AN AXIOMATIC CHARACTERIZATION OF THE VALUE FUNCTION FOR BIMATRIX GAMES

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ABSTRACT

In this paper we obtain an axiomatic characterization of the value function for the class of all bimatrix games satisfying the equivalency and efficiency properties.

1. Introduction : In Vilkas (1963), there exists a characterization of the value-function, defined on the class of all finite matrix games. In Tijs (1975) this result was extended to the class of all finite and semi-infinite matrix games. In a subsequent paper (Tijs (1981)), this result was further extended to the set of all determined two-person zero sum games.

Our purpose in this paper, is to obtain an axiomatic characterization of the value function on the class of all bi-matrix games satisfying the equivalency condition, which contains as a subclass the class of all matrix games. The method of characterization extends naturally to the class of all determined two-person games which satisfy the equivalency property.

2. Preliminaries : In this paper we consider (mixed extensions of) $m \times n$ bimatrix games (A,B) with $A = [a_{ij}]_{i=1}^{m} {}_{j=1}^{n}$ and $B = [b_{ij}]_{i=1}^{m} {}_{j=1}^{n}$ $(m \in \mathbb{N}, n \in \mathbb{N})$. With

$$\Delta_t = \left\{ x \in \mathbb{R}^t \Big/ \sum_{j=1}^{t} x_j = 1,\ x_j \geq 0 \text{ for all } j \in \{1,2,\ldots,t\} \right\} \ (t \in \mathbb{N})$$

(A,B) is a short notation for the two-person game $(\Delta_m, \Delta_n, K, L)$ in normal form with (mixed) strategy spaces Δ_m and Δ_n for player 1 and player 2, respectively and payoff functions

$$K : \Delta_m \times \Delta_n \to \mathbb{R} \text{ and } L : \Delta_m \times \Delta_n \to \mathbb{R}$$

respectively with $K(p,q) = pAq^t$ and $L(p,q) = pBq^t$ for all $p \in \Delta_m$ and $q \in \Delta_n$. The k-th pure strategy e_k for player 1 (or player 2) is defined as the vector $x \in \Delta_m$ (or Δ_n) with the k - th coordinate equal to 1. The set of completely mixed strategies is defined by

$$\Delta_t^{o} = \left\{ x \in \Delta_t / x_j > 0 \text{ for all } j \in \{1,2,\ldots,t\} \right\}.$$

A matrix game is identified with a bimatrix game $(A, -A)$.

A (Nash) equilibrium situation of a bimatrix game (A,B) is a pair $(\hat{p},\hat{q}) \in \Delta_m \times \Delta_n$ such that

$$\hat{p} A \hat{q}^T \geq pA\hat{q}^T \text{ and } \hat{p} B \hat{q}^T \geq \hat{p} B q^T \text{ for all } p \in \Delta_m,\ q \in \Delta_n. \quad (1)$$

In other words, unilateral deviation from an equilibrium situation does not pay. Nash (1951) proved that the set $E(A,B)$ of all Nash equilibria of a bimatrix game (A,B) is non-empty.

Two equilibrium situations (p,q) and (r,s) of a bimatrix game (A,B) are equivalent if $K(p,q) = K(r,s)$ and $L(p,q) = L(r,s)$. A bimatrix game (A,B) is said to possess the equivalency property if any two equilibrium situations are equivalent. All matrix games $(A,-A)$, in particular, satisfy the equivalency property.

A matrix game (A,B) is said to satisfy the <u>efficiency</u> property if $(\hat{p},\hat{q}) \in E(A,B)$ implies that there does not exist any other $(p,q) \in \Delta_m \times \Delta_n$ with $(K(p,q),L(p,q)) \geqq (K(\hat{p},\hat{q}),L(\hat{p},\hat{q}))$ with $(K(p,q),L(p,q)) \neq (K(\hat{p},\hat{q}),L(\hat{p},\hat{q}))$.

Let D denote the subclass of all bimatrix games satisfying the <u>equivalency property</u>, and the <u>efficiency property</u>. Define a function $V : D \to \mathbb{R}^2$ such that

$$V(A,B) = (K(\hat{p},\hat{q}),L(\hat{p},\hat{q})) \text{ where } (\hat{p},\hat{q}) \in E(A,B).$$

For a discussion of the concept of equivalency, one may refer to Szep and Forgo (1985).

