THREAT BARGAINING PROBLEMS WITH INCOMPLETE INFORMATION AND NASH'S SOLUTION

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ABSTRACT

In this paper we extend the framework of threat bargaining games to include those with incomplete information. In this set up we address ourselves to two significant problems: (1) Under what conditions would 'truthful' revelation of the disagreement payoffs be a Nash equilibrium of the threat bargaining game ? (2) Obtaining a characterization of the Nash bargaining solution without the Independence of Irrelevant Alternatives Assumption. Our framework of analysis is general enough to include within its purview the study of non-cooperative bargaining problems with incomplete information, played by Bayesian players, although the specific problem addressed to in this paper does not fall in that category.

1. Introduction :

In many bargaining situations parties cannot arrive at an agreement through negotiations but rather must make one irreversible claim , a decision on the basis of which may or may not be fulfilled. For example, imagine an arbitrator (or a mechanism) through whom a potential seller wishes to sell an item to a potential buyer. The seller states a price for the item which is supposed to convey the worth or value of the item to him. The buyer states a price for the item which is meant to convey the worth of the item to him. These stated values are the revealed worth of the object to the buyer and the seller respectively which may or may not be true. On the basis of these statements, the arbitrator decides on an outcome, which if acceptable to both parties, is implemented. Otherwise the object does not change hands.

In such problems, there is a definite incentive for the seller to overstate the true worth of the object to him; on the part of the buyer there is an incentive to understate the worth of the object. This is done in order to influence the arbitrated price of the object. Hence there is a definite incentive on the part of each player to misrepresent (in fact overstate) the current worth of his status quo point before trade takes place. Such situations are dealt with in the theory of threat bargaining problems, discussed in Lahiri [1988, 1989 a,b] ; Owen [1982].

In this paper we consider an additional complication in the context of threat bargaining problems. Following Anbar and Kalai [1978], we assume that each player has a belief regarding the acceptability to his opponent of an arbitrated outcome, which can be summarized by a probability distribution. Hence coupled with the strategic behaviour of the players in determining the final outcome, there is an uncertainty about the solution being acceptable to the opponent. Throughout we assume that the 'true' worths of the status quo point is known to either player. This is necessary is order to obtain the result we are after. However, bargaining proceeds on the basis of the stated worth of the status-quo point. It is like, allowing for a possibility of 'obvious misrepresentation' and getting by with it.

In this paper we show that if the beliefs of the players are uniformly distributed, then the only bargaining solution compatible with truthful revelation of the status-quo point is the Nash [1950] bargaining solution.

2. Definitions :

In a pure bargaining problem between a group of two participants there is a set of feasible outcomes, any one of which will result if it is specified by the unanimous agreement of all participants. In the event that no unanimous agreement is reached, a given disagreement outcome obtains. We shall assume that the utility space or the set of possible payoffs is R^2 i.e. a two person bargaining problem is a pair (H,d) of a subset H of R^2 and of a point d H. H is the feasible set, and d is the disagreement (or threat) point.

The class of bargaining problems we consider is given by the following definition:

Definition 1 :- The pair $\Gamma = (H,d)$ is a two-person fixed threat bargaining game if $H \subseteq R^2$ is compact, convex with non-empty interior, $d \in H$, and H contains atleast one element u such that $u \gg d$.

Definition 2 :- The set of two-person fixed threat bargaining games is denoted W.

For the purpose of this paper we define a solution to bargaining problems in W as follows:

Definition 3 :- A solution is a function $F : W \to R^2_+$ satisfying

(i) $F(H,d) \in H \;\; \forall \; (H,d) \in W$ (feasibility)

(ii) $y \in H, \;\; y \geq F(H,d)$ implies $y = F(H,d)$ (Pareto optimality)

(iii) $F(H,d) \geqslant d \ \forall (H,d) \in W$ (individual rationality)

(iv) If $(a_1, a_2) \gg 0$, $(b_1, b_2) \in R^2$, $H' = \{y \in R^2 / y_i = a_i x_i + b_i$, $i = 1,2$, $y = (y_1, y_2)$, $x = (x_1, x_2) \in H\}$ and $d_i' = a_i d_i + b_i$, $i = 1,2$, $d' = (d_1', d_2')$, then $F_i(H', d') = a_i F_i(H,d) + b_i$, $i = 1,2$.

(Independence with respect to affine utility transformations)

The conditions we impose on a solution to bargaining problems are standard and are satisfied by the more well known solutions to bargaining problems (e.g. Nash (1950), Kalai-Smorodinsky (1975)).

We now make an assumption which is satisfied by most familiar solutions to bargaining problems and which will be required significantly by us.

<u>Assumption</u> (FUD) :- Let $(H,d) \in W$ and $P(H_d) = \{(x_1, x_2) \in H / x = (x_1, x_2)$, $x_i \geqslant d_i$, $i = 1,2$ and $y \geqslant x_i$ $y \in H$ implies $y = x\}$

Then $\forall (x_1, x_2) \in P(H_d)$, $\exists$ $d_1' \geqslant d_1$, or $d_2' \geqslant d_2$ such that

(i) $F(H; d_1', d_2) = (x_1, x_2)$

or (ii) $F(H; d_1, d_2') = (x_1, x_2)$

(fullness through unilateral deviations).

