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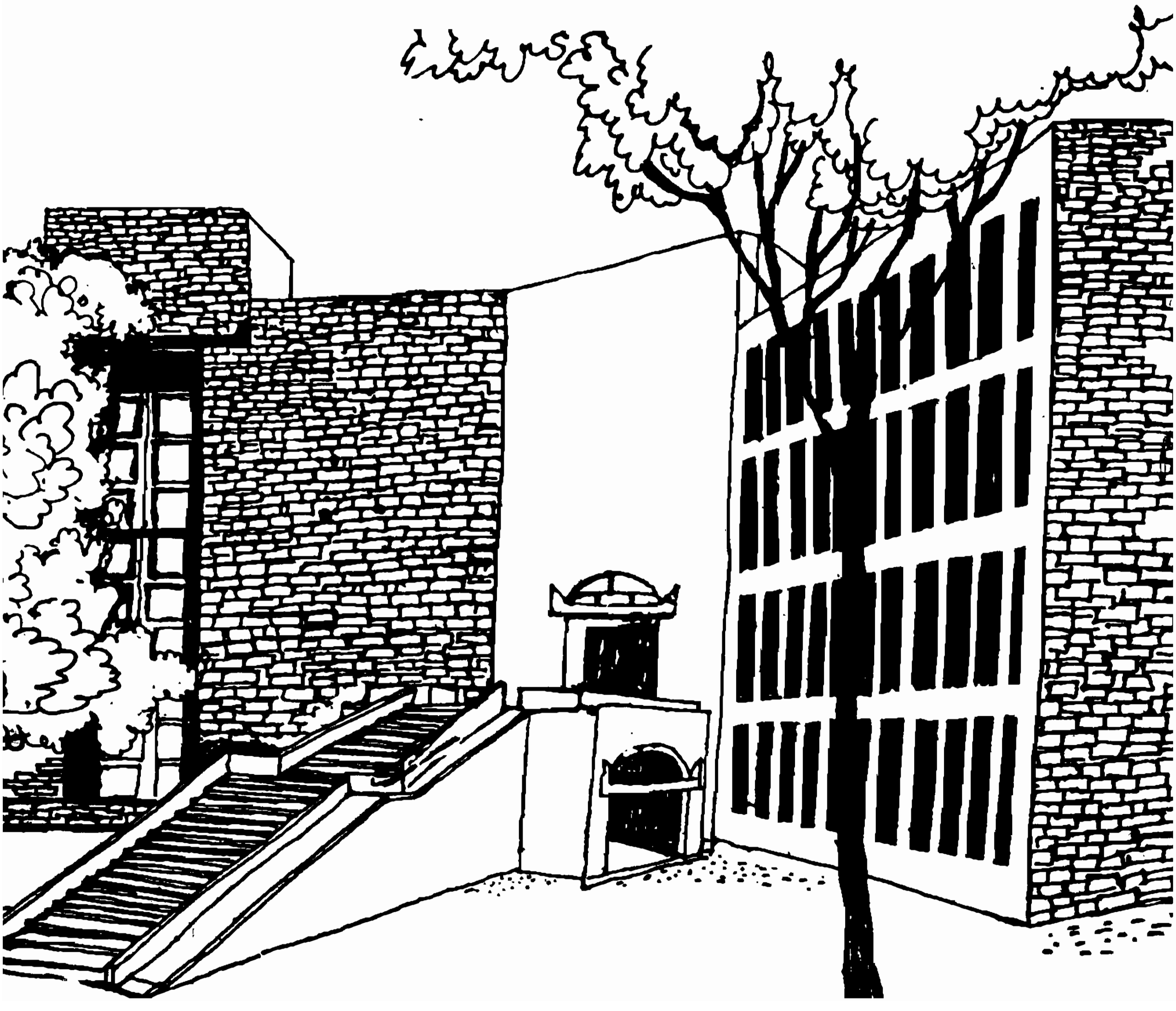

CONSISTENCY AND AN AXIOMATIC CHARACTERIZATION OF THE MARKET EQUILIBRIUM SOLUTIONS

By

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W P No.1335
October 1996

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September 1996

Abstract

In this paper, we present a unified theory for solutions to games of fair division, which are ordinal in nature and appear as non-symmetric variants of the equal income market equilibrium solution. We characterize the entire family of such solutions using consistency, converse-consistency, local-independence, individual rationality and a weak efficiency condition. This is all done in a variable population framework.

In the fixed population framework, we obtain an axiomatic characterization for the same family using monotonicity, individual rationality, local-independence, non-discrimination and another weak efficiency property.

1. Introduction:- Social choice in abstract spaces has been plagued with impossibility results. This has largely been due to the fact, that in abstract set theoretic settings it is almost impossible to impose enough structure on the problem, which would act as restrictions on the domain and lead to anything meaningful. Beleagured by such negative conclusions, economists in recent times have either given up social choice theory for lost or have chosen specific economic domains to hone their skills and arrive at possibility results. The motivation behind this paper, is of the latter category.

