ARBITRATION BY A BAYESIAN STATISTICIAN AND BOUNDED RATIONALITY

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W P No. 849
January, 1990

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

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ACKNOWLEDGEMENT

This paper owes its genesis to useful conversations and discussions that I had with Clifford Hildreth and Kevin Cotter over a period of time. I would like to express my gratitude to both of them.

ABSTRACT

In this paper we formalize the framework of an arbitration game, to accommodate a large class of situations where public decisions are implemented in a noncooperative setting. We then present a method of computing the equilibrium strategies of the players under assumptions of bounded rationality, so that the solutions correspond to what is observed in any realization of an arbitration game.

1. <u>Introduction</u> : - In many decision theoretic problems an arbitrator or a third party is responsible in settling disputes between two parties. Such for instance is the case when a judge is called upon for deciding which of two bargainers should win a dispute. There is a social value associated with resolving the dispute in favour of a bargainer. Variants of this problem have been studied by Ordover and Rubinstein (1983), P'ng (1983) and Samuelson (1983) where game-theoretic models of the legal process have been discussed. They study how incomplete information affects the decision to settle a dispute. The judge is not an active player in these models. In market settings, Grossman (1981), Milgrom (1981), and Farrell and Sobel (1983) study situations in which agents may disclose information, but cannot distort it. More recently Sobel (1985) presents an analysis where bargainers are not allowed to misrepresent their private information. This assumption is justified if the information revealed is verifiable, so that misrepresentation can be identified and punished.

In this paper we present a model where misrepresentation of the true characteristics of the bargainers is possible. The arbitration procedure is as follows: Each player (or bargainer) sends a signal to the arbitrator which encodes his true characteristics. Based on this information, the arbitrator attempts to infer the true characteristics of the players and announces a public decision which maximizes the expected social value. In other words, the arbitrator forms posterior beliefs about the true characteristics of the players and chooses an action which maximizes expected social value. This action in turn affects the individual expected welfare of the bargainers and since this decision is affected by the signal they send to the arbitrator, it is to their advantage to behave strategically in the choice of a signal. In choosing an appropriate signal, the players try to maximize their expected welfare given their own prior beliefs about the opponents' true characteristics. In order that the signalling rules are self enforceable, it is desirable to look for a Nash equilibrium of the resulting non-cooperative game.

In this paper, we follow Hildreth (1963), in modelling the behaviour of individuals, whether an arbitrator or a bargainer, as a Bayesian statistician, Compelling arguments for doing so, can be found in the same paper. Our solution technique relies heavily on the principle of first-order certainty equivalence as developed by Theil (1954) and Malinvaud (1969). This agrees with the assumption of bounded rationality of the players involved in the game.

2. **The Model** :- Consider a bargaining problem with two players and an arbitrator. Let $\Omega_i \subseteq R^{m_i}$ be a Borel subset of the m_i - dimensional Euclidean space, in which the true characteristics of player i belong. Thus the true characteristics of player 'i' is encoded in an m_i - dimensional vector, which is the private information of player i, for i = 1,2. Let $A \subseteq R^p$, be an open connected subset of the p-dimensional Euclidean space, consisting of all public decisions available to an impartial arbitrator.

A social value function is a function

$$W : A \times \Omega_1 \times \Omega_2 \to R,$$

which for each realization $(\theta_1, \theta_2) \in \Omega_1 \times \Omega_2$ of the true characteristics of the players and for each public decision $a \in A$, gives the value accruing to society.

Let $x_i \in R^{m_i}$, be the signal which player i communicates to the arbitrator. Based on the signals $(x_1, x_2) \in R^{m_1} \times R^{m_2}$, the arbitrator forms posterior beliefs about $(\theta_1, \theta_2) \in \Omega_1 \times \Omega_2$, which for the sake of simplicity, is assumed to be summarized by a conditional probability density function $\xi(\theta_1, \theta_2 \mid x_1, x_2)$ available to the arbitrator. If there is a tractable (for both the bargainers as well as the arbitrator) sufficient statistic, we may assume that x_1 and x_2 are precisely those. Their transmission to the arbitrator saves one step in his eventual decision process.

