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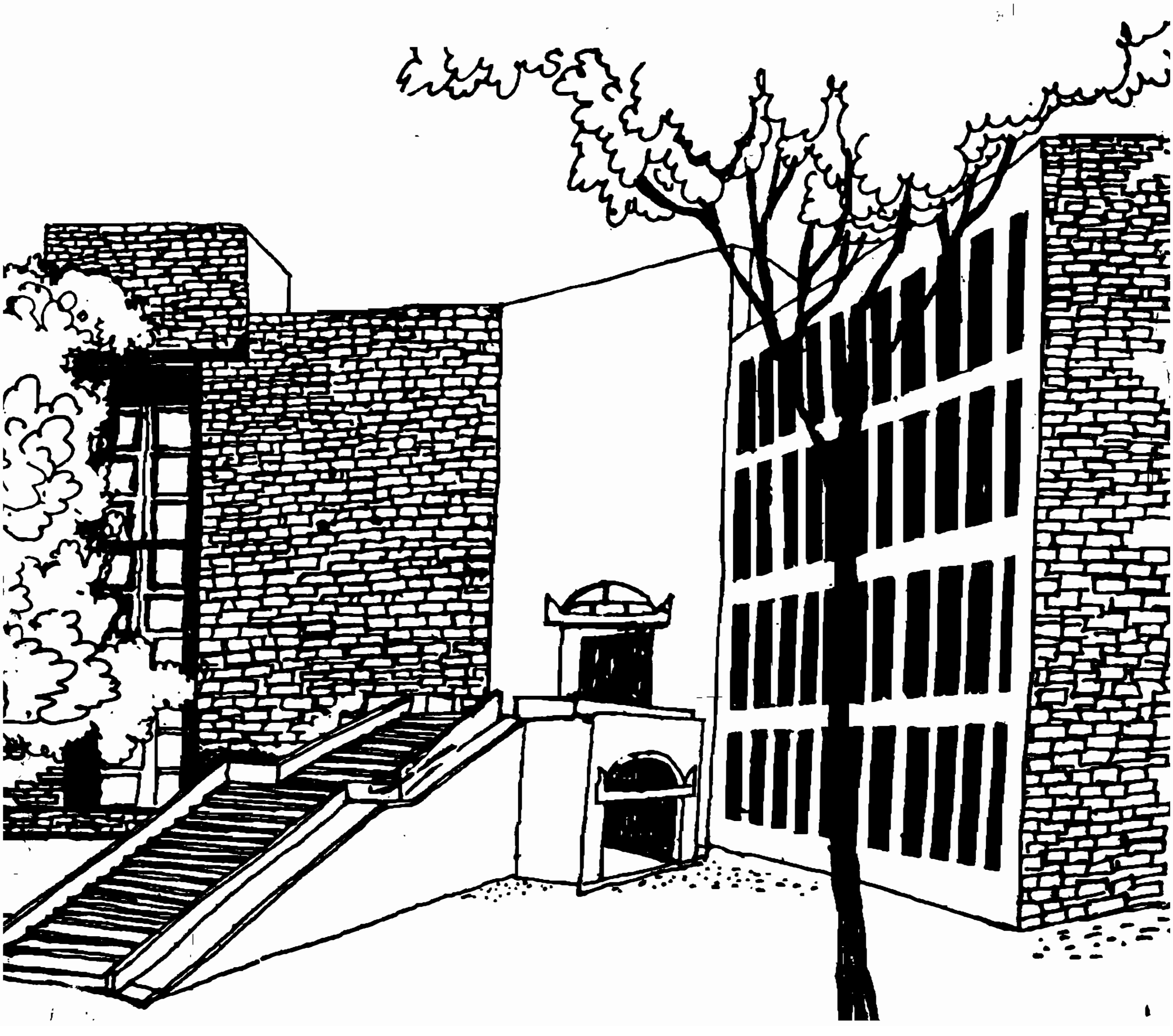

POPULATION MONOTONICITY AND THE CONSTRAINED EQUAL AWARDS SOLUTION FOR RATIONING PROBLEMS

By

Somdeb Lahiri

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INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD - 380 015
INDIA

Abstract

In this paper, we axiomatically characterize the Constrained Equal Awards Solution for Rationing Problems, using the axioms of No-Envy, Population Monotonicity, Resource Continuity and replication invariance.

Introduction :

The problems of allocating resources under the conditions of excess demand are all pervasive. Such problems arise quite naturally, when in the case of a market with a downward sloping demand curve and an upward sloping supply curve, the government chooses to impose a price ceiling on the commodity (which is below the price at which quantity demanded is equal to the quantity supplied i.e. the market equilibrium price). The study of this phenomena is one of the important aspects of disequilibrium economics, surveyed in various levels of generality by Benassy (1982) and Silvestre (1986).

