INDEPENDENCE OF IRRELEVANT TRANSFERS

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ABSTRACT

In this paper, we provide a partial geometric characterization of the Independence of Irrelevant Alternatives (IIA) Axiom, called Independence of Irrelevant Transfers (IIT) as also a characterization of the Nash Bargaining Solution without the IIA Axiom. The characterization has been motivated by the work of Shapley (1969) and Thomson (1981) to a very great extent and contributes to the growing literature on bargaining solutions without the IIA Axiom.

1. In this paper we shall follow Nash (1950) and define a *(two-person bargaining)* game as a pair (S,d) where

(i) S is a compact, convex subset of R^2

(ii) $d \in S$ and there exists $x \in S$ such that $x_i > d_i$ for $i = 1,2$

In interpreting a game (S,d) we have in mind two players who either agree on an $x \in S$ giving utility x_i to player i, or fail to agree in which case they end up with utilities d_i. The point d is called *status quo point* or *disagrement point* or *threat point.* Compactness of S, which is often satisfied if an underlying set of "physical" alternatives is finite, is required for mathematical convenience. Convexity of S may come from the use of Von-Neumann-Morgenstern utility functions defined on lotteries between underlying alternatives, or e.g. from the use of concave utility functions in division problems. Sometimes in addition we impose the following additional condition on a game :

(iii) $y \in S$ whenever $d \leqq y \leqq x$ and $x \in S$.

This last requirement often referred to as *comprehensiveness,* can be interpreted as *free disposibility of utility* for both players.

The definition of an *n-person bargaining game* is obtained by replacing 2 by n everywhere. We will be basically interested in two-person bargaining games although the result we establish in this paper as well as its significance extends effortlessly to n-person bargaining games.

Let B denote the set of all games satisfying (i) and (ii); let $\tilde{B}$ denote the set of all games satisfying (i), (ii) and (iii). A bargaining solution is a map $f: B \to R^2$ with $f(S,d) \in S$ for every $(S,d) \in B$ (feasibility).

For $(S,d) \in B$, we denote by $P(S)$ the Pareto optimal Subset of S:

$P(S) \equiv \{x \in S / \text{for all } y \in S, \text{ if } y \geq x \text{ then } y = x\}$,

and by $P(S,d) \equiv \{x \in P(S) : x \geq d\}$ the individually rational Pareto optimal set of (S,d). Further $S_d \equiv \{x \in S: x \geq d\}$, and $W(S) \equiv \{x \in S / y = (y_1, y_2),$ $y_i > x_i$, $i = 1,2$ implies $y \notin S\}$.

For reasons which are primarily technical we make the following blanket assumption:

Assumption :- $\forall (S,d) \in B$, and $x = (x_1, x_2) \in P(S,d)$, if there exists $(p_1, p_2) = p \in R^2_+$, $p_1 + p_1 = 1$ such that $p_1x_1 + p_2x_2 \geq p_1y_1 + p_2y_2$, $y = (y_1, y_2) \in S$, then $p_1 > 0$ and $p_2 > 0$.

Since $\tilde{B}$ is a subset of B, the above assumption applies for $\tilde{B}$ as well. It is like a regularity assumption.

In the above framework we have the following :

Definition 1 :- For every $(S,d) \in B$ and $0 < t < 1$, let $N^t(S,d)$ maximize the product $(x_1 - d_1)^t (x_2 - d_2)^{1-t}$ over S_d. We call $N^t : B \to R^2$ a (nonsymmetric) Nash Solution. We call $N \equiv N^{\frac{1}{2}}$ the Nash Solution.

The Nash solution N was first proposed by Nash (1950), and the non-symmetric Nash solutions N^t were proposed by Harsanyi and Selten (1972). Nash (1950) proposed the following properties for a solution f on B.

WPO (Weak Pareto Optimality) : $f(S,d) \in W(S)$ for every $(S,d) \in B$

IR (Individual Rationality) : $f_i(S,d) \geq d_i$, $i = 1,2$ for every $(S,d) \in B$

IAUT (Independence of positive affine utility transformations):

For all $a, b \in R^2$ with $a > 0$ and every $(S,d) \in B$, we have

$f(aS + b, ad + b) = a f(S,d) + b$. Here $ax \equiv (a_1x_1, a_2x_2)$ for $x \in R^2$, and $aT \equiv \{ax: x \in T\}$ for $T \subset R^2$.

SYM (Symmetry) : If $(S,d) \in B$ is symmetric, i.e. $d_1 = d_2$ and $S = \{(x_2, x_1) : x \in S\}$, then $f_1(S,d) = f_2(S,d)$.