In the above framework the arbitrator solves the following problem :

$$(1) \quad \max_{a \in A} \int_{\Omega_1 \times \Omega_2} W(a, \theta_1, \theta_2)\, \xi(\theta_1, \theta_2 \mid x_1, x_2)\, d\theta_1\, d\theta_2$$

and in response to the x_1 and x_2 communicated by players 1 and 2 respectively, announces a (x_1, x_2) which solves (1) as the public decision.

Each player i on the other hand has a utility function $u_i : A \times \Omega_i \to R$, and like a Bayesian has prior beliefs about the true characteristics of his opponent. Let $\xi_1(\cdot) : \Omega_2 \to R_+$ be the probability density function which summarizes player 1's beliefs about player 2's private characteristics and $\xi_2(\cdot) : \Omega_1 \to R_+$ be the probability density function which summarizes player 2's beliefs about player 1's private characteristics. As in common in noncooperative theory we assume that $u_1, u_2, W, \xi_1, \xi_2, \xi$ are common knowledge. Hence the ordered six-tuple $\Gamma = \langle u_1, u_2, W, \xi_1, \xi_2, \xi \rangle$ defines a non-cooperative game which for the purpose of this paper can be called an <u>arbitration game</u>.

Player i being aware of his true characteristics would need to choose a signalling rule $x_i : \Omega_i \to R^{m_i}$. We now need to define a non-cooperative equilibrium for this arbitration game and by dint of its self-enforceability we choose the natural analogue of the Nash equilibrium solution concept.

An equilibrium for $\Gamma = \langle u_1, u_2, W, \xi_1, \xi_2, \xi \rangle$ is an ordered pair $\langle x_1^*(\cdot), x_2^*(\cdot) \rangle$ such that

$$(2) \quad \int_{\Omega_2} u_1(a(x_1^*(\theta_1), x_2^*(\theta_2)), \theta_1) \xi_1(\theta_2)\, d\theta_2 \geq \int_{\Omega_2} u_1(a(x_1(\theta_1), x_2^*(\theta_2)), \theta_1) \xi_1(\theta_2)\, d\theta_2$$

$$\forall \theta_1 \in \Omega_1 \text{ and } \forall x_1 : \Omega_1 \to R^{m_1}$$

$$(3) \quad \int_{\Omega_1} u_2(a(x_1^*(\theta_1), x_2^*(\theta_2)), \theta_2) \xi_2(\theta_1)\, d\theta_1 \geq \int_{\Omega_1} u_2(a(x_1^*(\theta_1), x_2(\theta_2)), \theta_2) \xi_2(\theta_1)\, d\theta_1$$

$$\forall \theta_2 \in \Omega_2 \text{ and } \forall x_2 : \Omega_2 \to R^{m_2}$$

$$(4) \quad a : R^{m_1} \times R^{m_2} \to R^{p} \text{ solves (1) } \forall (x_1, x_2) \in R^{m_1} \times R^{m_2}$$

With this the formal specification of the problem and solution concept for this arbitration game (with signalling) is complete, we shall now turn to obtaining a solution to this problem.

3. <u>Solution</u> :- The solution to this arbitration game that we propose satisfies two countervailing conditions: it is reasonably precise, and it is easy to compute. The rationale for imposing these two conditions is that it lends credibility to a theory of rational behaviour described by complex maximization problems. Such is the merit of the certainty equivalence method. To implement this technique we make the following assumption.

<u>Assumption 1</u> :- $W : A \times \Omega_1 \times \Omega_2 \to R$ is thrice continuously differentiable in all arguments and strictly concave in 'a'.

<u>Assumption 2</u> :- $u_i : A \times \Omega_i \to R$, $i = 1,2$, are twice continuously differentiable.

<u>Assumption 3</u> :- Let $(\bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2))$ be the mean of the distribution specified by $\xi(\theta_1,\theta_2 \mid x_1,x_2)$; $\bar{\theta}_1$ be the mean of the distribution specified by $\xi_2(\theta_1)$ and $\bar{\theta}_2$ be the mean of the distribution specified by $\xi_1(\theta_2)$. Then:

$\bar{\theta}_i(\cdot) : R^{m_1} \times R^{m_2} \to R^{m_i}$ is a twice continuously differentiable function.