3. In this section we inspect some distinguished properties of the **value-function** $V : D \to \mathbb{R}^2$. For this purpose we need some definitions.

<u>Definition 1</u> : The <u>transpose</u> of a bimatrix-game (A,B) is the bi-matrix game (B^t,A^t) where B^t is the matrix transpose of B and A^t is the matrix transpose of A.

<u>Definition 2</u> : Let (A,B) be a bi-matrix game and S be a nonempty subset of $\{1,\ldots,m\}$. Then we say that S is <u>weakly sufficient</u> for player 1 in the game (A,B) if for each $i \notin S$, $1 \leq i \leq m$ and for each $q \in \Delta_n$, there exists a $p(i,q) \in \Delta(S)$ (possibly depending on i and q) such that

$$K(p(i,q),j) \geq a_{ij} \text{ for each } j \in \{1,\cdots,n\} \text{ such that } q_j > 0,$$
$$L(p(i,q),q) \geq L(p(i,q),j) \text{ for each } j \in \{1,\ldots,n\}$$

Here $\Delta(S) \subseteq \Delta_m$ such that $p \in \Delta(S)$ iff $p_k = 0 \; \forall k \notin S$.

<u>Definition 3</u> : Let (A,B) be a bi-matrix game and let T be a nonempty subset of $\{1,\ldots,n\}$. We say that T is sufficient for player II in the game (A,B) if T is sufficient for player I in the game (B^t,A^t).

<u>Theorem 1</u> :

(P.1) ["Objectivity"] Let (A,B) be a bi-matrix game and suppose that $m = 1 = n$ then $(A,B) \in D$ and $V(A,B) = (a_{11}, b_{11})$.

(P.2) ["Monotomicity"] Let $(A,B) \in D$ and $(A',B') \in D$ and suppose that $a'_{ij} = \min\{a_{ij}, t_1\}$, $b'_{ij} = \{b_{ij}, t_2\} \forall (i,j) \in \{1,\ldots,m\} \times \{1,\ldots,n\}$ where $t_k \leq V_k(A,B)$, $K = 1,2$. Then $V_1(A,B) \geq V_1(A',B')$ and $V_2(A,B) \geq V_2(A',B')$ where $V(A,B) = (V_1(A,B), V_2(A,B))$ and $V(A',B') = (V_1(A',B'), V_2(A',B'))$.

(P.3) ["Symmetry"] Let $(A,B) \in D$. Then $(B^t, A^t) \in D$ and $V_1(A,B) = V_2(B^t, A^t)$ $V_2(A,B) = V_1(B^t,A^t)$ where $V(A,B) = V_1(A,B), V_2(A,B))$ and $V_1(B^t,A^t) = (V_1(B^t,A^t), V_2(B^t,A^t))$.

(P.4) ["Sufficiency"] Let (A,B) [illegible] bimatrix [illegible] satisfying the efficiency property and $\emptyset \neq S \subset \{1,\ldots,m\}$ and let (A_S, B_S) be the bi-matrix game obtained by deleting the rows corresponding to indices not in S. Suppose that S is weakly sufficient for player I in the game (A,B). Then $(A_S,B_S) \in D$ iff $(A,B) \in D$ and

$$V(A,B) = V(A_S,B_S) \text{ if } (A_S,B_S) \in D.$$

Proof : (P.1) and (P.2) are obvious (P.3) follows from the fact that

$$pAq^t = q\ A^t\ p^t$$

and $$pBq^t = q\ B^t p^t$$

for each $(p,q) \in \Delta_m \times \Delta_n$.

Let us prove (P.4).

Take $\alpha \in \Delta_m$. Then there exists $\{i_1,\ldots,i_k\} \subseteq \{1,\ldots,m\}$ such that $\alpha_i = 0$ if $i \notin \{i_1,\ldots,i_k\}$. Since S is sufficient for player I, if $V(A,B) = (K(\hat{p},\hat{q}), L(\hat{p},\hat{q}))$ then there exists $p(i,\hat{q}) \in \Delta(S)$ for each $i \in \{i_1,\ldots i_k\}$ such that $K(p(i,\hat{q}),j) \geq a_{ij}$ for each $j \in \{1,\ldots,m\}$ such that $\hat{q}_j > 0$ and $L(p(i,\hat{q}),\hat{q}) \geq L(p(i,\hat{q}),j)$ for each $j \in \{1,\ldots,n\}$.

