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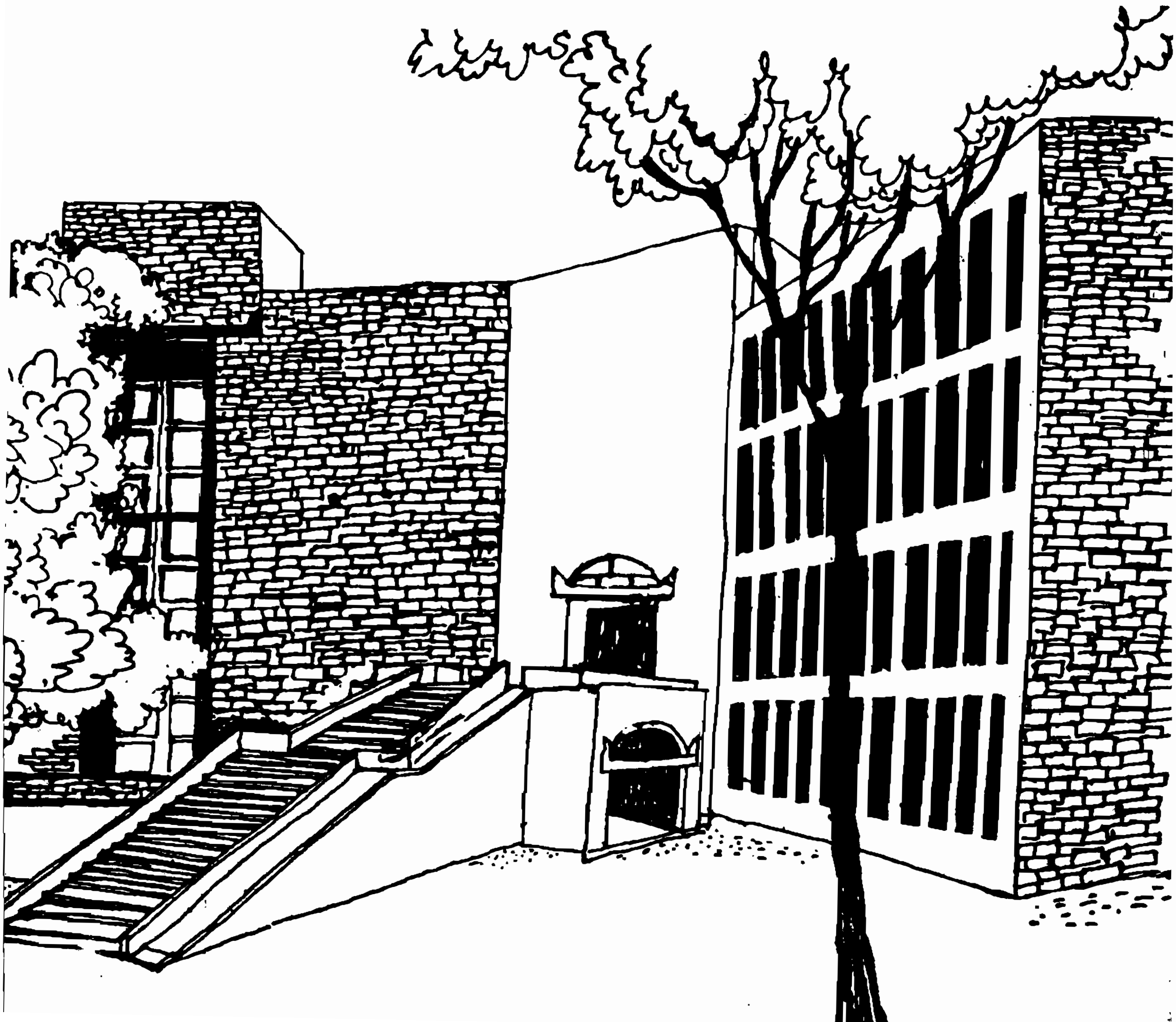

AN AXIOMATIC CHARACTERIZATION OF THE EQUAL INCOME MARKET EQUILIBRIUM CHOICE CORRESPONDENCE IN NON-CONVEX ECONOMIES

By

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ABSTRACT

A property of allocation rules (in problems involving fair division of infinitely divisible goods amongst a finite number of agents) that has received considerable attention in recent literature is consistency. The property may be described as follows: Apply a allocation rule to a problem within a chosen class of problems. Consistency would require that the restriction of any allocation chosen by the rule for that problem to any subgroup of agents is what this solution would recommend for the "reduced problem" obtained by imagining the departure of the members of the complementary group with what they receive, and re-evaluating the situation from the view point of those who remain.