Under the above conditions we have the following theorem:

<u>Theorem 1</u>:- Under assumption 1, given any $\delta > 0$, there exist functions $a: R^{m_1} \times R^{m_2} \to R^p$, $\epsilon : R^{m_1} \times R^{m_2} \to R_{++}$ and $\gamma : R^{m_1} \times R^{m_2} \to R_{++}$ such that if the Hessian matrix of the function $W(a, \bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2))$

with respect to 'a' is non-singular for all $(x_1,x_2) \in R^{m_1} \times R^{m_2}$

and if

$$\int \xi(\theta_1,\theta_2 \mid x_1,x_2)\, d\theta_1\, d\theta_2 > 1 - \eta(x_1,x_2) \quad \forall (x_1,x_2) \in R^{m_1} \times R^{m_2}$$

$$\|\bar{\theta}_1(x_1,x_2) - \theta_1\| < \epsilon(x_1,x_2)$$

$$\|\bar{\theta}_2(x_1,x_2) - \theta_2\| < \epsilon(x_1,x_2)$$

then,

$$\left| W(a(x_1,x_2), \bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2)) - \max_{a \in A} \int_{\Omega_1 \times \Omega_2} W(a,\theta_1,\theta_2)\, \xi(\theta_1,\theta_2 \mid x_1,x_2)\, d\theta_1\, d\theta_2 \right| < \delta$$

where $W(a(x_1,x_2), \bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2)) = \max_{a \in A} W(a, \bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2))$

Proof:- The proof of this theorem, relies on theorem 3, Ch.3 of Laffont [1989], by using the fact that given $(x_1,x_2) \in R^{m_1} \times R^{m_2}$, there exists a random variable e of dimension $(m_1 + m_2)$ with $E\, e = 0$, such that

$$(\theta_1, \theta_2) = (\bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2)) + e$$

where e is specified by $\xi(\theta_1, \theta_2 \mid x_1, x_2)$

<u>Theorem 2</u> :- Under assumption 1 and 3, the function $a: R^{m_1} \times R^{m_2} \to R^p$ is twice continuously differentiable, if the conditions of theorem 1 are also satisfied.

<u>Proof</u> :- By theorem 1, $D_a W(a(x_1,x_2), \bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2)) = 0$. By assumptions 1 and 3 the function $(x_1,x_2) \mapsto D_a W(a, \bar{\theta}_1(x_1,x_2), \bar{\theta}_2(x_1,x_2))$ is twice continuously differentiable and by a condition in theorem 1 (carried over to theorem 2),

$$D_a^2 W (a, \bar{\theta}_1 (x_1,x_2), \bar{\theta}_2 (x_1,x_2)) \text{ is globally non-singular.}$$

Hence, by the implicit function theorem, given $(x_1,x_2) \in R^{m_1} \times R^{m_2}$, there exists a neighbourhood $N(x_1,x_2)$ of (x_1,x_2) such that $\forall$ $(x_1', x_2') \in N(x_1,x_2)$,

$$D_a W(a(x_1',x_2'), \bar{\theta}_1 (x_1',x_2'), \bar{\theta}_2 (x_1',x_2')) = 0$$

By concavity of W in a,

$$W(a(x_1', x_2'), \bar{\theta}_1 (x_1', x_2'), \bar{\theta}_2(x_1',x_2') = \max_{a \in A} W(a, \bar{\theta}_1, (x_1',x_2'), \bar{\theta}_2(x_1',x_2'))$$

Further $a: N(x_1,x_2) \to R^p$ is twice continuously differentiable. This being true for all $(x_1,x_2) \in R^{m_1} \times R^{m_2}$, we obtain the desired result.

What the above two theorems indicate is that if the uncertainty embodied in the posterior distribution is "small" and the appropriate assumptions are satisfied, then the arbitrator needs merely to solve an equivalent deterministic optimization problem, to obtain the public decision which he would recommend to the bargainers. Note, for theorem 1, differentiability of W in θ_1 and θ_2 are not required. Thrice continuous differentiability of W in θ_1 and θ_2 is required for theorem 2.

An important objective of decision theory is to explore the world of bounded rationality. This is what the two theorems above precisely achieve in the case of an arbitrator who is a Bayesian statistician as well.

Significant to our analysis is the equilibrium for the arbitration game which results in a noncooperative setting. In the next theorem we attempt to characterize Nash equilibria of the game when the players are also characterized by bounded rationality.

