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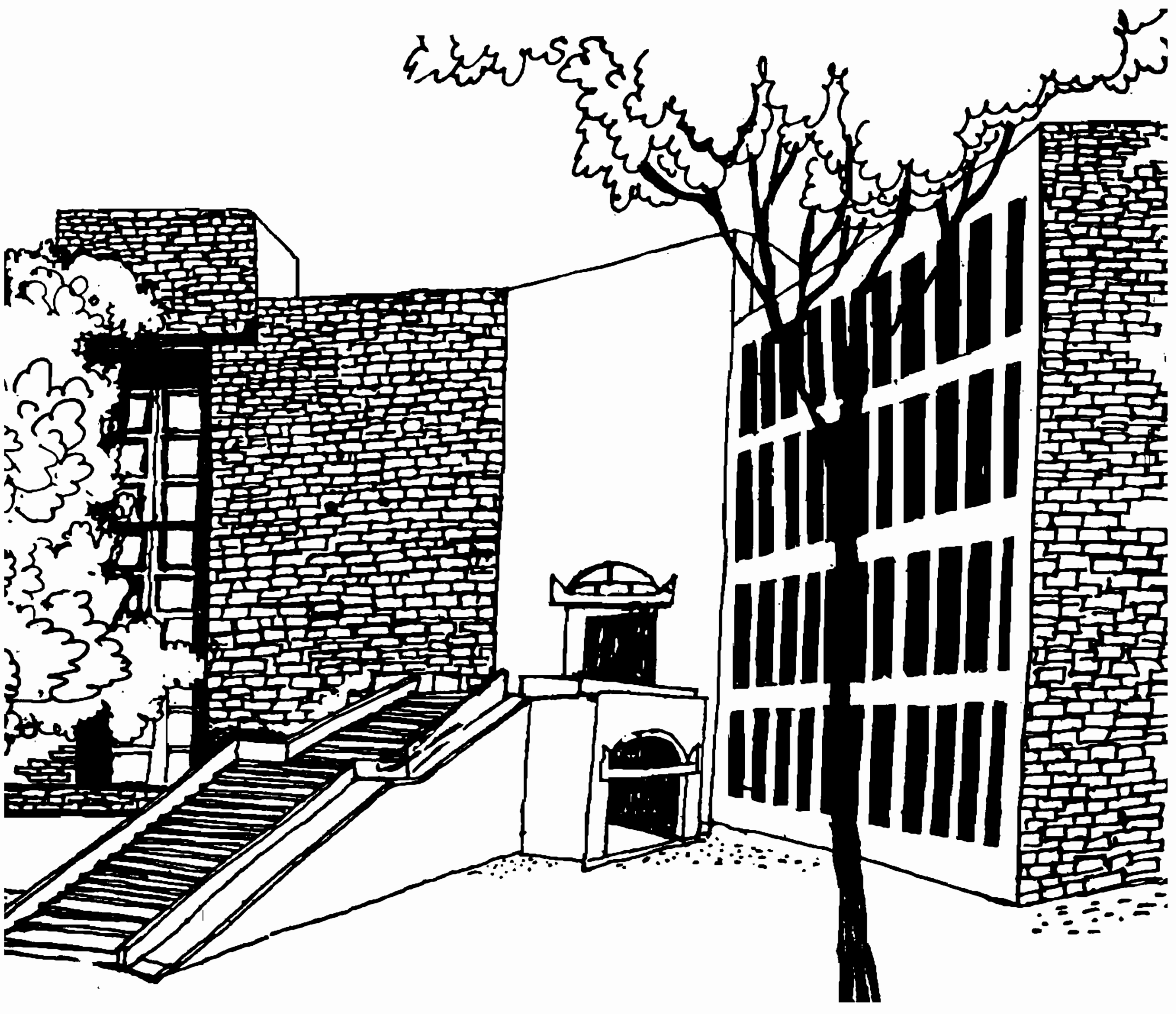

GENERALIZED DREZE-MULLER EQUILIBRIUM: SOME RATIONING SCHEME IN A DISTRIBUTION ECONOMY

By

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W P No. 1145
October 1993

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Abstract

Here we view rationing as an instrument available in the hands of a social planner to effect a more equitable distribution of goods. We introduce a separate currency (like a coupon) alongside money, and both money as well as this auxiliary currency are used in the purchase of commodities. It is precisely in this framework that we study existence, and fairness of alternative rationing schemes.