In Lahiri [1997] can be found an axiomatic characterization of the equal income market equilibrium choice correspondence in economies characterized by convex preferences, and using the properties of consistency and converse consistency. The earliest known work in a similar direction is the one by Thomson [1988] followed by Thomson [1994] where consistency plays a fundamental role. In the Lahiri [1997] paper, the second fundamental theorem of welfare economics plays a crucial role. Young [1993] provides a generalization of a Thomson [1988] result, where a characterization of sub-correspondences of the equal income competitive equilibrium allocations are available (for convex preferences) using consistency and a property called replication invariance. Simply put, replication invariance says that if we replicate an allocation problem, the solution outcomes also get replicated.

In this paper, we extend the Young [1993] result to economies where preferences may be non-convex. We invoke a property called Sigma optimality which Svennson [1994] uses rather persuasively to establish the existence of fair allocations in non-convex economies. The corresponding welfare theorem in terms of σ - optimality is used to characterize the equal income market equilibrium choice correspondence in terms of consistency replication invariance, and a property called quasi-local independence. The last property is implied by local independence - a property used repeatedly in Lahiri [1997].

For a general survey of the literature concerning problems of fair division, the reader should refer to Thomson and Varian [1985].

1. Introduction:- A property of allocation rules (in problems involving fair division of infinitely divisible goods amongst a finite number of agents) that has received considerable attention in recent literature is consistency. The property may be described as follows: Apply a allocation rule to a problem within a chosen class of problems. Consistency would require that the restriction of any allocation chosen by the rule for that problem to any subgroup of agents is what this solution would recommend for the "reduced problem" obtained by imagining the departure of the members of the complementary group with what they receive, and re-evaluating the situation from the view point of those who remain.

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In this paper, we extend the Young [1993] result to economies where preferences may be non-convex. We invoke a property called Sigma optimality which Svennson [1994] uses rather persuasively to establish the existence of fair allocations in non-convex economies. The corresponding welfare theorem in terms of σ - optimality is used to characterize the equal income market equilibrium choice correspondence in terms of consistency replication invariance, and a property called quasi-local independence. The last property is implied by local independence - a property used repeatedly in Lahiri [1997].

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2. The Model:- Let $\mathbf{R}$ denote the real line, $\mathbf{R}_+$ the set of non-negative reals and $\mathbf{R}_{++}$ the set of strictly positive

reals. Let $\mathbb{N}$ denote the set of all strictly positive integers. Given $\phi \neq X \subseteq \mathbf{R}$ and Q any non-empty finite set, let X^Q denote the set of all functions from Q to X.

Let P be the collection of all non-empty finite sets Q, where to avoid Russell's paradox we assume that the statements "Q belongs to Q" and "Q does not belong to Q" have no meaning. Given $Q \in P$ and $k \in \mathbb{N}$, let Q^k be the set Q x { 1, ..., k }. Clearly $Q^k \in P$.

Let there be $L \geq 2$, $(L \in \mathbb{N})$ infinitely divisible goods in the economy. The commodity space is $\mathbf{R}^L$ and the consumption set of any conceivable agent (consumer) is $\mathbf{R}^L_+$.

Any $Q(\in P)$, is an agent set.

An economy E is a pair $\langle (u^i)_{i \in Q}, \omega \rangle$ satisfying the following properties:-

(i) $Q \in P$, is the agent set

(ii) $\omega \in \mathbf{R}^L_{++}$, is the aggregate social endowment

(iii) $\forall\, i \in Q$, $u^i : \mathbf{R}^L_+ \to \mathbf{R}$ is a continuous and weakly increasing (i.e. $x^i, y^i \in \mathbf{R}^L_+, x^i \gg y^i$ implies $u^i(x^i) > u^i(y^i)$) utility function, which is continuously differentiable in $\mathbf{R}^L_{++}$.

(iv) $\forall\, i \in Q$, $x^i, y^i \in \mathbf{R}^L_+, x^i \in \mathbf{R}^L_{++}$ and $u^i(x^i) = u^i(y^i)$ implies $y^i \in \mathbf{R}^L_{++}$.

Let $\mathscr{E}$ be the set of all economies satisfying the above properties.

Given $E=\langle (u^i)_{i\in Q},\ \omega\rangle$, let

$A(E) = \{(x^i)_{i\in Q} \in \mathbf{R}_+^Q \ / \ \Sigma_{i\in Q} x^i = \omega\}$.

A (E) is the set of all feasible allocations for E.