The economic domain most popular in recent social choice theory is one of allocating a given bundle of resources among a finite number of agents. The study of such problems is also known as games of fair division.

There are basically two approaches to solving such problems. In one approach, pioneered by Nash (1950), we are interested in first choosing a vector of utilities for the agents, and then choosing an allocation, comformable with the utility vector. There is of course the underlying assumption of agents being equipped with utility functions, which expresses their preferences for consumption bundles. This approach, surveyed rather extensively in Thomson and Lensberg (1989), is relentlessly cardinal in spirit. The other approach to the problem (which we pursue in this paper), proceeds by operating some variant of the market mechanism from an initial distribution of income. This approach is inherelty ordinal. The distribution of income is part of the process of resolving the problem. One such method is equal division of income, as discussed in Thomson and Varian (1985). Interesting possibilities, with regard to distortions of utility function in such a scenario, have been studied in Lahiri (1991).

Till now, there is no unified theory, characterizing the market mechanism type solutions, for games of fair division. The equal-income method is treated as a district category all by itself, which is not amenable to non-symmetric generalizations. This is true, atleast in so far as the axiomatic theory of solutions to such problems go. Our purpose in this paper is to obtain an axiomatic characterization for the entire class of solutions consisting of the equal income market equilibrium solution as well as its non-symmetric variants. Our axiomatic characterization closely follows the axiomatic characterization of the Walras correspondence provided by van den Nouweland, Peleg and Tijs [1996].

A problem with the approach to the axiomatic characterizations for the Walras correspondence in the above cited paper, is that the correspondence may fail to be non-empty valued except under very stringent conditions. Further, the conventional model of pure exchange, though celebrated and venerated fails to lend itself to easy interpretation via consistency axioms. There is the recurrent problem of 'net indebtedness to the outside world', which crops up whenever we consider a reduction in the number of agents, with the departing agents getting their share of the Walrasian allocation.

In this paper, we use the axioms of consistency and converse consistency to characterize the market equilibrium solutions. Apart from the paper by van den Nouweland, Peleg and Tijs [1996], the other notable characterization theorems in similar lines of research can be found in Gevers [1986], Thomson [1988]. Thomson [1994], Nagahisa [1991], Nagahisa [1992], Nagahisa [1994] and Nagahisa and Suh [1995].

2. The Model:- We consider economies with arbitrary finite number of agents. Let N be a non-empty set of potential agents, whether finite or infinite (: if infinite N will be identified with the set of natural numbers). An agent set M is any non-empty finite subset of N. By $\mathbb{R}^l_+$ we denote the non-negative orthant of $\mathbb{R}^l$ and the strictly positive orthant of $\mathbb{R}^l$ is denoted by $\mathbb{R}^l_{++}$. For two vectors $x, y \in \mathbb{R}^l$ we denote $x \geq y$, $x > y$ and $x >> y$ according as $x-y \in \mathbb{R}^l_+$, $x-y \in \mathbb{R}^l_+ \setminus \{0\}$ and $x-y \in R^l_{++}$. The inner product x.y of two vectors x and $y \in \mathbb{R}^l$ is defined by $x.y = \sum_{i=1}^{l} x_i\, y_i$.

An economy is a list $E = \langle M; (u^i)_{i \in M}; \omega \rangle$ where M is an agent set, $\omega \in \mathbb{R}^l_{++}$ is the total initial endowment of goods in the economy and $u^i : \mathbb{R}^l_+ \to \mathbb{R}$ is the utility function of agent $i \in M$. An economy such as this is referred to in the introouction as a game of fair division. We will assume through-out the paper that the utility functions are continuous, quasi-concave and weakly increasing: given $E=<M;\ (u^i)_{i\varepsilon N};\ \omega>$ for $i\varepsilon M$ a utility function

$u^i:\mathbb{R}^l_+\to\mathbb{R}$ is said to be weakly increasing

if $\forall x, y\varepsilon\mathbb{R}^l_+, x\gg y\Rightarrow u^i(x)\geq u^i(y)$.

The class of all economics with continuous, quasi-concave and weakly increasing utility functions is denoted $\mathcal{E}$.

Given $E=<M;\ (u^i)_{i\varepsilon N};\ \omega>\varepsilon\mathcal{E}$, we define

$A(E)=\left\{(x^i)_{i\varepsilon N}\ \varepsilon\ (\mathbb{R}^l_+)^N / \sum_{i\varepsilon N} x^i=\omega\right\}$. A(E) is the set of

feasible allocations for $E\varepsilon\mathcal{E}$.

Let $W=\mathbb{R}^N_{++}$ i.e. the set of all functions from N to $\mathbb{R}_{++}$.