When there is excess demand, goods have to be rationed among the consumers and it is the study of these rationing rules (under the guise of a Talmudic bankruptcy problem) from a game theoretic perspective, which was pioneered by O'Neill (1982). Aumann and Maschler (1985) and Curiel, Maschler and Tijs (1988) provides axiomatic characterizations of several solutions to such rationing problems.

More recently Dagan (1996) provided an axiomatic characterization of the Constrained equal awards rule, which is the rule we intend to study in this paper. A fixed population and a variable population sequel to Dagan's (above cited) axiomatic characterizations are the axiomatic characterizations by Lahiri (1996 a,b). Basically, the constrained equal awards solution gives the agents with low demands what they want; each high demander gets an equal amount.

In a recent paper, Thomson (1995) studies the constrained equal awards rule (also called the uniform rule) in the framework of economies with single peaked preferences, and provides an axiomatic characterization using a property, called population monotonicity. In terms of preferences, our domains of bankruptcy problems are a strict subset of the domain of single peaked preferences, allowing only those single peaked preferences, whose portion above the horizontal axis form an isosceles triangle with the origin as a vertex. Thus results which hold for single peaked preferences may fail to hold, since proofs on the larger domain of single peaked preferences avail of the greater manouverability that the larger domain provides. Such is the case with Thomson's characterization where in addition to no-envy and population monitinicity, we now require replication invariance and continuity with respect to total resources. This is what we proceed to establish in this paper.

2. The Model :

Let ¶N be the set of natural numbers and let P be the set of all non-empty finite subsets of ¶N. Given $Q \in P$, a rationing problem for Q is an ordered pair $(d, S) \in$ ¶$R^{Q}_{++} \times$ ¶R_{+} such that $\sum_{i \in Q} d_i > S$. If d_i is interpreted as the demand of agent $i \in Q$ (the agent set that has been realized), then the above condition implies excess demand, where S is the total supply.

Let C^Q stand for the set of all rationing problems for Q and let $C = \bigcup_{Q \in P} C^Q$. Let $X = \bigcup_{Q \in P} ¶R^Q_{++}$.

Given $Q \in P$, and a rationing problem $(d, S) \in C^Q$ an allocation for (d, S) is a vector $x \in ¶R^Q_+$ such that $x_i \leq d_i \ \forall \ i \in Q$.

A solution on C is a function $F : C \rightarrow X$ such that $\forall$ (d,S) $\in$ C, F (d,S) is an allocation for (d,S).

The Contrained Equal Awards Solution denoted CEA $: C \rightarrow X$ is defined as follows : $\forall \ Q \in P$, $(d,S) \in C^Q$, CEA (d,S) = x implies

$$x_i = \min \{\lambda, d_i\}, \ i \in Q,$$

where $\lambda \geq o$ is so chosen that $\sum_{i \in Q} x_i = S$.

It is well known that such a λ exists uniquely given $(d,S) \in C$.

3. Properties of Solutions :

In this section we discuss some properties that we would like our solution to satisfy.

Resource Continuity : F is said to satisfy resource continuity if given $(d,S) \in C$ and $\epsilon > o$, there exists $\delta > o$ such that $|S' - S| < \delta$, $(d, S') \in C \rightarrow \|F(d,S) - F(d,S')\| < \epsilon$ where the norm is simply the Euclidean norm...

Resource Continuity is really a mind regularity assumption.

No-envy : F is said to satisfy no-envy if $\forall\ Q \in P$, and $(d, S) \in C^Q$, if $x = F(d, S)$, then $|d_i - x_i| \leq |d_i - x_j|$ $\forall\ i, j \in Q$.

What no-envy means is that one's own award should not be further away from one's own demand than the award received by some-one else. This property has been used by Lahiri (1996b) along with bilateral consistency and resource monotoricity to characterize the CEA solution.

Population Monotonicity : F is said to satisfy population monotinicity if $\forall\ Q \in P$ and $k \in$ ¶N $- Q$, $(d,S) \in b^Q$, $(d',S) \in b^{Q \cup k}$, if $d_i = d'_i$ $\forall\ i \in Q$, then $F_i(d',S) \leq F_i(d,S)$ $\forall\ i \in Q$.

Population mononotonicity says that the arrival of a new agent, should not increase the awards for existing agents. This assumption seems quite reasonable.

Replication-Invariance : F is said to satisfy replication invariance if $\forall\ Q \in P$ and $k \in$ ¶N, if $Q' \in P$ with $|Q'| = k|Q|$ and $i \in Q$ implies $(i,1),\ldots,(i,k) \in Q'$ such that for $(d,S) \in C^Q$ and $(d', kS) \in C^{Q'}$, $d'_{(i,j)} = d_i$, $j=1,\ldots,k$, $i \in Q$, then $x=F(d,S)$ implies $y_{(i,j)}=x_i$ $\forall\ i \in Q$, $j=1,\ldots,k$, where $y=F(d', kS) \in$ ¶$R^{Q'}_{+}$

The meaning of replication invariance is quite simple: if a rationing problem is replicated k times (i.e.) the available supply is multiplied k times and corresponding to each original agent there are now k agents with the same demand). then each replica in the replicated problem gets what the original agent in the original problem got. This assumption seems harmless.