Given E as above a feasible allocation $(x^i)_{i\in Q}$ is said to be Pareto Optimal, if there does not exist $(y^i)_{i\in Q} \in A(E)$ such that $u^i(y^i) > u^i(x^i)\ \forall i \in Q$. [This is actually the definition of Weak Pareto Optimality; however it is easy to show that for our kind of economies Weak Pareto Optimal allocations and Pareto Optimal allocations coincide.] Let P (E) denote the set of all Pareto Optimal allocations. It is easy to show that $P(E) \neq \phi$

Given E as above let

$IR(E) = \left\{ (x^i)_{i\in Q} \in A(E) \,/\, u^i(x^i) \geq u^i\left(\frac{\omega}{|Q|}\right) \forall\, i\in Q \right\}$. IR(E) is the set of all allocations for E which are Individually rational from equal division.

Given E as above and $k\in\mathbf{N}$, E^k denotes the k-replica of E where E^k is defined as follows:

$$E^k = \langle (u^{i,j})_{(i,j)\in Q^k}, k\omega \rangle \quad \text{where} \quad \forall (i,j) \in Q^k, u^{i,j} = u^i.$$

Given $E\in\mathcal{E}$ and $(x^i)_{i\in Q} \in A(E)$ (where Q is the agent set for E) and $k\in\mathbf{N}$, let $(y^{(i,j)})_{(i,j)\in Q^k}$ be defined as $y^{(i,j)} = x^i \forall (i,j) \in Q^k$. $(x^i)_{i\in Q} \in P(E)$ is said to be σ- optimal for E if $\forall k\in\mathbf{N}, (y^{(i,j)})_{(i,j)\in Q^k} \in P(E^k)$. Let $\sigma-P(E)$ denote the set of all σ - optimal allocations for E.

Given $E\in\mathcal{E}$, a feasible allocation $(\overline{x}^i)_{i\in Q}$ is said to be a price equilibrium if there exists $p\in\mathbf{R}^L_+\setminus\{0\}$ such that $\forall i\in Q$, $\overline{x}^i$ solves

$$u^i(x^i) \to \max$$

$$s.t.\ p.x^i \leq p.\overline{x}^i$$

$$x^i \in \mathbf{R}^L_+$$

The following result is due to Svennson [1994]:

Proposition 1:- Given $E\in\mathcal{E}$

(i) every price equilibrium allocation is σ-*cptimal*.

(ii) every σ-*cptimal* allocation is a price equilibrium allocation, with respect to a unique price $p\in\mathbf{R}^L_+\setminus\{0\}$

Let $\overline{\mathscr{E}} = \{E \in \mathscr{E} / P(E) = \sigma - P(E)\}$.

Let $V \subset \overline{\mathscr{E}}$ be given

A choice correspondence on V is a non-empty valued correspondence $F: V \to\to \bigcup_{Q \in P} (\mathbf{R}_+^L)^Q$ such that $\forall E \in V, F(E) \subset A(E)$.

A choice correspondence F on is said to be efficient if $F(E) \subset P(E) \, \forall E \in V$.

A choice correspondence F on V is said to be individually rational (from equal division) if $F(E) \subset IR(E) \, \forall E \in V$.

A choice correspondence F on V is said to be consistent if $\forall E = \langle (u^i)_{i \in Q}, \omega \rangle \in V, \forall \phi \neq M \subset Q$,

$\forall (x^i)_{i \in Q} \in F(E), E' = \langle (u^i)_{i \in M}, \Sigma_{i \in M} x^i \rangle \in V$ implies

$(x^i)_{i \in M} \in F(E')$.

A choice correspondence F on V is said to be replication invariant if

$\forall E = \langle (u^i)_{i \in Q}, \omega \rangle \in V,\quad \forall k \in \mathbf{N},\ (x^i)_{i \in Q} \in F(E)$ implies

$(x^{(i,t)})_{(i,t) \in Q^k} \in F(E^k)$ provided $E^k \in V$.

We now define the equal income market equilibrium choice correspondence G as follows:

Let $E = \langle (u^i)_{i \in Q}\ \omega \rangle \in \overline{\mathscr{E}}$. $(\overline{x}^i)_{i \in Q} \in A(E)$

is said to be an equal income market equilibrium allocation if there exists $p \in \mathbf{R}^L_+ \setminus \{0\}$ such that $\forall i \in Q$, $\overline{x}^i$ solves

$$u^i(x^i) \to \max$$

$$s.t.\ p.x^i \leq 1$$

$x^i \in \mathbf{R}^L_+$.

