, M.A.H. Dempster, J.K. Lenstra and A.H.G. Rinnooy Kan, eds., Reidel, Dordecht, pp. 35-73.

Leachman, R. C. and A. Gascon (1988) A Heuristic Scheduling Policy for Multi-Item, Single Machine Production Systems with Time-Varying, Stochastic Demands, *Management Science*, 34, pp. 377-390.

Leung, J. and T. L. Magnanti (1989) Valid Inequalities and Facets of the Capacitated Plant Location Problem, *Mathematical Programming*, 44, 271-291.

Leung, J., T. L. Magnanti and R.Vachani (1989) Facets and Algorithms for Capacitated Lot Sizing, *Mathematical Programming*, 45, 331-359.

Madigan, J. C. (1968) Scheduling a Multi-Product, Single-Machine System for an Infinite Planning Period, *Management Science*, 14, pp. 713-719.

Manne, A. S. (1958) Programming of Economic Lot Sizes, *Management Science*, 4, pp. 115-135.

Magnanti, T. L. and R. Vachani (1990) A Strong Cutting Plane Algorithm for Production Scheduling with Changeover Costs, *Operations Research*, 38, 456-473.

Mitsumori, S. (1972) Optimal Production Scheduling of Multicommodities in Flow Lines, *IEEE Transactions*, Systems, Man and Cybernetics, SMC-2, pp. 486-493.

Nahmias, S. (1989) *Productions and Operations Analysis*, IRWIN, Boston.

Padberg, M. W. and S. Hong (1980) On the Symmetric Traveling Salesman Problem: A Computational Study, *Mathematical Programming Study*, 12, pp. 78-107.

Padberg, M. W., T. J. Van Roy and L. A. Wolsey (1983) Valid Inequalities for Fixed Charge Problems, *Operations Research*, 33, pp.842-861.

Pochet, Y. (1988) Valid Inequalities and Separation for Capacitated Economic Lot-Sizing, *Operations Research Letters*, 7, pp.109-116.

Pochet, Y. and L. A. Wolsey (1988) Lot-Sizing Models with Backlogging: Strong Reformulations and Cutting Planes, *Mathematical Programming*, 40, pp. 317-335.

Rinnooy Kan, A. H. G. (1976) *Machine Scheduling Problems - Classification, Complexity and Computations*, Martinus Nijhoff, The Hague, Holland.

Schrage, L. (1982) The Multiproduct Lot Scheduling Problem, in *Deterministic and Stochastic Scheduling*, M.A.H. Dempster, J.K. Lenstra and A.H.G. Rinnooy, eds., Nato Advanced Study Institutes Series, D.Reidel, Holland.

Schroeder, R. G. (1989) *Operations Management Decision Making in the Operations Function*, McGraw-Hill Book Co., New York.

Stankard, M. F. and S. K. Gupta (1969) A Note on Bomberger's Approach to Lot Size Scheduling: Heuristic Proposed, *Management Science*, 15, 449-452.

Tenzer, N. (1969) Multi-Product Scheduling Problems with Time Independent Setup Costs, Ph.D. Thesis, School of Engineering and Applied Science, Columbia University.

Vachani, R. (1986) Polyhedral Structure of Fixed Charge Production Planning and Scheduling Problems, Ph.D. Thesis, Sloan School of Management, MIT.

Van Roy, T. J. and L. A. Wolsey (1985) Valid Inequalities and Separation for Uncapacitated Fixed Charge Networks, *Operations Research Letters*, 4, 105-112.

Van Roy, T. J. and L. A. Wolsey (1986) Valid Inequalities for Mixed 0-1 Programs, *Discrete Applied Mathematics*, 14, pp.199-213.

Van Roy, T. J. and L. A. Wolsey (1987) Solving Mixed Integer Programs by Automatic Reformulation, *Operations Research*, 35, pp. 45-57.

Vergin, R. C. and T. N. Lee (1978) Scheduling Rules for the Multiple Product Single Machine System with Stochastic Demand, *INFOR*, 16, pp. 64-73.

Wagner, H. M. and T. M. Whitin (1958) Dynamic Version of the Economic Lot-Size Model, *Management Science*, 5, pp. 89-96.

Wolsey, L. A. (1975) Faces of a Linear Inequality in 0-1 Variables, *Mathematical Programming*, 8, pp. 165-178.

Wolsey, L. A. (1989) Uncapacitated Lot-Sizing Problems with Start-Up Costs, *Operations Research*, 37, pp. 741-747.

Zangwill, W.I. (1966) A Deterministic Multi-Period Production Scheduling Model with Backlogging, *Management Science*, 13, pp. 105-119.

Zangwill, W. I. (1969) A Backlogging Model and Multi-Echelon Model of a Dynamic Economic Lot Size Production System - A Network Approach, *Management Science*, 15, pp. 506-527.