Given $E=<M;\ (u^i)_{i\varepsilon N};\ \omega>\ \varepsilon\ \mathcal{E},\ x\varepsilon A(E)$ is said to be a

<u>w-market equilibrium</u> for E is there exists

$p\varepsilon\mathbb{R}^l_+\setminus\{0\}\ with\ \sum_{i\varepsilon N} w^i=p.\omega,$ such that $\forall i\varepsilon M, x^i$ solves the following programming

problem:-

$u^i\ (y)\to$ max

$s.t.p.y\leq w^i$

$y\varepsilon\mathbb{R}^l_+$

In Lahiri [1995] we show that $\forall E\varepsilon\mathcal{E}$ and all $w\varepsilon W$ a w-market equilibrium exists for E. If $w \in W$ is such that $w_i = w_j \ \forall \ i, j \in M$, then a w-market equilibrium for E is also called an equal income market equilibrium for E.

Given $E = <M; (u^i)_{i \in M}; \omega> \in \mathscr{E}$, an $x \in A(E)$ is said to be efficient if there does not exist $y \in A(E)$ with $u^i(y^i) \geq u^i(x^i) \ \forall i \in M$ with at least one strict inequality. If x is efficient in E, then it is also referred to as Pareto Optimal in E. Given $E \in \mathscr{E}$, the set of efficient allocations is non-empty since A(E) is compact and the utility functions are continuous.

First Fundamental Theorem of Welfare Economics:- Given $E \in \mathscr{E}$ if x is a w - market equilibrium for E with $w \in W$, then x is efficient.

Second Fundamental Theorem of Welfare Economics:- Given $E = <M; (u^i)_{i \in M}; \omega> \in \mathscr{E}$ if x is efficient in E and

$x^i \in \mathbb{R}^l_{+}, \forall \ i \in M$ then there exists $w \in W$ such that x is a

w-market equilibrium for E.

These results are established in Malinvaud [1993].

Easily verifiable observation:- If $E = <M; (u^i)_{i \in M}; \omega > \in \mathscr{E}$, x is a w - market equilibrium for E for some $w \in W$ and $p \in \mathbb{R}^l_{+} \setminus \{0\}$ is the associated price vector (in the definition of a w - market equilibrium) then $p.x^i = w^i \ \forall \ i \in M$.

Given $E \in \mathscr{E}$ and $w \in W$, let $G^w(E)$ be the set of all w-market equilibrium for E; let G(E) be the set of all equal income market equilibrium for E.

A domain is any non-empty subset of $\mathscr{E}$.

We now consider two specific domains:

$$\mathscr{E}_1 = \{ \langle M; (u^i)_{i\in M}; \omega \rangle \in \mathscr{E} / \forall\, i\in M,\ \forall x\in \mathbb{R}^l_{++},\ y\in \mathbb{R}^l_{+},\ u^i(x) = u^i(y) \text{ implies } y\in \mathbb{R}^l_{++} \}$$

Let $E = \langle M;\ (u^i)_{i\in M};\ \omega \rangle \in \mathscr{E}$. E is said to be smooth if $\forall i\in M, \forall x\in R^l_{++}$, there exists a vector $p\in R^l_{+}$ (unique upto multiplication by a positive scalar) such that $p.y \geq p.x$ whenever $u^i(y) \geq u^i(x)$ and $y\in \mathbb{R}^l_{+}$.

Let $\mathscr{E}_2 = \{ E \in \mathscr{E} / E \text{ is smooth} \}$. Let $\mathscr{E}_0 = \mathscr{E}_1 \cap \mathscr{E}_2$.

Proposition 1:- Let $E = \langle M; (u^i)_{i\in M};\ \omega \rangle \in \mathscr{E}_1$. Then, given

$x\in \mathbb{R}^l_{++}$ and $y\in \mathbb{R}^l_{+} \setminus \mathbb{R}^l_{++}$ and $i\in M, u^i(x) > u^i(y)$.

Proof:- Towards a contradiction assume $u^i(y) \geq u^i(x)$.

Let $e\in \mathbb{R}^l_{++}$ be the vector with all co-ordinates equal to one.

By weak monotonicity $u^i(y+e) > u^i(y)$.

Let $z(t) = x + t(y+e-x), t\in [0,1]$.

$z(t) \in \mathbb{R}^l_{++} \forall t\in [0,1]$.

$u^i(z(0)) = u^i(x) \leq u^i(y) < u^i(y+e) = u^i(z(1))$.

Hence, by the intermediate value theorem for continuous functions, there exists $t\in [0,1)$

such that $u^i(z(t)) = u^i(y)$.

Since $z(t) \in \mathbb{R}^l_{+}$ we must have $y \in \mathbb{R}^l_{++}$. This contradiction establishes the proposition.

Q.E.D.

Proposition 2:- Let $E = \langle M;\ (u^i)_{i\in M};\ \omega \rangle \in \mathcal{E}_1$ and x be a w-market equilibrium for E for some $w \in W$. Then $x^i \in \mathbb{R}^l_{+} \forall i \in M$.