1. Introduction

:- Rationing as a device for attaining fair allocations of resources, has a long standing of repute both in theory as well as in practice. It could be an explicit quantity constraint, as for instance in equal division of some or all of the available goods amongst the individuals (Lahiri (1993b)) or the introduction of an auxiliary currency which along with money forms the unit of purchase of commodities. This latter mode of rationing has been studied by Dreze and Müller (1980), Baumol (1986), Polterovich (1993) and Lahiri (1993a). Except for Baumol, the other endeavours have been a contribution to reconciling market clearing with transactions at disequilibrium prices.

Here we view rationing as an instrument available in the hands of a social planner to effect a more equitable distribution of goods. Hence our view is similar to that of Baumol. In Lahiri (1993b) for instance, we were concerned with equal division of some goods. Hence, no one would envy the consumption of the rationed goods of any other agent. In this paper, we introduce a separate currency (like a coupon) alongside money, and both money as well as this auxiliary currency are used in the purchase of commodities. It is precisely in this framework that we study existence, and fairness of alternative rationing schemes.

2. The Model

:- Our model consists of l goods and n consumers. The aggregate initial endowment of the goods are summarized in a vector $\Omega \in \mathbf{R}^l_{++} \equiv \{x \in \mathbf{R}^l / x^j > 0 \ \forall \ j=1,\ldots,l\}$. The money income of agent i is $w_i > 0$ and his preferences over alternative consumption bundles are summarized in a utility function $u_i : \mathbf{R}^l_+ \to \mathbf{R}$ $(\mathbf{R}^l_+ \equiv \{x \in \mathbf{R}^l / x^j \geq 0 \ \forall_j = 1,\ldots,l\})$. Throughout our analysis we will assume that (i) u_i is continuous; (ii) u_i is strongly monotonic i.e. $x \geq y$, $x \neq y$, $x, y \in \mathbf{R}^l_+ \Rightarrow u_i(x) > u_i(y)$; (iii) u_i is quasi-concave i.e. $\forall \ x, y \in \mathbf{R}^l_+, \forall t \in [0,1]$, $u_i(tx+(1-t)y) \geq \min(u_i(x), u_i(y))$. Together the above assumptions imply, u_i is semi-strictly quasi-concave i.e. $\forall x, y \in \mathbf{R}^l$, $\forall t \in (0,1)$, $u_i(x) > u_i(y) \Rightarrow u_i(tx+(1-t)y) > u_i(y)$.

Let $R \subset \{1,\ldots,l\}$ be a subset of commodities with fixed prices $(\hat{p}_j)_{j \in R}$ such that $p_j \geq 0 \ \forall \ j \in R$ and $\Sigma_{j \in R} \hat{p}_j \Omega_j < \Sigma^n_{i=1} w_i = w$. (To make the analysis non-trivial we will assume $\Sigma_{j\in R} \hat{p}_j \Omega^j < w$). Let $\alpha = w - \Sigma_{j\in R} \hat{p}_j \Omega^j$. α is the aggregate money income that is spent on the flexible price markets. A flexible price vector (to be realized on the market without fixed prices) is a vector $(p_j)_{j\in R^c}$ such that $p_j \geqq 0 \; \forall \; j \in R^c$ and $\Sigma_{j\in R^c} p_j \Omega^j = \alpha$. Let Δ_1 be the set of flexible price vectors.

Let us assume that each consumer is endowed with $v_i > 0$ units of coupons, which have to be used alongside money for all transactions. A vector of coupons prices of the commodities is a vector $(a_j)_{j=1}^{l}$ such that $a_j \geqq 0 \; \forall \; j \in \{1,\ldots,l\}$ and $\Sigma^l_{j=1} a_j \Omega^j = \Sigma^n_{i=1} v_i \equiv v > 0$. Let Δ_2 be the set of coupons price vectors.