This assumption requires that unilateral deviation from the given disagreement payoffs yield any Pareto Optimal and individually rational outcome. As mentioned earlier this property is satisfied by all the more well known solutions to bargaining problems, including some of those which may not satisfy some of the conditions of Definition 3 (e.g. the Proportional Solution of Kalai [1977 a]).

Our analysis requires the notion of a true bargaining problem, which in view of the above and following Anbar and Kalai (1978) may be defined as follows:

Definition 4 :- A true bargaining problem H is a compact, convex subset of the unit square containing (0,0), (1,0) and (0,1).

The interpretation of such a bargaining game is that the true disagreement point of the players have been set equal to (0,0) and the game has been normalized in such a way that the utility demands of the players belong to the closed interval $[0,1]$. Let us call the set of all true bargaining problems $\overline{W}$.

Every member $H \in \overline{W}$ defines uniquely a monotone non-increasing concave function $\phi_H : [0,1] \to [0,1]$ by $\phi_H(x_1) = \max\{x_2 / (x_1,x_2) \in H\}$. Conversely every monotone non-increasing concave function $\phi : [0,1] \to [0,1]$ such that $\phi(0) = 1$ determines uniquely a set $H_\phi \in \overline{W}$ by $H_\phi = \{(x_1,x_2) : 0 \leq x_1 \leq 1,\ 0 \leq x_2 \leq \phi(x_1)\}$. For every such function ϕ we define the (generalized) inverse $\phi^{-1} : [0,1] \to [0,1]$ by $\phi^{-1}(x_2) = \max\{x_1/(x_1,x_2) \in H_\phi\}$.

Let G_i be the distribution function on $[0,1]$ which summarizes the belief of player i about player $j \neq i$ (i's opponent) accepting a utility outcome, $i = 1,2$.

The non-cooperative game we have in mind is the following. The underlying <u>true bargaining problem</u> $H \in \overline{W}$ being given, each player i announces a disagreement utility d_i. The pair (H,d), $d = (d_1,d_2)$ is a fixed threat bargaining problem in W. Based on the information announced by the players the arbitrator using a solution F selects an outcome $F(H,d)$ which each player accepts with a definite probability given by G_1 and G_2 respectively. In the event that the outcome is rejected, by any one or both the players, the participants settle down for their true disagreement payoffs $0 = (0,0)$.

Let $(d_1,d_2) \in H$ be the announced disagreement pay offs of the respective players. If F is the solution being used by the arbitrator, the expected payoff of player 1 is

$$P_1(d_1,d_2) = F_1(H;\, d_1,d_2).\; G_1\,(F_2(H;\, d_1,d_2)).$$

The expected payoff of player 2 is

$$P_2\,(d_1,d_2) = F_2(H;\, d_1,d_2).\; G_2\,(F_1(H;\, d_1,d_2)).$$

<u>Definition 5</u> :- A threat bargaining problem with incomplete information equipped with a solution F is an ordered triplet (H, F, G) where

(i) $H \in \overline{W}$, is a true bargaining problem

(ii) $F : W \rightarrow R^2$ is a bargaining solution

(iii) $G = (G_1, G_2)$ is a pair of probability distribution function on $[0,1]$.

The notion of an equilibrium that we adopt in this paper is given by the following definition.

Definition 6 :- An equilibrium for a threat bargaining game with incomplete information equipped with a solution F, i.e. (H,F,G) is an ordered pair $(d_1^*, d_2^*) \in H$ such that

(i) $P_1(d_1^*, d_2^*) \geqslant P_1(d_1, d_2^*) \;\forall\; d_1 \in (0,1)$

(ii) $P_2(d_1^*, d_2^*) \geqslant P_1(d_1^*, d_2) \;\forall\; d_2 \in (0,1)$.

This is the familiar Nash equilibrium which by dint of its self enforceability finds a distinguished placed as a solution concept. In the case of threat bargaining problems, the relationship between a Nash equilibrium and well known solutions to bargaining problems have been studied in Lahiri [1988, 1989 b].

3. Main Theorems :- Our quest in this section is to obtain conditions under which truthful revelation of disagreement utility will be guaranteed by a bargaining solution. In the economics literature, problems in a related spirit are known as problems of incentive compatibility. There it is usually assumed that true worth is not known at all. Such problems can also be studied in our framework, although that is not the purpose of our immediate analysis.

