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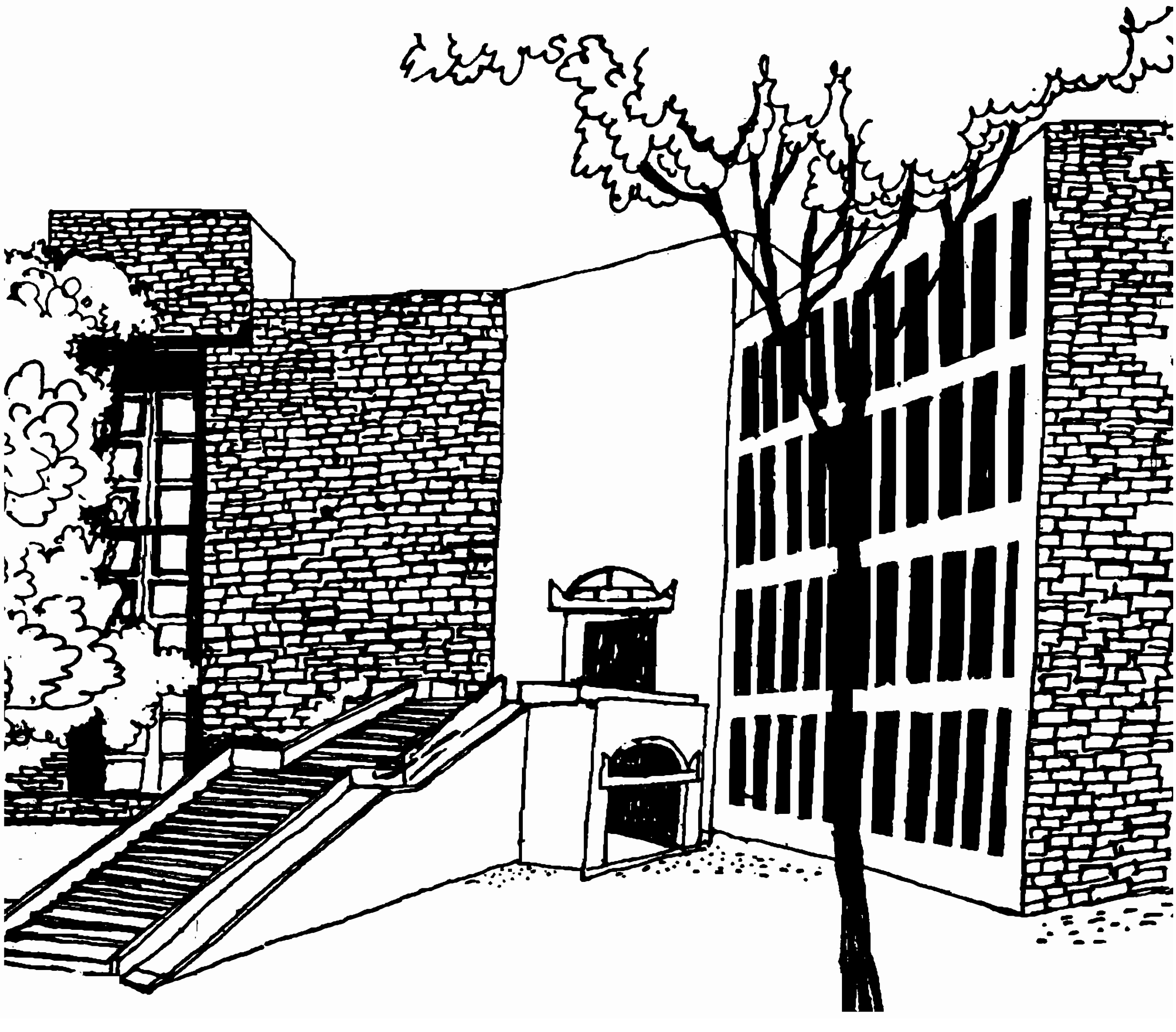

MID-POINT DOMINATION AND A SOLUTION TO CHOICE PROBLEMS ON NON-CONVEX DOMAINS

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Abstract

In this paper we axiomatically characterize a solution to choice problems on non-convex domains, which satisfies mid-point domination and other intuitively reasonable properties.

1. Introduction: The theory of choice is concerned with the problem of choosing an alternative from a feasible set of alternatives. Beginning with the work of Nash (1950), choice theory in Euclidean spaces (i.e. feasible sets are subsets of a finite dimensional Euclidean space) has developed many solution concepts, which have been surveyed rather elegantly in Moulin (1988). Some properties of solution concepts to choice problems find lucid description in Thomson (forthcoming) with related developments receiving analytical expression in Lahiri (forthcoming).