A <u>coupon's equilibrium</u> is a triple $(p^0, a^0, x^0) \in \Delta_1 \times \Delta_2 \times (\mathbb{R}^l_+)^n$ such that

(i) $\Sigma^n_{i=1} x^0_i = \Omega$

(ii) $\forall \; i = 1,\ldots,n$, x^0_i solves

$$u_i(x_i) \to \max$$
$$\text{s.t. } \Sigma_{j\in R} \hat{p}_j x^j_i + \Sigma_{j\in R^c} p^0_j x^j_i \leqq w_i$$
$$\Sigma^l_{j=1} a^0_j x^j_i \leqq v_i$$
$$x^j_i \geqq 0 \quad \forall \; j = 1,\ldots,l.$$

We consider the alternative possibility of agents being able to sell their coupons at market determined prices i.e. something like a black market for coupons.

A <u>coupon's equilibrium with sales is a four-tuple</u> $(p^0, a^0, c^0, x^0, s) \in \Delta^1 \times \Delta^2 \times \mathbb{R}_+ \times (\mathbb{R}^l_+)$ such that

(i) $\Sigma^n_{i=1} x^0_i = \Omega$

(ii) $\forall \; i = 1,\ldots,n$, x^0_i solves

$$u_i(x_i) \to \max$$
$$\text{s.t. } \Sigma_{j\in R} \hat{p}_j x^j_i + \Sigma_{j\in R^c} p^0_j x^j_i + c^0 s_i \leqq w_i$$
$$\Sigma^l_{j=1} a^0_j x^j_i \leqq v_i + s_i$$
$$x^j_i \geqq 0 \quad \forall \; j = 1,\ldots,l$$

(iii) $s = (s_i)^n_{i=1}$ satisfies $s_i \geqq -v_i$ and $\Sigma^n_{i=1} s_i = 0$

2. Existence and Fairness of a coupon's Equilibrium :- We begin with the following lemma whose proof is a simple consequence of the semi-strict quasi concavity of each u_i.

<u>Lemma 1</u> :- If $x^0_i \ll \Omega$

$$\left.\begin{array}{l} u_i(x_i) \to \max \\ \text{s.t. } \Sigma_{j\in R}\hat{p}_j x^j_i + \Sigma_{j\in R^c} p_j x^j_i \leq w_i \\ \Sigma^l_{j=1} a_j x^j_i \leq v_i, \; x^j_i \geq 0 \; \forall \; j=1,\dots,l \\ x^j_i \leq \Omega^j+1 \; \forall \; j=1,\dots,l \end{array}\right\} \qquad (*)$$

then x^0_i solves

$$\begin{array}{l} u_i(x_i) \to \max \\ \text{s.t. } \Sigma_{j\in R}\hat{p}_j x^j_i + \Sigma_{j\in R^c} p_j x^j_i \leq w_i \\ \Sigma^l_{j=1} a_j x^j_i \leq v_i, \; x^j_i \geq 0 \; \forall \; j=1,\dots,l \end{array}$$

where $p\in\Delta^1$ and $a\in\Delta^2$.

<u>Proof</u> :- Appeal to the semistrict quasi-concavity of u_i.

For each $p\in\Delta^1$ and $a\in\Delta^2$ let $z_i(p,a)=\{x\in\mathbb{R}^l_+ / x$ solves $(*)\}$ and $z(p,a)=\Sigma^n_{i=1} z_i(p,a)$. The correspondence $z:\Delta^1 \times \Delta^2 \to\to \mathbb{R}^l_+$ is non-empty valued, convex valued, uppersemicontinuous and compact-valued (see Nikaido (1967)). Further $z(p,a)\subseteq Q \; \forall (p,a)\in\Delta^1\times\Delta^2$ where $Q=\{x\in\mathbb{R}^l_+ \; x\leq(n+1)\Omega\}$.

<u>Theorem 1</u> :- Under the above assumptions a coupons equilibrium exists.