Theorem 1 :- Let G_1, G_2 be distributed uniformly, i.e.

$$G_i(x) = x \;\forall\; 0 \leq x \leq 1, \; i = 1,2$$

Then (0,0) is an equilibrium of the threat bargaining game with incomplete information (H, F, G) equipped with a solution F if and only if F is the Nash bargaining solution i.e.

$$F(S,d) = \underset{\substack{(x_1,x_2) \in S \\ x_1 \geq d_1,\; x_2 \geq d_2}}{\arg\max} (x_1-d_1)(x_2-d_2) \;\forall\; (S,d) \in W$$

Proof :- Since G_1 and G_2 are uniformly distributed,

$$P_1(d_1,d_2) = F_1(H;d_1,d_2) \cdot \varphi_H(F_1(H;d_1,d_2))$$

$$P_2(d_1,d_2) = F_2(H;d_1,d_2) \cdot \varphi_H^{-1}(F_2(H;d_1,d_2)).$$

Observe that by property (i) of a solution $F_2(H;d_1,d_2) = \varphi_H(F_1(H;d_1,d_2))$ and $F_1(H;d_1,d_2) = \varphi_H^{-1}(F_2(H;d_1,d_2))$.

Suppose $F = (F_1, F_2)$ is the Nash bargaining solution.

$P_1(H;\, 0,0) = F_1(H;\, 0,0) \cdot \varphi_H(F_1(H;0,0)) \geq x_1 \varphi_H(x_1) \,\forall\, 0 \leq x_1 \leq 1$

by definition of the Nash bargaining solution.

Since $P_1(H;\, d_1,\, 0) = x_1 \varphi_H(x_1)$ for some $x_1 \in [0,1]$,

we get

$P_1(H;0,0) \geq P_1(H;\, d_1,0) \,\forall\, d_1 \in [0,1]$

By a similar argument it follows that

$P_2(H;0,0) \geq P_2(H;0,d_2) \,\forall\, d_2 \in [0,1]$

Hence $(0,0)$ is an equilibrium for (H, F, G).

Conversely suppose that $(0,0)$ is an equilibrium for (H,F,G), but F is not the Nash bargaining solution. Let $(x_1^*, \varphi_H(x_1^*))$ be the Nash bargaining solution outcome for $H \in \overline{W}$. By assumption (F U D) and without loss of generality $\exists\; d_1' \geq 0$, such that

$F(H;\, d_1',\, 0) = (x_1^*, \varphi_H(x_1^*))$

Hence

$P_1(H;\, d_1',\, 0) = x_1^* \varphi_H(x_1^*) > F_1(H;0,0)\, \varphi_H(F_1(H;0,0)) =$

$P_1(H;0,0)$

contradicting that $(0,0)$ is an equilibrium. Hence the theorem.

<u>Q.E.D.</u>

A characterization of the family of nonsymmetric Nash bargaining solution (see, Harsanyi and Selten [1972], Kalai [1977 b]) is embodied in the following theorem.

<u>Theorem 2</u> :- Let, $G_1(x) = x^k$, $0 \leq x \leq 1$ and $G_2(x) = x^{1/k}$, $0 \leq x \leq 1$, $k > 0$, be the distribution functions embodying the beliefs of the two players. Then $(0,0)$ is an equilibrium for the threat bargaining game with incomplete information (H,F,G) equipped with a solution F if and only if, F is a non-symmetric Nash bargaining solution i.e. $F(S; d_1,d_2) = \arg\max\limits_{\substack{(x_1,x_2)\in S \\ x_1 \geq d_1,\ x_2 \geq d_2}} (x_1-d_1)^k (x_2-d_2)\ \forall\ (S,d) \in W.$

<u>Proof</u> :- Analogous to the proof of Theorem 1.

4. <u>Conclusion</u> :- Now we shall briefly summarize all that we have achieved in this paper.

To begin with we have extended the framework of threat bargaining games to include within it the study of threat bargaining games with incomplete information. This extension adds a dose of realism to our analysis, in that we immediately obtain the seeds of a general model in which to study the theoretical properties of what in the literature of economics and operations research has come to be known as bargaining under incomplete information (played by Bayesian players). The problem of truthful revelation of the net value to the players of an item to be traded can be studied within this framework.

Second, we achieve a characterization of the Nash bargaining solution (or more generally, the family of non-symmetric Nash bargaining solutions) without the debatable Independence of Irrelevant Alternatives assumption, a problem which has come to occupy a central place in bargaining game theory (see Lahiri [1989 c]). We show that provided the beliefs of the players in a threat bargaining game with incomplete information are distributed uniformly, the only bargaining solution compatible with truthful revelation of disagreement payoffs is the Nash bargaining solution.

Third, our framework is perfectly general in that given definition - 3 (of a solution to bargaining problems) our result continues to hold in the class W of bargaining problems. In this case a true bargaining problems would be $(H,d) \in W$ and truthful revelation of disagreement payoffs would imply agent i announcing d_i as his disagreement pay off. The support of the distribution function of the beliefs of player i would naturally be $\left[d_j, \max\left\{x_j/(x_i,x_j) \in H\right\}\right]$ where $j \neq i$. $\bar{W}$ was invoked merely for notational convenience and ease of analysis.

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