Theorem 3 :- Suppose that the arbitration game $\Gamma = \langle u_1, u_2, W, \xi_1, \xi_2, \xi \rangle$ has a equilibrium. Let $\bar{\theta}_1$ and $\bar{\theta}_2$ be the means of $\xi_2(\theta_1)$ and $\xi_1(\theta_2)$ respectively. Suppose that, the Hessian matrices of the functions $u_1(a, \theta_1)$ and $u_2(a, \theta_2)$ with respect to 'a' are nonsingular for all $\theta_1 \in \Omega_1$ and $\theta_2 \in \Omega_2$. Further suppose Assumptions 1,2 and 3 hold and the problems

(a) $\max_{x_1 \in R^{m_1}} u_1(a(x_1,x_2), \theta_1)$, $x_2 \in R^{m_2}$, $\theta_1 \in \Omega_1$

and

(b) $\max_{x_2 \in R^{m_2}} u_2(a(x_1,x_2), \theta_2)$, $x_1 \in R^{m_1}$, $\theta_2 \in \Omega_2$

have unique solutions. Then given $\delta > 0$ there exists $x_1^* : \Omega_1 \to R^{m_1}$, $x_2^* : \Omega_2 \to R^{m_2}$, and real numbers $\epsilon_1 > 0$, $\epsilon_2 > 0$, $\eta_1 > 0$, $\eta_2 > 0$ such that

whenever $\int_{\|\theta_1 - \bar{\theta}_1\| < \epsilon_1} \xi_2(\theta_1)\, d\theta_1 > 1 - \eta_1$

and $\int_{\|\theta_2 - \bar{\theta}_2\| < \epsilon_2} \xi_1(\theta_2)\, d\theta_2 \quad 1 - \eta_2$

hold, we have

(c) $\left| u_1(a(x_1^*(\theta_1), x_2^*(\bar{\theta}_2)), \theta_1) - \max_{x_1 \in R^{m_1}} \int_{\Omega_2} u_1(a(x_1, x_2^*(\theta_2), \theta_1)\ (\theta_2) d\theta_2 \right| < \delta$

where

$u_1(a(x_1^*(\theta_1), x_2^*(\bar{\theta}_2), \theta_1) = \max_{x_1 \in R_+^{m_1}} u_1(a(x_1, x_2^*(\bar{\theta}_2)), \theta_1)$

for all $\theta_1 \in \Omega_1$

and

(d) $$\left| u_2(a(x_1^*(\bar{\theta}_1), x_2^*(\theta_2), \theta_2) - \max_{x_2 \in R^{m_2}} \int_{\Omega_1} u_2(a(x_1(\theta_1), x_2, \theta_2) \quad _2(\theta_1)\, d\theta_1 \right| < d$$

where

$$u_2(a(x_1^*(\bar{\theta}_1), x_2^*(\theta_2), \theta_2) = \max_{x_2 \in R^{m_2}} u_2(a(x_1^*(\bar{\theta}_1), x_2, \theta_2)$$

__Proof__ :- The proof of this theorem is analogous to that of Theorem 1, where we had appealed to Theorem 3,Chapter 3 of Laffont [1989]. For convenience a version of this theorem and its proof is being relegated to the appendix.

Q.E.D.

Thus under the assumptions of Theorems 1, 2 and 3, an equilibrium of the arbitration game is easily computed:

__Step 1__:- Solve,

$$O_a\,(a, \bar{\theta}_1(x_1), \bar{\theta}_2(x_2)) = 0$$

for 'a' in terms of x_1 and x_2. Let the solution be

$$a\,(x_1, x_2)$$

Step 2 :- Solve the following system of equations :

$$D_a u_1 (a(x_1,x_2(\bar{\theta}_2)),\theta_1) \cdot D_{x_1} a(x_1,x_2(\bar{\theta}_2)) = 0$$

$$D_a u_2(a(x_1(\bar{\theta}_1), x_2, \bar{\theta}_2) \cdot D_{x_2} a (x_1(\bar{\theta}_1),x_2) = 0$$

simultaneously for each $\theta_1 \in \Omega_1$, to obtain $x_1^* : \Omega_1 \rightarrow R^{m_1}$; and the system of equations

$$D_a u_2(a(x_1(\bar{\theta}_1), x_2, \theta_2) \cdot D_{x_2} a(x_1(\bar{\theta}_1), x_2) = 0$$

$$D_a u_1(a(x_1,x_2(\bar{\theta}_2), \bar{\theta}_1) \cdot D_{x_1} a (x_1,x_2(\bar{\theta}_2)) = 0.$$

simultaneously for each $\theta_2 \in \Omega_2$ to obtain $x_2^* : \Omega_2 \rightarrow R^{m_2}$. The ordered pair $\langle x_1^* (\cdot), x_2^* (\cdot) \rangle$ is the computed equilibrium of the arbitration game Γ , under assumptions of bounded rationality.