<u>Proof</u> :- Define the functions $f: \Delta^1\times\Delta^2\times Q\to\Delta^1$ and $g: \Delta^1\times\Delta^2\times Q\to\Delta^2$ as follows:

$$f_j(p,a,z) = \frac{p_j + \max(0, z^j-\Omega^j)\,/\,\Omega^j}{1+\Sigma_{j\in R^c}[\max(0,z^j-\Omega^j)]\cdot\frac{}{\alpha}}, \quad j\in R^c$$

$$g_j(p,a,z) = \frac{a_j + \max(0, z^j-\Omega^j)\,/\,\Omega^j}{1+\Sigma^l_{j=1}[\max(0,z^j-\Omega^j)]\cdot\frac{}{v}}, \quad j=1,\dots,l$$

Let $A: \Delta^1\times\Delta^2\times Q\to\to\Delta^1\times\Delta^2\times Q$ be defined as follows:

$A(p,a,z)=(f(p,a,z))x(g(p,a,z))xz(p,a)$

A is a non-empty valued, compact valued, convex valued and uppersemicontinuous correspondence. Thus by Kakutani's fixed point theorem, $\exists(p^0,a^0,z^0)\in \Delta^1 x \Delta^2 x Q$ such that $(p^0,a^0,z^0)\in A(p^0,a^0,z^0)$. It is easy to check now that (p^0,a^0,z^0) is a coupon's equilibrium.

Q.E.D.

Corollary to Theorem 1 :- If (p^0,a^0,x^0) is a coupon's equilibrium then

(i) $\Sigma_{j\in R} \hat{p}_j x^{0j}_i + \Sigma_{j\in R^c} p^0_j x^{0j}_i = w_i \ \forall i=1,\dots,n$

(ii) $\Sigma^l_{j=1} a^0_j x^{0j}_i = v_i \ \forall \ i=1,\dots,n$

Proof :- Follows easily by observing that $p^0\in\Delta^1$ and $a^0\in\Delta^2$.

Q.E.D.

We say that an allocation $x\in(\mathbb{R}^l_+)^n$ is Pareto efficient if

(i) $\Sigma^n_{i=1}\bar{x}_i \leqq \Omega$

(ii) $\not\exists\ y\in(\mathbb{R}^l_+)^n$ with $\Sigma^n_{i=1} y_i \leqq \Omega$ and $u_i(y_i)\geqq u_i(\bar{x}_i) \ \forall \ i=1,\dots,n$ with at least one strict inequality.

The following example reveals that a coupon's equilibrium need not be Pareto efficient:

Example :- $n=2$, $l=2$, $u_1(x,y)=x^{1/2}y$, $u_2(x,y)=xy^{1/2}$, $w=(2,2)$, $w_1=2$, $w_2=2$. Let $\hat{p}$=fixed price of $x=1/2$; p^0=flexible price of $y=3/2$. Let $a^0_x=3/2$, $a^0_y=1/2$ be the respective prices of x and y on the coupon's market and $v_1=v_2=2$. Let $(x^0_1,y^0_1)=(x^0_2,y^0_2)=(1,1)$. Then $(p^0,a^0,((x^0_1,y^0_1),(x^0_2,y^0_2)))$ is a coupons equilibrium. However the allocation where agent 1 gets 2/3 units of good x and 4/3 units of good y and agent 2 gets the rest makes both agents better off. Hence the coupon's equilibrium allocation is not Pareto efficient.

We now come to a concept of fairness, which is ultimately why such rationing schemes are introduced. The concept is due to Pazner and Schmeidler (1974) and is defined as follows: An allocation - price pair $(x^0,a^0)\in(\mathbb{R}^l)^n x\Delta_2$ is said to be coupon's income fair if $\Sigma^l_{j=1}x^{0j}_i a^0_j = \Sigma^l_{j=1} x^{0j}_k a^0_j \ \forall\ i,k\in\{1,\dots,n\}$.

If $v_i = v_k = \frac{v}{n}$ $\forall$ i,k∈{1,...,n}, then a coupon's equilibrium is coupon's income-fair. Hence, in some currency such an equilibrium allocation is income-fair and has the merits normally associated with an income fair allocation.

3. Existence, Efficiency and Fairness of a Coupons Equilibrium with Sales :- Suppose now that a market for coupons is introduced with one unit of coupon costing 'c' units of money with c>0. Then the budget constraint of agent i becomes

$$\Sigma_{j\in R} \hat{p}_j x^j_i + \Sigma_{j\notin R} p_j x^j_i + \Sigma^l_{j=1} (ca_j) x^j_i \leq w_i + cv_i$$

where $(p_j)_{j\notin R} \in \Delta_1$, and $(a_j)^l_{j=1} \in \Delta_2$.

