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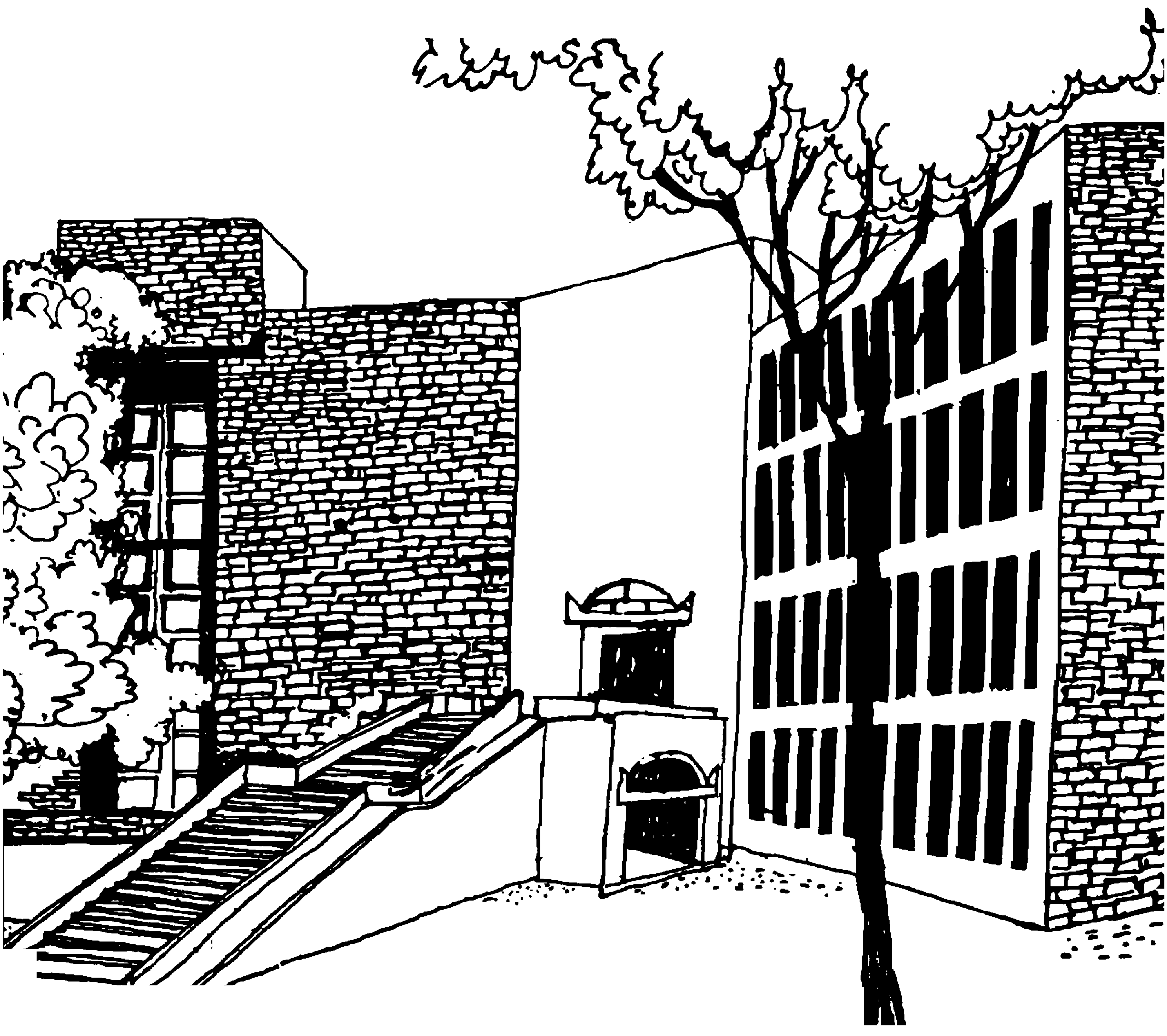

REVEALED PREFERENCE AND UTILITARIANISM IN MULTIATTRIBUTE CHOICE PROBLEMS

By

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Abstract

In this paper we have investigated two related issues : (a) representability of a choice function; (b) characterization of a representable choice functions, namely the utilitarian choice function. Probably the most well-known representable choice function is the one due to Nash (1950), which is characterized by properties similar to those which characterize the utilitarian choice function, except that instead of shift invariance we have scale invariance for Nash's solution. Our proof of representability is simpler and easier to comprehend than most other proofs existing in the literature.