Proof:- Let E be as above and let x be a w-market equilibrium for E, $w \in W$. Since $w^i > 0$, if $p \in \mathbb{R}^l \setminus \{ 0 \}$ is the price vector associated with x, then there exists $z \in \mathbb{R}^l_{+}$ $with\ p.z \le w^i$. If $x^i \in \mathbb{R}^l \setminus \mathbb{R}^l_{+}$, then by proposition 1, $u^i(x^i) < u^i(z)$, contradicting that x is a w-market equilibrium.

Thus, $x^i \in \mathbb{R}^l_{+} \forall\ i \in M$.

Q.E.D.

Let $E = \langle M;\ (u^i)_{i\in M};\ \omega \rangle \in \mathcal{E},\ x \in A(E)\ and\ \phi \ne S \subset M$. The reduced economy of E with respect to S and x, denoted $E^{s,x}$ is defined as $\langle S;\ (u^i)_{i\in S};\ \omega - \sum_{i\in M\setminus s} x_i \rangle$, provided $\omega - \sum_{i\in M\setminus S} x^i \in \mathbb{R}^l_{++}$.

If $E \in \mathcal{E}_1\ and\ x$ is a w-market equilibrium for E then $E^{s,x}$ is well defined whenever $\phi \ne S \subset M$. This is because for $x \in A(E),\ x^i \in \mathbb{R}^l_{+} \forall\ i \in M(E),\ where\ M(E)$ is the agent set for E.

It is easy to see that given $E \in \mathcal{E}_1$, with agent set M, if x is a w-market equilibrium for E, for some $w \in W$ then given $\phi \neq S \subset M, x^S$ is a w-market equilibrium for $E^{S,x}$.

We close this section with the definition of a solution and a proposition. A solution F on $\mathcal{E}_0$ is a set valued mapping such that $\forall E \in \mathcal{E}_0,\ \phi \neq F(E) \subset A(E)$.

Proposition 3:- Given

$\phi \neq M \subset N,\ M\ finite,\ x \in (\mathbb{R}^l_+)^M,\ p \in \mathbb{R}^l_+,\ \omega \in \mathbb{R}^l_+,\ w \in W$ with

$p.\omega = \sum_{i \in M} w_i\ and\ \sum_{i \in M} x^i = \omega,$ there exists $E = \langle M;\ (u^i)_{i \in M};\ \omega \rangle \in \mathcal{E}_0$

such that, x is the unique w-market equilibrium for E (with associated price vector being p).

Proof:- Let $\alpha^i_j = P_j\, x^i_j\ /\ w^i,\ i \in M,\ j = 1,\dots,l.$

For $i \in M,\ y^i \in \mathbb{R}^l_+,\ let\ u^i\ (y^i) = \prod_{j=1}^{l} (y^i)^{\alpha^i_j}.$ [see Madden (1986)].

It is easy to see that x is a w-market equilibrium for $E = \langle M; (u^i)_{i \in M};\ \omega \rangle \in \mathcal{E}_0$

Let $z \in \mathbb{R}^l$ be any other w-market equilibrium for E with associated price vector q.

$\therefore \alpha^i_j = q_j\ z^i_j / w_i,\ i \in M,\ j = 1,\dots,l$

If $z \neq x,\ there\ \exists\ i \in M,\ j \in \{1,\dots,l\},$ such that $z^i_j \neq x^i_j$. Without loss of generality assume

$z^i_j > x^i_j$.

Then $q_j < p_j$.

Thus $z_j^k > x_j^k \; \forall \; k \in M$.

$$\therefore \sum_{k \in M} z_j^k > \sum_{k \in M} x_j^k = \omega_j.$$

Hence z is not feasible for E contradicting that z is a w-market equilibrium for E.

Q.E.D.

3. Characterizations of the Market Equilibrium Solutions:-

For $w \in W$, we define G^w to be the w-market equilibrium solution, where for each $E \in \mathcal{E}_0, G^w(E)$ is the set of all w-market equilibria for E. We define G to be the equal-income market equilibrium solution on $\mathcal{E}_0$ in a similar fashion.

A solution F on $\mathcal{E}_0$ is said to be consistent(CONS) if $\forall E \in \mathcal{E}_0$, $\phi \neq S \subset M(E)$ (:the agent set for E) and $x \in F(E)$:

(i) $E^{s,x} \in \mathcal{E}_0$

(ii) $x^s \in F(E^{s,x})$.

It is easy to see that G^w is consistent $\forall w \in W$

A solution F on $\mathcal{E}_0$ is said to satisfy Converse Consistency(COCONS) if for every $E \in \mathcal{E}$ with $|M(E)| \geq 3$ and for every $x \in A(E)$ that is efficient, if whenever $\phi \neq S \subset\subset M(E)$ we have (a) $E^{s,x} \in \mathcal{E}_0$ and (b) $x^s \in F(E^{s,x})$, then we also have $x \in F(E)$.

Lemma 1:- Given $w \in W, G^w$ is converse consistent.

Proof:- Let $E = \langle M; (u^i)_{i \in M}; \omega \rangle \in \mathcal{E}_0$ with $|M| \geq 3$.