Let $\Delta_3 = \{(\psi_j)^l_{j=1} / \psi_j \geq 0 \ \forall\ j\in\{1,...,l\}$ and $\Sigma^l_{j=1} \psi_j \Omega^j = x + cv\}$.

For $(\psi_j)^l_j = \psi$, let $x_i (\in \mathbb{R}^l_+) \in d_i(\psi)$ if and only if x_i solves

$$\left.\begin{aligned} & u_i(y) \to \max \\ & \text{s.t. } \Sigma_{j\in R} \hat{p}_j y^j + \Sigma^l_{j=1} \psi_j y^j \leq w_i + cv_i \\ & y^j \geq 0 \ \forall\ j=1,...,l \\ & y^j \leq \Omega^j + 1 \ \forall\ j=1,...,l. \end{aligned}\right\} \quad (**)$$

The following lemma is immediate:

Lemma 2 :- If $x_i << \Omega^j$ solves (**) then x_i solves

$$\begin{aligned} & u_i(y) \to \max \\ & \text{s.t. } \Sigma_{j\in R} \hat{p}_j y^j + \Sigma^l_{j=1} \psi y^j \leq w_i + cv_i \\ & y^j \geq 0 \ \forall\ j=1,...,l. \end{aligned}$$

Proof :- Follows from the semi-strict quasi concavity of u_i, i=1,....,n.

Q.E.D.

Let $d(\psi) = \Sigma^n_{i=1} d_i(\psi)$, for $\psi \in \Delta_3$. Then $d: \Delta_3 \to\to Q$ is a non-empty valued, convex valued, compact valued, uppersemicontinuous correspondence.

<u>Theorem 2</u> :- Under the above assumptions there exists a coupon's equilibrium with sales

<u>Proof</u> :- Let $f : \Delta_3 \times Q \to \Delta_3$ be defined as follows:

$$f_j(q, z) = \frac{q_j + \max(0, z^j - \Omega^j)}{1 + \Sigma^l_{j=1}[\max(0, z^j - \Omega^j)] \cdot \frac{\Omega^j}{\alpha + cv}}, \quad j = 1, \ldots, l$$

where $c > 0$ is a fixed real number. Let $h(q, z) \equiv \{f(q, z)\} \times d(q)\ \forall (q, z) \in \Delta_3 \times Q$. The range of h is $\Delta_3 \times Q$. h is non-empty valued, convex valued, compact valued and uppersemicontinuous. Thus by Kakutani's fixed point theorem, there exists $(q^0, z^0) \in \Delta_3 \times Q$ such that $(q^0, z^0) \in h(q^0, z^0)$. Now consider the system of linear equations:

$$\hat{p}_j + ca_j = q^0_j \quad \forall j \in R$$

$$p_j + ca_j = q^0_j \quad \forall j \in R^c$$

$$(p_j)_{j \in R^c} \in \Delta_1,\ (a_j)^l_{j=1} \in \Delta_2.$$

The total number of equations are $l+2$, of which l are linearly independent. The total number of unknowns are where $l + |R^c|$ where $|R^c| \equiv$ cardinality of R^c. Hence the system of equations has at least one solution. In fact corresponding to each $(p^0_j)_{j \in R^c} \in \Delta_1$ with $p_j \leq q^0_j\ \forall\ j \in R^c$, one can find a unique $a^0 \in \Delta_2$. Hence (p^0, a^0, c, z^0) is a coupon's equilibrium with sales (by Lemma 2).

Q.E.D.

The following theorem asserts the Pareto efficiency of a coupon's equilibrium with sales.

<u>Theorem 3</u> :- A coupon's equilibrium with sales is Pareto efficient.

<u>Proof</u> :- Suppose (p^0, a^0, c^0, x^0) is a coupon's equilibrium with sales which is not Pareto efficient. Then, there exists $\bar{x} \in (\mathbb{R}^l_+)^n$ such that

(i) $\Sigma^{n}_{i=1}\bar{x}_i \leqq \Omega$

(ii) $u_i(\bar{x}_i) \geqq u_i(x^0_i)$ $\forall$ $i=1,\ldots,n$ with at least one strict inequality.