The following proposition is due to Svennson [1994].

Proposition - 2:- Every $E \in \overline{\mathscr{E}}$ possesses an equal income market equilibrium allocation.

In view of the above we can define G(E) to be the set of equal income market equilibrium allocations for $E \in \overline{\mathscr{E}}$. G is called the equal income market equilibrium choice correspondence. Further given $E \in \overline{\mathscr{E}}$, $(x^i)_{i \in Q} \in G(E)$ implies $x^i \in \mathbf{R}^L_+ \ \forall i \in Q$.

The following proposition is easily verified:

Proposition 3:- G satisfies efficiency, Individual Rationality from equal division, Consistency and Replication Invariance.

3. The Main Result:- We now present a generalization of a theorem due to Young [1993].

Theorem 1:- Let F be a choice correspondence on $\overline{\mathscr{E}}$ which satisfies Efficiency, Individual Rationality from equal division, Consistency and Replication Invariance. Then $\forall E \in \overline{\mathscr{E}}, F(E) \subset G(E)$.

Proof:- Assume the conditions of the theorem for F. Let $E = \langle (u^i)_{i \in Q}, \omega \rangle \in \overline{\mathscr{E}}$ and towards a contradiction assume,

$(\overline{x}^i)_{i \in Q} \in F(E) \setminus G(E)$. By proposition 1 and since, $F(E) \subset P(E)$,

there exists $p \in \mathbf{R}^L_+ \setminus \{0\}$ such that $\forall i \in Q, \overline{x}^i$ solves

$$u^i(x^i) \to \max$$

$$s.t. \quad p.x^i \leq p.\overline{x}^i$$

$$x^i \in \mathbf{R}^L_+.$$

Without loss of generality, we may assume $p.\omega = |Q|$. We would be done if we could show $p.\bar{x}^i \leq \frac{p.\omega}{|Q|}\ \forall\ i\epsilon Q$. Assume $p.\bar{x}^i > \frac{p.\omega}{|Q|}$ for some $i\epsilon Q$. Thus there exists $j\epsilon Q$ such that $p.\bar{x}^j < \frac{p.\omega}{|Q|}$. Thus $p.\bar{x}^j < p.\bar{x}^i$.

By the smoothness assumption on preferences, there exists $\bar{\lambda}\ \epsilon\ (0,1)$ sufficiently small so that $u^j(\bar{x}^j+\lambda(\bar{x}^i-\bar{x}^j)) > u^j(\bar{x}^j)\ \forall\ \lambda\ \epsilon\ (0,\bar{\lambda})$. Choose $k\ \epsilon\ \mathbf{N}$ such that $\frac{1}{1+k} < \bar{\lambda}$ and consider E^k. Let $\bar{y}^{(n,m)} = \bar{x}^n\ \forall\ (n,m)\epsilon Q^k$. By Replication Invariance $(\bar{y}^{(n,m)})_{(n,m)\ \epsilon\ Q^k}\ \epsilon\ F(E^k)$. Let $S\subset Q^k$ be defined as follows:

$$S = \{(i,1)\} \cup \{(j,m)\ :\ m = 1,\ldots,k\}.$$

Let $E' = \langle (u^n)_{n\epsilon S}, k\bar{x}^j + \bar{x}^i \rangle$.

By Consistency, $((\bar{y}^{(j,m)})_{m=1}^{k}, \bar{x}^{i}) \in F(E')$.

But $$\frac{k\bar{x}^{j} + \bar{x}^{i}}{k+1} = \bar{x}^{j} + \frac{1}{k+1}(\bar{x}^{i} - \bar{x}^{j}).$$

Hence $$u^{j}\left(\frac{k\bar{x}^{j} + \bar{x}^{i}}{k+1}\right) > u^{j}(\bar{x}^{j}) = u^{j}(\bar{y}^{(j,m)}) \ \forall m = 1, \ldots, k.$$

This contradicts Individual Rationality from equal division and proves the theorem.

Q. E. D.

As a corollary to the above theorem it easily follows that the largest choice correspondence which satisfies Efficiency, Individual Rationality from equal division, Consistency and Replication Invariance is G.

The theorem we have proved above is actually an extension of the Thomson [1988] result and not so much an extension of the Young [1993] result, given its present form. However, that is because our domain of problems consists of fair division problems as considered in Lahiri [1997].