Our characterization of the utilitarian choice function is both elegant and concise. The meaningfulness of the properties characterizing this solution, should enhance its appeal as a choice function.

1. Introduction :- In Peters and Wakker (1991) can be found a theory of rational choice in multiattribute choice problems, which answers the question: When is a choice function, a maximizer of a real valued function defined on the nonnegative orthant of an Euclidean space. They draw on the consumer demand theory in microeconomics and answer the question along the lines of Richter (1971), Varian (1982) and Pollack (1990).

However, Sondermann (1982) has provided an elementary treatment of revealed preference and the initial purpose of this paper is to obtain a similar treatment for multiattribute choice problems.

Subsequently, we turn to a study of the utilitarian choice function and obtain a new characterization result for the same. This solution is well-known in the literature on social choice and independent characterizations of the same can be found in Myerson (1981) and Thomson (1981). We propose yet another characterization in this paper.

2. Multiattribute Choice Problems :- A multiattribute choice problem is an ordered pair (S,c) where $0\in S\subseteq \mathbb{R}^n_+$ and $c\in \mathbb{R}^n_+$ for some $n\in \mathbb{N}$ (the set of natural numbers). The set S is called the set of feasible attribute vectors and the point c is called a target point. Similar constructs have been studied in Chun and Peters (1991), Chun and Thomson (1992), Bossert (1992a), Bossert (1992b), Lahiri (1993a,b,c), Abad and Lahiri (1993).

We shall consider the following class $\mathcal{C}$ of admissible multiattribute choice problems : $(S,c)\in\mathcal{C}$ if and only if

(i) S is compact and convex

(ii) S satisfies minimal transferability : $x\in S$, $x_i > 0 \Rightarrow \exists y\in S$ with $y_i < x_i$ and $y_j > x_j\ \forall j \neq i$.

(iii) S is comprehensive : $x\in S$, $0\leq y\leq x \Rightarrow y\in S$

(Here for $x,y\in \mathbb{R}^n$, $x\geq y$ means $x_i \geq y_i\ \forall i\in\{1,\ldots,n\}$; $x>y$ means $x\geq y$, $x \neq y$; $x>>y$ means $x_i > y_i\ \forall\ i=1,\ldots,n$.)

A similar class of problems can be found in Moulin (1988), in the analysis of axiomatic models of bargaining.

A domain in any subset D of $\mathcal{Q}$.

A (multiattribute) choice function on $D(\subseteq\mathcal{Q})$ is a function $F:D \to \mathbb{R}^n$ such that $\forall (S,c)\in D$, $F(S,c)\in S$.

Let $F:D \to \mathbb{R}^n$ be a choice function. Two important properties often required of a choice function are the following :

(P.1) Anonymity :- For any permutation σ of $N=\{1,\dots,n\}$, $\forall (S,c)\in D$, $F(\sigma(S), \sigma(c))=\sigma(F(S,c))$, (where for any vector $x\in\mathbb{R}^n_+$, write $\sigma(x)$ for the vector whose ith component $\sigma(x)_i = x_{\sigma(i)}$), whenever $(S,c)\in D$ implies $(\sigma(S), \sigma(c))$ belongs to D.

(P.2) Efficiency :- $\forall (S,c)\in D$, $x\in S$, $x \geqq F(S,c) \Rightarrow x=F(S,c)$

An important domain, studied traditionally in axiomatic bargaining is the following:

$\mathcal{Q}_u \equiv \{(S,c)\in\mathcal{Q} / c=u(S)$ where $u_i(S)=\max\{x_i / x\in S\}\}$.

For obvious reasons problems $(S,c)\in\mathcal{Q}_u$ will be denoted simply by S.

A property that turns out to be important in characterizing a choice function which we shall study later is the following:

(P.3) Nash's Independence of Irrelevant Alternatives (NIIA) :- $\forall S,S'\in\mathcal{Q}_u$: $\{S\subseteq S'$ and $F(S')\in S\} \Rightarrow \{F(S)=F(S')\}$.