Suppose $x \in A(E)$ be efficient and (a) $E^{S,x} \in \mathcal{E}_0$,

(b) $x^S \in G^w(E^{S,x}) \forall \phi \neq S \subset\subset M$. Clearly $x^i \in \mathbb{R}^l_+ \forall i \in M$. By the second fundamental theorem of welfare economics, there exists $p \in \mathbb{R}^l_+ \setminus \{0\}$ such that p supports u^i at $x^i, i \in M$. Since $E \in \mathcal{E}_0$, this p is unique upto multiplication by a positive scalar. Now $x^i \in G^w(E^{\{i\},x})$ implies that there exists $p^i \in \mathbb{R}^l_+ \setminus \{o\}$ which is a scalar multiple of p such that p^i supports u^i at x^i. Clearly $p^i.x^i = w^i$. We need to show $p^i = p^j$. Let $S = \{i, j\}$. Thus $(x^i, x^j) \in G^w(E^{S,x})$ implies that there exists $p^S \in \mathbb{R}^l_+ \setminus \{0\}$ (which is a scalar multiple of p^i and p^j) such that p^S supports u^i at x^i and u^j at x^j. Thus

$p^S.x^i = w^i$ and $p^S.x^j = w^j$. Thus $p^i = p^j = p^S$. Let $p^i = p', i \in M$. Thus $p'.x^i = w^i \forall i \in M$ and p' supports u^i at $x^i, i \in M$. Thus, $x \in G^w(E)$.

Q.E.D.

Remark:- In the definition of COCONS, $|M(E)| \geq 2$ (as is usually the case) will not do in showing that G^w is converse consistent. This is because $G^w(E) = \{\omega\}$, whenever $E = \langle M; (u^i)_{i \in M}; \omega \rangle$ with $|M| = 1$, whatever $w \in W$.

The next condition is due to Nagahisa (1991).

A solution F on $\mathcal{E}_0$ is said to satisfy local independence (LI) if $\forall E=\langle M;\ (u^i)_{i\in M};\ \omega\rangle$, if $x\in F(E)$ with $x^i\in \mathbb{R}^l_{++}$ and

$p^i \in \mathbb{R}^l_{++}$ supports u^i at x^i, then $x\in F(\bar{E})$ where

$\bar{E}=\langle M;\ (v^i)_{i\in M};\ \omega\rangle\in\mathcal{E}_0$ is such that $v^k=u^k\ \forall\ k\neq i$ and

p^i supports v^i at x^i.

It is easily seen that G^w satisfies LI on $\mathcal{E}_0$.

The next condition we invoke is a type of individual rationality condition.

Given $w\in W$, a solution F on $\mathcal{E}_0$ is said to satisfy w-Individual Rationality (w-IR) if

$$\forall E = \langle M;\ (u^i)_{i\in M};\ \omega\rangle \in \mathcal{E}_0,\ \forall\ x \in F(E),\quad u^i\ (x^i) \geq u^i \left(\frac{w^i}{\sum_{k\in M} w^k}\ \omega\right)\forall\ i \in M.$$

In a similar context Young [1993] calls $\frac{w^i}{\sum_{k\in M} w^k}\ \omega$ agent i's entitlement in E.

A solution F on $\mathcal{E}_0$ is said to satisfy <u>Individual Rationality from Equal Division (IRED)</u> if

$$\forall E=\langle M;\ (u^i)_{i\in M};\ \omega\rangle\in\mathcal{E}_0,\quad \forall\ x \in F(E), u^i\ (x^i) \geq u^i \left(\frac{\omega}{|M|}\right)\forall\ i \in M.$$

The final property we invoke is a mild efficiency condition as in van den Nouweland, Peleg and Tijs [1996].

A solution F on $\mathscr{E}_0$ is said to satisfy Pareto Optimality for Two Agent Economies(PO(2)) if $\forall E=<M; (u^i)_{i\in M}; \omega>\varepsilon\mathscr{E}_0$ with $|M|=2, x\varepsilon F(E)$ implies x is efficient in E.

We are now in a position to state and prove the main theorem of this paper.

Theorem 1:- The only solution on $\mathscr{E}_0$ to satisfy PO(2), w-IR, LI, COCONS and CONS is G^w.

As an easy consequence of this theorem we have the following:

Theorem 2:- The only solution on $\mathscr{E}_0$ to satisfy PO(2), IRED, LI, COCONS and CONS is G.

We will now prove Theorem 1.

Proof of Theorem 1:- That G^w satisfies the properties listed in Theorem 1 is clear. Hence let F be a solution on $\mathscr{E}_0$ satisfying the listed properties and let $E=<M; (u^i)_i\varepsilon M; \omega> \varepsilon \mathscr{E}_0$.

We will show that $F(E)=G^w(E)$.

If $|M| = 1, then\ G^w(E) = \{\omega\} = F(E)$.