In Lahiri (1996b) we show that CEA satisfies no-envy. The other axioms are easier to verify for the CEA solution. In the rest of the paper, we shall be concerned with proving the converse proposition that if a solution satisfies all the properties listed above, then it must coincide with CEA. Inspite of our best attempts to the contrary we have not been able to dispense with either replication invarance or resourse continuity. Whether it is possible to do away with any one or both of these two axioms is an open question, which we leave for future research.

4. The Main Theorem :

In this section we prove the main therorem of this paper, which states that the only solution to satisfy no-envy, population monotonicity, resource continuity and replication invariance is the CEA solution.

Theorem 1 :

The only solution to satisfy no-envy, population monotonicity, replication invariance and resourse continuity is CEA.

Proof :

That CEA satisfies the above properties has been discussed earlier. Hence, let us establish the converse. Thus, suppose F is a solution which satisfies the desired properties and towards a contradiction assume that there exists $Q \in P$, $(d, S) \in C^Q$ such that $F(d,S) \neq CEA\ (d,S)$. Thus there exists $i, j \in Q$ such that

$$x_i < d_i,\ x_i \neq x_j$$

where $x = F\ (d,S)$.

By no-envy, we must have

$$x_i < d_i \leq 2d_i - x_i \leq x_j \leq d_j$$

If we keep the available supply fixed at S, and simply replicate each agent 'k' times, then by no-envy, each agent of the same type gets the same amount. By population monotonicity and no-envy, we must have

either $x^k_i \leq x_i < d_i \leq 2\ d_i - x_i \leq 2d_i - x^k_i \leq x^k_j \leq x_j \leq d_j$ (1)

or $x^k_i = x^k_j$ (2)

where x^k_i is the common amount that a type i agent gets in the replicated problem (where the supply remains) fixed.

If (1) holds $\forall k$, then

$$kx^k_j \geq k\ (2d_i - x_i) > S$$

for $k \in ¶N$ sufficiently large.

Hence for a sufficiently large replication, (2) holds.

Since i and $j \in Q$ were arbitrarily chosen, we get that there exists $k^* \in ¶N$, such that if each agent is replicated k* times and the supply is held fixed at S, then F(d′, S) = CEA (d′, S′) where d′ is as defined in the statement of the replication invariance property.

However, by replication invariance,

$$F_{(i,l)}\ (d', k^*S) = F_i\ (d, S)\ \forall\ i \in Q,\ l=1,...k^*$$

where (i,l) is the l^{th} agent of type i (i.e. the l^{th} replica of agent i in the original problem).

Thus, there exists i, j $\in$ Q such that

$$x_i < d_i \leq 2d_i - x_i \leq x_j \leq d_j$$

$$\text{and } x_i^{k^*} = x_j^{k^*} < C_i$$

As the total resources are increased form S to k*S, the individual awards of type i and type j agents change from $x_i^{k^*}$ to x_i and $x_j^{k^*}$ to x_j respectively. By resource continuity, there exists S′ > S, S′ < k*S such that if y_i is what a type i agent gets at S′ and y_j, is what a type j agent gets as S′, then $y_i < y_j < d_i$

Thus no-envy is easily seen to be violated; infact, i envies j.

This contradiction establishes the theorem.

Q.E.D.

References :

1. R. J. Aumann and M. Maschler (1985) : "Game theoretic analysis of a bankruptry problem from the Talmud", Journal of Economic Theory 36 : 195-213.

2. J. P. Benassy (1982) : "The Economics of Market Disequilibrium", San Diego : Academic Press.

3. I. J. Curiel, M. Maschler and S. H. Tijs (1988) : "Bankruptcy Games", Z. op. Res. 31 : A143-A159.

4. N. Dagan (1996) : "New Characterizations of old bankruptcy rules", Soc. Choice and Welfare 13 : 51-59.

5. S. Lahiri (1996 a) : "An Axiomatic Characterization of The Constrained Equal towards solution for Rationing Problems", mimeo.

6. S. Lahiri (1996 b) : "The Constrained Equal Awards Solution for claims Problems". Indian Institute of Management, Ahmedabad, Working Paper No:1315.

7. B. O'Neill (1822) : "A problem of rights arbitration from the Talmud", Math. Social Sciences 347-371.

8. J. Silvestre (1986) : "Elements of Fixed - Price Micro-economics", Ch.9 in Larry Samuelson (ed.) Micro Economic Theory, Martinus - Nijhoff.

9. W. Thomson (1995) : "Population - monotonic solution to the problem of fair division when preferences are single peaked", Econ. Theory 5, 229-246.