Young [1993] considers economies where the income of each agent is also part of the economic environment and then obtains an axiomatic characterization like the one presented here, for sub-correspondences of the market equilibrium correspondence, when preferences are convex. Since an easy adjustment of the Svennson [1994] proof of the existence of equal income market equilibrium for non-convex economies yields a proof of the existence of market equilibrium for arbitrary income distribution (: to be precise, in the proof, the expenditure functions have to be divided by the income, and a fixed point of the associated function has to be sought), the nonemptiness of the market equilibrium correspondence is guaranteed. The rest of the argument, leading to a direct generalization of the Young [1993] result to accommodate non-convex preferences, proceeds exactly like the proof of Theorem 1, provided above.

For the ensuing discussion we consider a sub-domain of given $E = \langle (u^i)_{i\in Q}, \omega\rangle \in \overline{\mathcal{E}}$, say that E is strictly monotonic in $\mathbf{R}^L_{++}$ if $\forall\, i \in Q$, $\forall x^i, y^i \in \mathbf{R}^L_{++}$, $x^i > y^i$ implies $u^i(x^i) > u^i(y^i)$. Note we require x^i, y^i to belong to $\mathbf{R}^L_{++}$.

Let, $\mathcal{E}^{*} = \{ E \in \overline{\mathcal{E}} \;/\; E \text{ is strictly monotonic} \}$.

Clearly if $E \in \mathcal{E}^{*}$ and p is a market equilibrium price vector for E, then $p \in \mathbf{R}^{L}_{++}$.

The following proposition a general version of which has been stated and proved in Lahiri [1997] will be required in the sequel:

Proposition 4:- Let $(x^{i})_{i \in Q} \in (\mathbf{R}^{L}_{++})^{Q}$, $\omega \in \mathbf{R}^{L}_{++}$, $p \in \mathbf{R}^{L}_{++}$ with $p \cdot \omega = |Q|$. Then there exists $E \in \mathcal{E}^{*}$ such that $\{(x^{i})_{i \in Q}\} = G(E)$.

We now invoke the following property:

A choice correspondence F on $\mathcal{E}^{*}$ is said to satisfy Quasi-

Local Independence if $\forall E = \langle (u^i)_{i\in Q}, \omega \rangle$, $E' = \langle (v^i)_{i\in Q}, \omega \rangle \in \mathcal{E}^*$

with $u^i = v^i \; \forall \, i \neq j$, if $(x^i)_{i\in Q} \in F(E) \cap P(E')$ and $p^j \in \mathbf{R}^L_{++}$ is normal to the tangent at x^j to both $\{ y^j \in \mathbf{R}^L_+ / u^j(y^j) \geq u^j(x^j) \}$ and

$\{ y^j \in \mathbf{R}^L_+ / v^j(y^j) \geq v^j(x^j) \}$ then $(x^i)_{i\in Q} \in F(E')$.

The difference between local independence and quasi-local independence is that, in the latter $(x^i)_{i\in Q}$ is required to belong to $P(E')$ as well. Note that the above definition is perfectly general, in that all the utility functions could be different. This is obtained, by changing one utility function at a time and applying Quasi-local Independence at each stage.

__Proposition 5__:- G satisfies Quasi-Local Independence on $\mathcal{E}^*$.

__Proof__:- Easy.

The following theorem is the main result of the paper.

Theorem 2:- The only choice correspondence on $\mathscr{E}^*$ to satisfy Efficiency, Individual Rationality from Equal Division, Consistency, Replication Invariance and Quasi-Local Independence is G.

Proof:- Let F be a choice correspondence which satisfies the properties and let $E = \langle (u^i)_{i\in Q}, \omega \rangle \in \mathscr{E}^*$. By theorem 1, $F(E) \subset G(E)$. Let $(x^i)_{i\in Q} \in G(E) \subset \mathbf{R}^L_{++}$. Let $p \in \mathbf{R}^L_{++}$ be the associated unique price vector. By Proposition 4, there exists $E' = \langle (v^i)_{i\in Q}, \omega \rangle \in \mathscr{E}^*$ satisfying the hypothesis of Proposition 4 for all i such that $\{ (x^i)_{i\in Q} = G(E')$. Since $F(E') \subset G(E')$ and $F(E') \neq \phi$, $F(E') = \{ (x^i)_{i\in Q} \}$. Now $(x^i)_{i\in Q} \in P(E)$ and E and E' (with roles reversed) satisfy all the conditions of Quasi-Local Independence. Thus $(x^i)_{i\in Q} \in F(E)$. Thus F(E)=G(E).

Q.E.D.

We have thus been able to obtain a characterization theorem for the equal income market equilibrium correspondence on a suitable domain of possibly non-convex economies.

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