If $|M| = 2, let\ M = \{i,k\} \subset N\ and\ x\ \varepsilon\ F(E)$. By PO(2), x is efficient in E. Hence there exists $p\ \varepsilon\ \mathbb{R}^l_+ \setminus \{0\}$, which supports $u^i\ at\ x^i\ and\ u^k\ at\ x^k$; this follows from the second fundamental theorem of welfare economics. By CONS, $x^i\ \varepsilon\ F\left(E^{\{i\},x}\right)\ and\ x^k\ \varepsilon\ F\left(E^{\{k\},x}\right)$ with both $E^{\{i\},x}\ and\ E^{\{k\},x}$

being well defined. Thus $x^i, x^k \in \mathbb{R}^l_+$ and $p \in \mathbb{R}^l_+$ with p being unique upto multiplication by a positive scalar. Choose p such that $p.\omega = W^i + W^k$. Let us show that

$p.x^i = W^i$ and $p.x^k = W^k$. By (W - IR) $u^i(x^i) \geq u^i\left(\frac{W^i}{W^i+W^k}\omega\right)$ and

$u^k(x^k) \geq u^k\left(\frac{W^k}{W^i+W^k}\omega\right)$. By choosing $\bar{E} = \langle\{i,k\}; (v^i, v^k); \omega\rangle \in \mathcal{E}_0$, with p supporting

v^i at x^i and v^k at x^k but with the new indifference curves of v^i and v^k through x^i and x^k being flatter than the indifference curves of u^i and u^k through x^i and x^k, we ensure that $\frac{p.W^i}{W^i+Wk}\omega \leq p.x^i$ and $\frac{p.W^k}{W^i+W^k}\omega \leq p.x^k$. For this we need to appeal to LI and W-IR of F. Since

$x^i + x^k = \omega = \frac{W^i\omega}{W^i+W^k} + \frac{W^k\omega}{W^i+W^k}$, we must have

$W^i = .p.x^i$ and $W^k = p.x^k$. Thus

$x \in G^W(E)$. So $F(E) \subset G^W(E)$ if $|M| = 2$.

Now, let $x \in G^W(E)$ with $E = \langle M; (u^j)_{j \in M}; \omega\rangle \in \mathcal{E}_0$, $|M| = 2$. Clearly $x^i, x^k \in \mathbb{R}^l_+$ where $M = \{i,k\}$. Now $x \in G^W(E)$ implies there exists $p \in \mathbb{R}^l_+$ with $p.W = W^i + W^k$, such that p supports u^i at x^i and u^k at x^k. As in proposition 3, let $\bar{E} = \langle M; (v^i, v^k); \omega\rangle \in \mathcal{E}_0$ with $G^W(\bar{E}) = \{x\}$, and p the associated price vector. Thus p supports v^i at x^i and v^k at x^k. Now

$F(\bar{E}) \subset G^w(\bar{E}), F(\bar{E}) \neq \phi$ implies $F(\bar{E}) = \{x\}$. By LI, $x \in F(E)$.

Thus $F(E) = G^w(E)$ if M the agent set for E has two elements.

The rest of the proof is by induction on the number of elements in M(E), the agent set for E.

Suppose $F(E) = G^w(E)$ $\forall E \in \mathcal{E}_0$ with $|M(E)| \leq n$, $n \geq 2$. Let

$E = \langle M; (u^i)_{i \in M}; \omega \rangle \in \mathcal{E}$ with $|M| = n+1$. Let $x \in G^w(E)$. Thus x is

efficient in E with $x^i \in \mathbb{R}^l_+$ $\forall i \in M$. By CONS of

G^w, $\phi \neq S \subset\subset M$ implies $x^S \in G^w(E^{S,x})$. By the induction

hypothesis, $G^w(E^{s,x}) = F(E^{s,x})$. By COCONS of F, $x \in F(E)$. Thus $G^w(E) \subset F(E)$.

Now, let $x \in F(E)$. By CONS of F, $\phi \neq S \subset\subset M$, implies

$\bar{X^s} \in F(E^{s,x})$. Taking S = {i}, we get $x^i \in \mathbb{R}^l_+$ $\forall i \in M$.

Let $p^i \in \mathbb{R}^l_+$ be the unique vector with $p^i . x^i = w^i$, which

supports u^i at x^i. Let $S = \{i, k\}$. $x^s \in F(E^{s,x}) = G^w(E^{s,x})$, the latter equality following by the induction hypothesis. Thus, there exists a unique $p \in \mathbb{R}^l_+$ with $p.x^i = W^i$, $p.x^k = W^k$, such that p supports u^i at x^i and u^k at x^k. Thus $p^i = p = p^k$. Thus $x \in G^w(E)$.

<u>Q.E.D.</u>

Conclusion:- In the above two theorems we axiomatically characterize the entire family of market equilibrium solutions.

It may be tempting to conjecture that the domain $\mathcal{E}_2$ is a subset of $\mathcal{E}_1$ *and so* $\mathcal{E}_0 = \mathcal{E}_2$. This is not in general true unless the utility functions are assumed to be strictly increasing. In general, $\mathcal{E}_0$ is a strict subset of both $\mathcal{E}_1$ *and* $\mathcal{E}_2$.