In Lahiri (1994a) and references therein can be found a theory of choice problems with a target. Briefly such problems append a target point to each choice problem and the solution to such a problem depends on both the feasible set of attribute vectors as well as the target point. Lahiri (1994b) contains a possible developments of the theory of such problems.

The purpose of this paper is to axiomatically characterize a solution to such problems which satisfies a property called mid-point domination (due to Thomson (forthcoming)). The relative egalitarian solution due to Kalai-Smorodinsky (1975) also satisfies mid-point domination. In fact on the subdomain of bargaining problems (i.e. choice problems without an explicit target point), our solution coincides with the relative egalitarian solution. Thus our solution extends the relative egalitarian solution onto a larger domain.

In Lahiri (1994a), we consider one particular extension of the relative egalitarian solution and provide an axiomatic characterization of the extended solution. In this paper we present yet another extension of the relative egalitarian solution.

Traditionally, choice theory has been developed on domains which admit only convex feasible sets. However, Kaneko (1980) and Herrero (1989) are notable exceptions where attempts have been made to extend choice theory to non-convex domains. Our axiomatic characterization of the proposed solution also takes place on non-convex domains. However our reasons for the choice of non-convex domains is different from the reasons implicit in the earlier mentioned works. In previous axiomatic characterizations, a full-blown axiomatic characterization on convex domains has been extended rather ingeniously to non-convex domains. As far as our work is concerned, no such parallel exists. In particular our proof is not valid on convex domains. Whether that is a property specific to our proof or a property inherent in our axiomatic characterization, is as yet a matter of conjecture which only future research can decide.

2. The Model: A choice problem (with a target) is an ordered pair (S,c) where $0 \in S \subseteq \mathbb{R}^n_+$ for some $n \in \mathbf{N}$ (the set of natural numbers). S is called the feasible set of alternatives and c is called the target point (or claims point). We shall follow Thomson and Lensberg (1989) in considering the following class $\mathcal{C}$ of admissible choice problems: $(S,c) \in \mathcal{C}$ if and only if

(i) S is nonempty and compact;

(ii) S is strictly comprehensive: (a) $0 \leq y \leq x \in S \Rightarrow y \in S$; (b) $x \in S$, $y \in S$, $y > x \Rightarrow \exists z \in S$ such that $z >> x$.

A domain is any subset of $\mathcal{C}$.

Let D be a domain. A choice function is a function $F: D \rightarrow \mathbb{R}^n_+$ such that $\forall (S,c) \in D$, $F(S,c) \in S$.

The domain of bargaining problems is defined thus: $(S,c) \in \mathcal{C}_u$ if and only if $c = u(S)$ where,

$u_i(S) = \max\{x_i / x \in S\}$, $i \in \{1,\ldots,n\}$.

For obvious reasons, we suppress c whenever $(S,c) \in \mathcal{C}_u$ and simply write $S \in \mathcal{C}_u$, with the interpretation being obvious.

In this paper we consider the following domain, when it comes to our characterization theorem:

$$\mathcal{C} = \{(S,c) \in \mathcal{C} / c >> \frac{u(S)}{n} \text{ and } (S - \{\frac{u(S)}{n}\}) \cap \mathbb{R}^n_+ \text{ is convex, non-empty}\}.$$

Let D be a domain and $F: D \rightarrow \mathbb{R}^n_+$ be a choice function.

(a) F is said to satisfy efficiency if $\forall (S,c) \in D$, $y \in S, y \geq F(S,c) \Rightarrow y = F(S,c)$.

(b) F is said to satisfy scale invariance if $\forall (S,c) \in D$ and $a \in \mathbb{R}^n_{++}$, $F(aS,ac) = aF(S,c)$ whenever $(aS,ac) \in D$.

(Here for $a \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n_+$, $ax \in \mathbb{R}^n_+$ with

$ax=(a_1x_1,\dots,a_nx_n); aS=\{ax/x\in S\})$.

(c) F is said to satisfy <u>symmetry above mid-point</u> if $\forall(S,c)\in D$ with $\left(S\left(\frac{u(S)}{n}\right), c-\frac{u(S)}{n}\right)\in D$, $F_i(S,c)-\frac{u_i(S)}{n} = F_j(S,c)-\frac{u_j(S)}{n}$

$\forall i,j\in\{1,\dots,n\}$, whenever $c_i-\left(\frac{u_i(S)}{n}\right) = c_j-\frac{u_j(S)}{n})\ \forall i,j\in\{1,\dots,n\}$

and $x\in S\left(\frac{u(S)}{n}\right)$ implies $\pi(x)\in S\left(\frac{u(S)}{n}\right)$ where, $\pi:\{1,\dots,n\}$

$\rightarrow\{1,\dots,n\}$ is a permutation. (Here $S\left(\frac{u(S)}{n}\right) = \left\{S-\left(\frac{u(S)}{n}\right)\right\}\cap\mathbb{R}^n_+$.