4. <u>Conclusion</u> :- In this paper we have formalized the framework of an arbitration game, to accomodate a large class of situations where public decisions are implemented in a noncooperative setting. We have then presented a method of computing the equilibrium strategies of the players under assumptions of bounded rationality, so that the solutions correspond to what is observed in any realization of an arbitration game. Such arbitration games arise naturally in many bargaining situations as discussed in Kalai [1988].

Appendix

Consider the following model

$y = g\ (x, \varepsilon), \quad x \in R^n, \quad y \in R^m,$

where the vector e has a probability density function $f\ (e,x)$ that can depend on the instruments but $\mathrm{E}\, e = 0$ for all x. The objective function is

$$u\ (x,y,e)$$

so that the optimization problem for the decision maker can be written as

$$\max_x \int_{R^p} u(x,g(x,e),\ e)\ f(e,x)de \qquad (1)$$

The certainty problem associated with this is written

$$\max_x u\ (x,g(x,0),0). \qquad (2)$$

In order to vary uncertainty in the neighbourhood of the certainty problem, we introduce the parameter ξ in the following way:

$$\max_x \int_{R^p} u\ (x,g(x,\xi e),\ \xi e)\ \ f(e,x)\ de,$$

which is equivalent to (1) if $\xi = 1$ and to (2) if $\xi = 0$. The problem exhibits the first order certainty equivalence property if the solution to (2) is equal (to the first order in ξ) to the solution to (1) in the neighbourhood of $\xi = 0$.

<u>Theorem</u> :- If $u(x,y,e)$ and $g(x,e)$ are twice differentiable, if $f(e,x)$ is twice differentiable in x, and if a unique optimal solution to the associated certainty problem exists, we will have first-order certainty equivalence if the Hessian matrix of the function $u(x, g(x,0),0)$ is nonsingular.

<u>Proof</u> :- See proof of Theorem 3, Chapter 3 of Laffont [1989].

References :

1. Farrell, J., and J. Sobel (1983) : Voluntary Disclosure of Information. Draft.

2. Grossman, S. (1981) : The Information Role of Warranties and Private Disclosure about Product Quality. Journal of Law and Economics, 24, 461-483.

3. Hildreth, C. (1963) : Bayesian Statisticians and Remote Clients. Econometrica 31: 422-438.

4. Kalai, E. (1988) : Solutions to the bargaining problems. (In L. Hurwiz, D. Schmeidler and H. Sonenschein (ed): Social Goals and Social Organizations; essays in memory of Eliza Pazner).

5. Laffont, J.J. (1989) : The Economics of Uncertainty and Information. The MIT Press, Cambridge (Mass), London (England).

6. Malinvaud, E. (1969) : Décisions en face de l'aléatoire et situation certaine approximativement équivalents. Cahiers du Séminaire de la Société d' Econometrie 11: 37-50.

7. Milgrom, P.R. (1981) : Good News and Bad News: Representation Theorems and Applications. Bell Journal of Economics, 12, 380-91.

8. Ordover, J.A., and A. Rubinstein (1983) : On Bargaining, Settling, and Litigating: A Problem in Multistage Games with Imperfect Information. Technical Report 83-07, New York University, April.

9. P'ing, I.P.L. (1983) : Strategic Behaviour in Suit, Settlement, and Trial. Bell Journal of Economics, 14, 539-50.

10. Samuelson, W (1983) : Negotiation vs. Litigation. Discussion Paper, Boston University School of Management, April.

11. Sobel, J. (1985) : Disclosure of evidence and resolution of disputes: Who should bear the burden of proof ? (In Alvin E. Rath (ed) Game theoretic models of bargaining. Cambridge University Press.)

12. Theil, H. (1954) : Econometric Models and Welfare Maximization. Weltwirtschaftigeches Archiv 72: 60-83.