In a recent paper, Dagan [1996] characterizes the Walrasian allocation correspondence, in classes of exchange economics with smooth and convex preferences, by means of consistency, converse - consistency, Pareto optimality, envy freeness and neutrality. In yet another axiomatic characterization in the same paper, the fact that the solution is a core selection, atleast for two agent problems is used. However, as in Van den Nouweland, Peleg and Tijs [1996], there is the problem of indebtedness to the outside world, skilfully circumvented with the help of a "net trade vector".

There may be other axiomatic characterizations of the family of market equilibrium solutions. However, with our axiomatic characterization, a new begining has been made in the study of ordinally invariant solutions to games of fair division. The equal income market equilibrium solution is now viewed as a member of a larger family consisting of its non-symmetric analogues as well.

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Appendix

Here we assume that the agent set M is fixed and obtain axiomatic characterizations of the family of market equilibrium solutions.

Let $\phi \neq M \subset N$, with $|M| \geq 2$. Throughout the appendix M remains the fixed agent set under consideration.

Let $\mathcal{E}^M = \{E \in \mathcal{E} / E = <M;\ (u^i)_{i \in M};\ \omega>\}$.

Since M remains fixed here, whenever $E \in \mathcal{E}^M$ we suppress M and write $E = <(u^i)_{i \in M};\ \omega>$. Clearly $\mathcal{E}^M \subset \mathcal{E}$. Let

$\mathcal{E}_3^M = \{E = <(u^i)_{i \in M};\ \omega> / \exists\ p \in \mathbb{R}^l_{++}$ such that

$\forall\ i \in M,\ u^i(x^i) = H_i(p.x^i)\ \forall\ x^i \in \mathbb{R}^l_+$, where $H_i : \mathbb{R}_+ \to \mathbb{R}$ is a strictly increasing function$\}$.

Let $\mathcal{E}_1^M = \{E \in \mathcal{E}_1 /\ E \in \mathcal{E}^M\} = \mathcal{E}_1 \cap \mathcal{E}^M$

$\mathcal{E}_2^M = \{E \in \mathcal{E}_2 /\ E \in \mathcal{E}^M\} = \mathcal{E}_2 \cap \mathcal{E}^M$

We may call an economy in $\mathcal{E}_1 \cap \mathcal{E}_2$ extra smooth. Finally let $\mathcal{E}_4^M = \mathcal{E}_3^M \cup \{\mathcal{E}_1^M \cap E_2^M\}$.

We postulate now the following property for a solution F on $\mathcal{E}_4^M$ called non-discrimination of Pareto indifferent solutions:

<u>Non-Discrimination of Pareto Indifferent Solutions (ND)</u>:-

$\forall E \in \mathcal{E}_4^M,\ \forall\ x,\ y \in A(E),\ x \in F(E)$ and

$u^i(y^i) = u^i(x^i)\ \forall\ i \in M$, with $E = <(u^i)_{i \in M};\ \omega>$ implies $y \in F(E)$.

The above property is discussed in Nagahisa (1991). It is easy to see that G^w satisfies $ND \; \forall \; w \in W$.

Observe now, that since we are going to dispense with CONS and COCONS, the definition of w outside M does not matter.

The following property due to Gevers [1986] is called Monotonicity:

Monotonicity (MON):- F is said to satisfy Monotonicity on

$\mathcal{E}_4^M$ $if \; \forall E^1 = <(u^i)_{i \in M};\, \omega>$ and $E^2 = <(v^i)_{i \in M};\, \omega>$ both belonging to $\mathcal{E}_4^M$, $\forall \; x \in A(E^1)$ whenever $\forall \; i \in M$, and $y^i \in \mathbb{R}_+^l$, $u^i(x^i) \geq u^i(y^i)$ implies $v^i\,(x^i) \geq v^i\,(y^i)$, then $x \in F(E^1)$ implies $x \in F(E^2)$.

Lemma 1:- Let $E \in \mathcal{E}_4^M$, $x \in A(E)$, $p \in \mathbb{R}_{++}^l$. If $\forall \; i \in M$, $p.x^i \geq p\frac{w^i}{\sum_{j \in M} w^j}\omega$,

then $\forall i \in M$, $p.x^i = \frac{p.w^i}{\sum_{j \in M} w^j}\omega$, where $w \in W$.

Proof:- Obvious.

Lemma 2:- Suppose $E = <(u^i)_{i \in M};\, \omega> \in \mathcal{E}_3^M$ and F on $\mathcal{E}_4^M$ satisfies ND and w-IR. Then

$$F(E) = \left\{ x \in A(E) \,/ \forall \, i \in M,\, p.x^i \geq \frac{p.w^i}{\sum_{j \in M} w^j}\omega \right\} \quad \text{where} \quad u^i(x^i) = p.x^i \; \forall \; x^i \in \mathbb{R}_+^l$$

$i \in M$ and $p \in \mathbb{R}_{++}^l$.