3. Representable choice functions on $\mathcal{Q}_u$:- We now try to impose conditions on a choice function $F:\mathcal{Q}_u \to \mathbb{R}^n$ such that there exists a realvalued function $f:\mathbb{R}^n_+ \to \mathbb{R}$ and $\forall S\in\mathcal{Q}_u$,

$$F(S)= \arg\max_{x\in S} f(x).$$

Such choice functions will be called representable choice functions.

Given a choice function $F:\mathcal{Q}_u \to \mathbb{R}^n$ we define a binary relation R_F on $\mathbb{R}^n_+$ as follows : xR_Fy ("x is directly revealed preferred to y") if there is an $S\in\mathcal{Q}_u$ with $x=F(S)$, $y\in S$.

We now postulate the following axiom which is essentially sufficient for the representability of choice functions:

Strong Axiom of revealed Preference (SARP) :- R_F is acyclic; i.e. $x^1 R_F x^2 R_F \dots R_F x^k$ implies not $x^k R_F x^1$ where $x^1, \dots, x^k \in\mathbb{R}^n_+$ and are necessarily distinct.

Let H_F denote the <u>transitive hull</u> of R_F that is, $xH_F y$ if and only if $xR_F\ x^1 R_F x^2 \ldots R_F y$ for some finite (possibly empty) sequence $x^1, \ldots, x^k$ in $\mathbf{R}^n_+$. Then the Strong Axiom is equivalent to : H_F is irreflexive.

We now prove the following theorem, which is essentially a slight modification of Sondermann (1982).

<u>Theorem 1</u> :- Let $F:\ell_u \to \mathbf{R}^n$ be a choice function satisfying the following connectedness property:

$\forall x,y \in \mathbf{R}^n_+$ such that $x, y \in \{F(S) : S \in \ell_u\}$ and $xR_F y$ there exists $U \in \ell_u$ and $t \in (0,1)$ such that $tx+(1-t)y=F(U)$, and $\{x,y\} \cap$ interior $(U) \neq \emptyset$. Then the Strong Axiom of Revealed Preference implies that F is a representable choice function.

<u>Proof</u> :- (Almost as in Sondermann (1982)) : The topology of $\mathbf{R}^n_+$ has a countable base of open sets, say $\{0_k\}_{k \in \mathbf{N}}$ For $x=F(S), S \in \ell_u$ define $N(x)=\{k \in \mathbf{N} : x \in 0_k$ or $wH_F x$ for some $w \in 0_k\}$ and $f(x) = \Sigma_{k \in N(x)} 2^{-k}$. For $x \in \mathbf{R}^n_+ \setminus \{F(S):S \in \ell_u\}$ set $f(x)=-1$. For $x \neq y$, let $xR_F\ y$. Clearly, by transitivity of H_F, $N(x) \supseteq N(y)$, hence $f(x) \geq f(y)$. If $y \notin \{F(S):S \in \ell_u\}$, then $f(x) \geq 0 > -1 = f(y)$. Otherwise by hypothesis there exists $t \in (0,1)$, $U \in \ell_u$ such that $z=tx+(1-t)y=F(U)$ and $\{x,y\} \cap$ interior $(U) \neq \emptyset$.

Let $x=F(S)$. $xR_F\ y \Rightarrow y \in S$; thus by convexity of S, $z \in S$. Further $z \neq x$ since $t \in (0,1)$. Thus $xR_F z$. By the Strong Axiom of Revealed Preference we have not $zR_F x$ i.e. $x \notin U$. Thus $y \in$ interior (U). Hence $\exists\ k \in \mathbf{N}$ such that $y \in 0_k \subseteq$ interior (U) and $z \notin 0_k$ i.e. $xR_F\ zR_F\ w\ \forall w \in 0_k$. By the Strong Axiom of Revealed Preference $\forall w \in 0_k$, not $wH_F\ x$ i.e. $k \in N(y) \setminus N(x$. Thus $f(x)>f(y)$. This completes the proof.

Q.E.D.