(d) F is said to satisfy <u>strong restricted monotonicity</u> if $\forall(S,c),(T,c)\in D$ with $S\subseteq T$ and $u(S)=u(T)$, $F(S,c)\leq F(T,c)$.

We refer to property (d) as <u>strong restricted monotonicity</u> to distinguish it from restricted monotonicity defined either on ℓ_u as in Moulin (1988) or on ℓ as defined in Lahiri (1994a).

(e) F is said to satisfy <u>mid-point domination</u> if $F(S,c)\geq\frac{u(S)}{n}$ $\forall(S,c)\in D$.

The choice function we propose to characterize axiomatically is the following:

$$G(S,c)=\frac{u(S)}{n}+t(S,c)\left[c-\frac{u(S)}{n}\right]$$

where $t(S,c)=\max\{t\in\mathbb{R}_+/\frac{u(S)}{n}+t\left[c-\frac{u(S)}{n}\right]\in S\}$.

3. The Main Theorem: In this section, we state and prove the main characterization theorem.

<u>Theorem</u>: The only choice function which satisfies properties (a), (b), (c), (d) and (e) on ℓ is G.

<u>Proof</u>: It is easy to see that G satisfies the above mentioned properties. Hence let us assume that $F:\ell\rightarrow\mathbb{R}^n_+$ is a choice function satisfying the above properties and let $(S,c)\in\ell$.

If $\frac{u(S)}{n}$ is an efficient point (i.e. $y\geq\frac{u(S)}{n}$, $y\in S\Rightarrow y=\frac{u(S)}{n}$) then by (a) and (e), $F(S,c)=\frac{u(S)}{n}=G(S,c)$.

Thus assume $\frac{u(S)}{n}$ is not an efficient point of S.

By property (b) i.e. scale invariance we may assume that $c-\frac{u(S)}{n}$ has all co-ordinates equal. Hence $G(s,c)-\frac{u(S)}{n}$ has all co-ordinates equal.

Let $(T,c)\in D$ be constructed as follows: Choose $\bar{\alpha}$ such that $\bar{\alpha}=\max\{\alpha\in\mathbb{R}_+/$convex comprehensive hull $[(\alpha,0,\ldots,0),\ldots,(0,\ldots,0,\alpha),\ G(S,c)-\frac{u(S)}{n})]\subseteq S\left(\frac{u(S)}{n}\right)\}$.

Such a $\bar{\alpha}$ exists by the same methods as in Moulin (1988) (proof of Theorem 3.2). Let, $T=S\cap$ Convex comprehensive hull $[(\bar{\alpha}+\frac{u_1(S)}{n},0,\ldots,0),\ldots,(0,\ldots,0,\bar{\alpha}+\frac{u_n(S)}{n}),(u_1(S),0,\ldots,0),\ldots,(0,\ldots,0,u_n(S)),$ $G(S,c)]$.

Clearly $u(T)=u(S)$ and $T\subseteq S$. thus by property (d), $F(S,c)\geq F(T,c)$.

Now $T\left(\frac{u(T)}{n}\right)$ is a symmetric set and $c-\frac{u(T)}{n}=c-\frac{u(S)}{n}$ has all co-ordinates equal. Hence by property (c), $F(T,c)-\frac{u(T)}{n}$ has all co-ordinates equal.

By the efficiency of $G(S,c)$ in T, $F(T,c)=G(S,c)$.

By the efficiency of $G(S,c)$ in S, $F(S,c)\geq G(S,c)$ implies $F(S,c)=G(S,c)$.

Q.E.D.

4. Conclusion: We have thus managed to axiomatically characterize the choice function G, albeit, on a domain somewhat larger than the one used in received choice theory. Our domain is however smaller than the ones used in earlier expositions of choice theory on non-convex domains. Whether our axiomatization is valid on a domain restricted further is still an open question.

Notations: $\mathbb{R}^n$ stands for Euclidean n-dimensional space

$\mathbb{R}^n_+$ $\{x=(x_1,\ldots,x_n)\in\mathbb{R}^n/x_i\geq 0, i=1,\ldots,n\}$

For $x,y\in\mathbb{R}^n$, $x\geq y \Leftrightarrow x_i\geq y_i \forall i\in\{1,\ldots,n\}$

$x>y \Leftrightarrow x\geq y$ and $x\neq y$

$x>>y \Leftrightarrow x_i>y_i \forall i\in\{1,\ldots,n\}$.

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