Proof:- Suppose F is w-individually rational. Then

$$F(E) \subset \left\{ x \in A(E) \,/ \forall \, i \in M,\, p.x^i \geq \frac{p.w^i}{\sum_{j \in M} w^j}\omega \right\}.$$

Now, let $x \in A(E)$ with $p.x^i \geq \frac{p.w^i}{\sum_{j \in M} w^j} \omega \; \forall i \in M$.

By Lemma 1, $p.x^i = \frac{p.w^i}{\sum_{j \in M} w^j} \omega \; \forall i \in M$.

Thus all allocations $y \in A(E)$ with $p.y^i \geq \frac{p.w^i}{\sum_{j \in M}} w^j \omega \; \forall i \in M$, are Pareto indifferent.

Since $F(E) \neq \phi$, $x \in F(E)$.

Hence the lemma.

Q.E.D.

Proposition 1:- Let F be any solution on $\mathscr{E}_+^n$ which satisfies MON, W-IR and ND. Then

$G^w(E) \subset F(E) \; \forall \; E \in \mathscr{E}_+^n$.

Proof:- Consider any $E \in \mathscr{E}_+^n$ and any $x \in G^w(E)$.

Thus $p.x^i = \frac{p.w^i}{\sum_{j \in M} w^j} \omega$ where $p \in \mathbb{R}_+^l$ is the w-market equilibrium price vector associated with

x (so that in particular, $p.\omega = \sum_{j \in M} w^j$). .

Now, $p.x^i \geq p.y^i \Rightarrow u^i(x^i) \geq u^i(y^i) \; \forall \; i \in M$ since $x \in G^w(E)$.

Let $v^i(z^i) = p.z^i \; \forall \; z^i \in \mathbb{R}_+^l, \; i \in M$ and $E' = \langle (v^i)_{i \in M}; \omega \rangle$.

By lemma 2, $x \in F(E')$.

Hence, by monotonicity, $x \in F(E)$.

Q.E.D.

In order to obtain the converse inclusion, we need a weak version of Pareto Optimality:

<u>Binary Efficiency</u>:- A solution F on $\mathcal{E}_4^N$ is said to satisfy Binary Efficiency (BE) if

$\forall\, E = \langle (u^i)_{i\in M}, \omega\rangle \in \mathcal{E}_4^N,\ \forall\, x\in F(E),\ \forall i,j\in M,$ the following is true:

there does not exist

$y^i \in \mathbf{R}^l_+,\ y^j \in \mathbf{R}^l_+,\ y^i + y^j = x^i + x^j,\ u^i(y^i) \geq u^i(x^i), u^i(y^i) \geq u^j(x^j)$ with

atleast one strict inequality.

Once again Binary Efficiency being a weaker version of Pareto Optimality is seen to be easily satisfied by G^w, *whenever* $w \in W$.

<u>Proposition 2</u>:- If a solution F on $\mathcal{E}_4^N$ satisfies (BE), (MON), (N,D) (W-IR) and (LI) then

$F(E) = G^w(E)\ \forall\, E\in\mathcal{E}_4^N.$

Proof:- In view of proposition 1, we have to show that $F(E) \subset G^w(E)\ \forall\, E\in\mathcal{E}_4^N$ and in view of Lemma 2, that too for $E\in\mathcal{E}_1^N \cap \mathcal{E}_2^N$. Thus let $E = \langle (u^i)_i \in M, \omega\rangle$ and $x\in F(E)$. Since F satisfies (w-IR) and $E \in \mathcal{E}_1^N \cap \mathcal{E}_2^N, x^i \in \mathbf{R}^l_{++}\ \forall\, i\in M$. By (BE) and the second fundamental theorem of welfare economics, given $i,j\in M$, there exists $p\in\mathbf{R}^l_{++}$ such that p supports u^i *at* x^i and u^j *at* x^j. If $|M| = 2$ then this is enough; if $|M| > 2$, then by taking $i, k\in M,\ k \neq j$, we get $p' \in \mathbf{R}^l_{++}$ such that p' supports u^i, *at* x^i and u^j *at* x^j. Since all utility functions are smooth, p' must be a positive multiple of p. Thus we can choose $p \in \mathbf{R}^l_{++}$, with

$p.\omega = \sum_{i \in N} w^i$ such that p supports u^i at x^i, whatever $i \in M$. By (LI) and (w-IR) as applied

to F in Theorem 1 (in the main body of the paper) we get $p.x^i = p.w^i \; \forall \; i \in M$. Thus

$x \in G^w(E)$. This proves the proposition.

Q.E.D.

This appendix which has been added mainly for completeness, reproduces results reported in Gevers [1986] (i.e. Lemmas 1, 2 and Proposition1) and Nagahisa [1991] (i.e. Proposition 2) in our context, without significant variation in the style of presentation.