IIA (Independence of Irrelevant Alternatives) : For all (S,d), $(T,d) \in B$ with $S \subset T$ and $f(T,d) \in S$, we have $f(T,d) = f(S,d)$.

Nash (1950) proved the following theorem.

Theorem 1 :- The Nash solution $N : B \to R^2$ is the unique solution with the properties WPO, IAUT, SYM and IIA.

The IIA-property is the most debated property in the literature on the Nash bargaining solution (see e.g. Kalai and Smorodinsky (1975)). What the IIA property says is that if the set of underlying alternatives shrinks while the original solution alternative is still available, then the new solution alternative should be the originally available solution alternative. Two other properties for a solution f on B are defined as follows :

SIR (Strong Individual Rationality)) : $f(S,d) > d$ for every $(S,d) \in B$

PO (Pareto Optimality) : $f(S,d) \in P(S)$ for every $(S,d) \in B$

The following two theorems are proved in Roth (1979), deKoster et al (1983), Binmore (1987).

Theorem 2 :- $\mathcal{N}^{p} = \{N^t : 0 < t < 1\}$ is the family of all solutions with the properties SIR, IAUT, and IIA.

There exist many variations of the above bargaining theorems that involve IIA or properties in the same spirit. Kalai (1977(a)) defines a solution called the proportional solution which satisfies IIA. As discussed in Lahiri (1989), inspite of serious criticisms of the IIA axiom, it remains a criterion that most solutions in practise are required to obey. The rationale for this is probably the following :

A bargaining game can be viewed as a decision problem in which the decision maker consists of the two bargainers as a group, and in which the decision or compromise is the point assigned by some solution f. In this context one might expect that the "decision maker" would maximize certain "preferences;" formally, we say that the binary relation $\succsim$ on R^2 represents f if for every game (S,d) there is a unique point z with $z \succsim x$ for all x in S, and z = f (S,d). In light of this we may state as in Peters and Wakker (1987).

Theorem 3 :- There exists a binary relation $\succsim$ on R^2 representing f if and only if f satisfies IIA.

This brings out the significance of the IIA axiom. However, the unreasonableness of the IIA axiom has given rise to a spate of alternative axioms and the formulation we shall propose in this paper is one such, which at the same time highlights the simple geometry of the IIA axiom. Our formulation is motivated by the work of Shapley (1969), where he suggests a method of selecting a set of self-justifying weights for a generalized utilitarian social welfare function. This method also underlies the definition of the "modified Shapley value."

2. In Thomson (1981) we find an alternative to the IIA axiom, called the independence of irrelevant expansions (IIE) property.

IIE (Independence of irrelevant expansions) : For every $(S,d) \in B$ there exists a vector $p \in R^2_+$ with $p_1 + p_2 = 1$ such that :

(i) $p.(x-d) = p.(f(S,d)-d)$ is the equation of a supporting line of S at $f(S,d)$,

(ii) for all $(T,d) \in B$ with $S \subset T$ and $p.(x-d) \leq p.(f(S,d)-d)$ for all $x \in T$, we have $f(T,d) \geq f(S,d)$.

($x.y$ denotes the inner product $x_1 y_1 + x_2 y_2$, for $x, y \in R^2$)

The property we shall be suggesting is similar to IIE and hence it is desirable that what IIE suggests be understood. Contrary to IIA, IIE says something about the way in which a game may be expanded without essentially changing the solution. A non-symmetric Nash solution N^t ($0 < t < 1$) satisfies IIE with equality in (ii), i.e. $f(S,d) = f(T,d)$; p is the normal vector of the supporting line separating a game from the hyperbola which is the level set of the maximal non symmetric Nash product. We now have the following theorem due to Thomson (1981).

Theorem 4 :- $\mathcal{N}^c$ is the family of all solutions with the properties IR, PO, IAUT, and IEE.

The property we suggest both as a geometric characterization of IIA as well as an alternative to it rests on the supporting hyperplane theorem (see Rockafellar (1970), Section 11), by which if $f(S,d) \in B$ and $x \in P(S)$ then there exists $p \in R^2_+$, $p_1 + p_2 = 1$, such that $p \cdot x \geq p \cdot y$ for all $y \in S$. This holds since S is assumed to be compact and convex for all $(S,d) \in B$.

Given $p \in R^2_{++}$, $p_1 + p_2 = 1$, $x \in R^2$ and $d \in R^2$ with $p \cdot d < p \cdot x$, we denote $S(p, x, d) \equiv \{y \in R^2 / p \cdot y \leq p \cdot x \text{ and } y \geq d\}$

Hence $(S(p,x,d),d)$ is a game in B, infact in $\tilde{B}$.