If $u_i(\bar{x}_i) > u_i(x^0_i)$, then $\Sigma_{j\in R}\hat{p}_j \bar{x}^j_i + \Sigma_{j\in R^c} p^0_j \bar{x}^j_i + \Sigma^l_{j=1} a^0_j \bar{x}^j_i > w_i + c^0 v_i$.

If $u_i(\bar{x}_i) = u_i(x^0_i)$, then it is easy to see that

$$\Sigma_{j\in R}\hat{p}\, x^j_i + \Sigma_{j\in R^c} p^0_j \bar{x}^j_i + \Sigma^l_{j=1} a^0_j x^j_i \geqq w_i + c^0 v_i.$$

Thus $\Sigma_{j\in R}\hat{p}_j \Omega^j + \Sigma_{j\in R^c} p^0_j \Omega^j + \Sigma^l_{j=1} a^0_j \Omega^j > \Sigma^n_{i=1}[w_i + c^0 v_i]$, which is a contradiction.

Hence x^0 is Pareto efficient.

Q.E.D.

In some senses a coupon's equilibrium with sales is superior to a coupon's equilibrium:

Theorem 4 :- Let (p^0, a^0, c^0, x^0) be a coupon's equilibrium with sales and (p^*, a^*, x^*) be a coupon's equilibrium. Then $u_i(x^*_i) \geqq u_i(x^0_i)$ $\forall$ $i=1,\ldots,n$.

Proof :- Since x^*_i is feasible for the ith agent's problem with $s_i = 0$, $u_i(x^0_i) \geqq u_i(x^*_i)$.

Q.E.D.

Hence every agent finds a coupon's equilibrium with sales, superior to a coupon's equilibrium (without sales). However, we can make a much stronger statement about a coupon's equilibrium with sales when c^0 and v_i, $i=1,\ldots,n$ are suitably chosen.

Following Baumol (1986), we say that an allocation $\bar{x}\in(\mathbb{R}^l)^n$ such that $\Sigma^n_{i=1}\bar{x}_i \leqq \Omega$ in fair if $\forall$ $i,k\in\{1,\ldots,n\}$, $u_i(\bar{x}_i) \geqq u_i(\bar{x}_k)$. The following theorem is thus significant:

Theorem 5 :- If $c^0 > 0$ and v_i, $i=1,\ldots,$ are such that $w_i + c^0 v_i = w_k + c^0 v_k$ $\forall$ $i,k\in\{1,\ldots,n\}$, a coupon's equilibrium with sales is fair.

Proof :- If (p^0, a^0, c^0, x^0) be the corresponding coupon's equilibrium with sales, then $\forall$ i, $k\in\{1,\ldots,n\}$

$$\Sigma_{j\in R}\ \hat{p}_j x^{0j}_k + \Sigma_{j\in R}\ p_j x^{0j}_k + \Sigma^n_{j=1}\ c^0 a^0_j x^{0j}_k = w_k + c^0 v_k = w_i + c^0 v_i .$$

Hence, $u_i\ (x^0_i) \geq u_i\ (x^0_k)$.

Q.E.D.

Observe we do not need to redistribute incomes in order to obtain a fair allocation. All that we need to do is to distribute coupons among the agents in such a way that the aggregate money income with coupons being saleable is the same for all agents. This is a significant result, because it declares a coupon's equilibrium with sales as superior to a coupon's equilibrium without sales both on grounds of efficiency and distributive justice. It is also much easier to implement than an equal income market equilibrium. We do not confiscate wealth from the wealthy. We simply compensate the poor with additional coupons, so that they can acquire the same wealth as the rich after trading the coupons on the market.

4. Conclusion :- The special case with R={1,...,l}, corresponds to a Dreze-Muller equilibrium with and without sales of coupons. Hence our solutions are a generalization of earlier solution concepts.

In sum, we observe that a coupon's equilibrium with sales is both fair and efficient. Further its easy implementability makes it a more popular egalitarian device than income transfers.

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