The above result could have been proved on any domain $D(\subseteq \ell_u)$, so long as it is closed under the operation required in the hypothesis of Theorem 1. Such a domain is ℓ^0_u defined as follows :

Given $S \subseteq \mathbf{R}^n$, let $P(S)=\{x \in S / y \geq x, y \in S \Rightarrow y=x\}$. P(S) is called the <u>set of efficient points of S</u> . We define

$\mathcal{C}^0_u = \{S \in \mathcal{C}_u \,/\, x, y \in S,\ t \in (0,1) \Rightarrow tx + (1-t)y \notin P(S)\}$.

4. The Utilitarian Choice Function :- On $\mathcal{C}^0_u$ the utilitarian choice function $F_{ut} : \mathcal{C}^0_u \to \mathbb{R}^n$ is defined as follows :

$$F_{ut}(S) = \arg\max_{x \in S} (\Sigma^n_{i=1} x_i)$$

It is easy to see that on $\mathcal{C}^0_u$ the utilitarian choice function is well defined. Myerson (1981) has established the additivity of the utilitarian choice function i.e. if one adds two sets in $\mathcal{C}^0_u$, then the utilitarian choice for the sum is the sum of the utilitarian choices for each. Myerson proceeds to show that additivity, anonymity and efficiency essentially characterize the utilitarian choice function on $\mathcal{C}^0_u$. It can also be observed that F_{ut} satisfies the conditions of theorem 1.

Moulin (1988) proposes a characterization of the utilitarian choice function by considering a class of unbounded choice problems. We propose a characterization on $\mathcal{C}^0_u$ itself, since as much of multiattribute choice theory suggests, a domain should necessarily be a subset of $\mathcal{C}$. We consider the following property for a choice function $F : \mathcal{C}^0_u \to \mathbb{R}^n$.

(P.4) Shift Invariance :- For $S \in \mathcal{C}^0_u$ and $a \in \mathbb{R}^n_+$, let $S(a) = \{x - a : x \in S\} \cap \mathbb{R}^n_+$. ($S(a) \in \mathcal{C}^0_u$ since $a \in \mathbb{R}^n_+$). If $S \in \mathcal{C}^0_u$ and $0 \leqq a \leqq F_{ut}(S)$, then

$$F(S(a)) = F(S) - a.$$

Theorem 2 :- There is only one choice function on $\mathcal{C}^0_u$ which satisfies anonymity, efficiency NIIA and shift invariance. It is the utilitarian choice function.

Proof :- Let $S \in \mathcal{C}^0_u$. That F_{ut} satisfies the above properties is clear. If $S = \{0\}$, then $F(S) = 0 = F_{ut}(S)$. Hence suppose $S \neq \{0\}$ and let $x^* = F_{ut}(S)$. Let $\bar{\lambda} = \sup \{\lambda \geqq 0 / x^* - \lambda e \in \mathbb{R}^n_+\}$ where e is the vector in $\mathbb{R}^n_+$ with all coordinates being equal to unity. Let $a = x^* - \bar{\lambda} e \leqq x^*$. $a \in \mathbb{R}^n_+$ by the definition of $\bar{\lambda}$. It is easily observed that $F_{ut}(S(a)) = \bar{\lambda} e$. Let T be the smallest symmetric set in $\mathcal{C}^0_u$ containing $S(a)$. Clearly $\bar{\lambda} e \in T$; infact $\bar{\lambda} e \in P(T)$. By efficiency and anonymity, $F(T) = \bar{\lambda} e \in S(a)$. (Observe $\bar{\lambda} e \in P(S(a))$). By NIIA, $F(S(a)) = \bar{\lambda} e$, thus completing the proof.

Q.E.D.

5. Conclusion :- In this paper we have investigated two related issues : (a) representability of a choice function; (b) characterization of a representable choice functions, namely the utilitarian choice function. Probably the most well-known representable choice function is the one due to Nash (1950), which is characterized by properties similar to those which characterize the utilitarian choice function, except that instead of shift invariance we have scale invariance for Nash's solution. Our proof of representability is simpler and easier to comprehend than most other proofs existing in the literature.

Our characterization of the utilitarian choice function is both elegant and concise. The meaningfulness of the properties characterizing this solution, should enhance its appeal as a choice function.

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