We shall now mention the property which we propose as an alternative to as well as a partial characterization of IIA.

IIT (Independence of Irrelevant Transfers) :- Given $(S,d) \in B$, and $x \in P(S)$, if $f(S(p,x,d),d) \in S$ for some $p \in R^2_{++}$, $p_1 + p_2 = 1$, then $f(S,d) = f(S(p,x,d),d)$.

The intuition behind IIT is clear. Consider the weights $p = (p_1, p_2)$ which play the role of conversion rates that transform individual utilities into some universal unit. In particular, these weights act also as rates of transformation or rates of exchange between individual utilities. Every choice of weights $p = (p_1, p_2)$ defines a system of transferable utility between the individuals where the ratio of the weights determine the rate at which utility side payments between the individuals are to be made. Once utility is transferable we can construct from the given game (S,d), a simpler game $(S(p,x,d),d)$ and apply our solution concept to this game. In general the solution will not be feasible for the underlying game. In such a case

$p = (p_1, p_2)$ fails to justify itself , in the sense that these rates of utility transfers, do not correctly reflect the realities of the underlying situation at $x \in P(S)$. Therefore, a new set of rates for utility transfer must be examined and the process must be repeated until a set of weights, say $p^* = (p_1^*, p_2^*)$, is found having the property that the solution to the associated simplified game is feasible for the game (S,d). Such weights are self justifying, and the alternatives in the simplified game which do not coincide with the solution are irrelevant from our stand point. Let us make the following assumption about our bargaining solution.

(CONT.) (Continuity) : The solution $f : B \to R^2$ is continuous i.e. if there exists a sequence $\{(S^k, d^k)\}_{k=1}^{\infty}$ of games belonging to B such that $\lim_{k\to\infty} S^k = S \subset R^2$ in the Hausdorf topology and $\lim_{k\to\infty} d^k = d \in R^2$ and $(S,d) \in B$, then $\lim_{k\to\infty} f(S^k, d^k) = f(S,d)$

It is easily established by appealing to Brouwer's fixed point theorem, that if $f: B \to R^2$ satisfies (P O), (I R) and (CONT), then there exists $x^* \in P(S)$ and $p^* \in R^2_{++}$, $p_1^* + p_2^* = 1$ such that $f(S(p,x,d),d) = x^* \in S$. In view of this we can state and prove the following main theorem of our analysis:

Theorem 5 :- Let $f: B \to R^2$ be a solution satisfying PO, IR and CONT. Then IIA implies and is implied by IIT.

Proof :- Suppose f satisfies IIA. Then IIT is immediate.

Conversely, suppose f satisfies PO, IR, CONT and IIT.

Let (S,d) and (T,d) be two games belonging to B and $f(T,d) \in S$.

We have to show that $f(S,d) = f(T,d)$.

By PO, IR and CONT., there exists $x^* \in P(T)$ and $p^* \in R^2_{++}$, $p^*_1 + p^*_2 = 1$ such that $x^* = f(S(p^*, x^*, d),d) \in T$

By IIT, $f(T,d) = f(S(p^*, x^*, d),d) \in T$

But $f(T,d) \in S$ implies $x^* = f(S(p^*, x^*, d),d) \in S$.

$x^* \in P(T)$ and $S \subset T$ implies $x^* \in P(S)$

Hence by IIT, $x^* = f(T,d) = f(S(p^*, x^*, d),d) = f(S,d)$

The above theorem provides a simple geometric explanation of the IIA axiom whenever a solution satisfies PO,IR and CONT. It says that consider the supporting hyperplanes to a game at each Pareto optimal point and apply the given solution to the simplified game described earlier. If for some such simplified game the Pareto optimal point generating it coincides with the solution to the simplified game then it is the solution to the original game. It turns out that this is what IIA is all about.

Using Theorem 5 we can state and prove a corollary which provides a characterization of Nash's (1950) bargaining solution, and its asymmetric extensions without the IIA axiom.

Corollary :- $\mathcal{N}^0$ is the family of all solutions with the properties SIR, PO, IAUT, CONT and IIT.

Proof :- That N^t, $0 < t < 1$ satisfies the condition is fairly straight forward. The converse follows by appealing to Theorem 5 and Theorem 2 above.

3. There are many other characterizations of the Nash solution without the IIA axiom, notably that of Kalai (1977 b), Van Damme (1986), Peters and Van Damme (1987), Peters (1986), Binmore (1984). Our characterization is yet another with the additional desirable property that it provides a geometric explanation of IIA as well.

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