Let $\bar{\alpha} = \sum_{l=1}^{k} \alpha_{i_l}\, p(i,\hat{q})$. Then $\bar{\alpha} \in \Delta(S)$ and

$$K(\bar{\alpha},j) = \sum_{l=1}^{k} \alpha_{i_l} K(p(i_l,\hat{q}),j) \geq \sum_{l=1}^{k} \alpha_{i_l} a_{i_l j} = K(\alpha,j) \text{ for each}$$

$j \in \{1,\ldots,n\}$, such that $\hat{q}_j > 0$

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$$L(\bar{\alpha},\hat{q}) = \sum_{l=1}^{k} \alpha_{i_l} L(p(i_l,\hat{q}),\hat{q}) \geqslant \sum_{l=1}^{k} \alpha_{i_l} L(p(i_l,\hat{q}),j) = L(\bar{\alpha},j)$$

for each $j \in \{1,\ldots,n\}$.

Thus, $K(\bar{\alpha},\hat{q}) \geqslant K(\alpha,\hat{q})$

and $L(\bar{\alpha},\hat{q}) \geqslant L(\bar{\alpha},q) \; \forall q \in \Delta_n$.

Let $\alpha = \hat{p}$ and $\bar{\alpha} = \hat{\alpha}$. Then $K(\hat{\alpha},\hat{q}) \geqslant K(\hat{p},\hat{q}) \geqslant K(p,\hat{q}) \; \forall p \in \Delta_m$

Thus $(A,B) \in D$ implies $(A_S,B_S) \in D$

Since $\hat{\alpha} \in \Delta(S) \subseteq \Delta_m$ and $(\hat{p},\hat{q}) \in E(A,B)$ we get

$$K(\hat{p},\hat{q}) \geqslant K(\hat{\alpha},\hat{q}) \geqslant K(\hat{p},\hat{q}) \Rightarrow K(\hat{\alpha},\hat{q}) = K(\hat{p},\hat{q})$$

Since $(A,B) \in D$, $L(\hat{\alpha},\hat{q}) = L(\hat{p},\hat{q})$ on observing that $(\hat{\alpha},\hat{q}) \in E(A,B)$

Conversely suppose, $(A_S,B_S) \in D$ and let $(\hat{\alpha},\hat{q}) \in E(A_S,B_S)$.

Since for every $(\alpha,q) \in \Delta_m \times \Delta_n$, there exists $\bar{\alpha}$ such that

$$K(\bar{\alpha},\hat{q}) \geqslant K(\alpha,\hat{q})$$
$$L(\bar{\alpha},\hat{q}) \geqslant L(\bar{\alpha},q)$$

we get

$$K(\hat{\alpha},\hat{q}) \geqslant K(\bar{\alpha},\hat{q}) \geqslant K(\bar{\alpha}(\alpha,\hat{q}),\hat{q}) \geqslant K(\alpha,\hat{q}) \; \forall \alpha \in \Delta_m$$

and $L(\hat{\alpha},\hat{q}) \geqslant L(\hat{\alpha},q) \; \forall q \in \Delta_n$.

$\therefore \; (\hat{\alpha},\hat{q}) \in E(A,B)$

Further, $\hat{\alpha} \in \Delta(S) \subseteq \Delta_m \Rightarrow K(\hat{\alpha},\hat{q}) \geqslant K(\hat{p},\hat{q}) \geqslant K(\hat{\alpha},\hat{q})$

$$\Rightarrow K(\hat{\alpha},\hat{q}) = K(\hat{p},\hat{q})$$

Since $(\hat{\alpha},\hat{q}) \in E(A,B)$, we must have $L(\hat{\alpha},\hat{q}) = L(\hat{p},\hat{q})$

$\therefore \; (A,B) \in D$

Also, in both cases $V(A,B) = V(A_S,B_S)$.

Note : To establish $(A,B) \in D \Rightarrow (A_S,B_S) \in D$ in (4) we do not require V to satisfy the efficiency property. Equivalency property alone suffices. It is to establish that $(A_S,B_S) \in D \Rightarrow (A,B) \in D$ in (4) that we require both equivalency and efficiency.

4. The following theorem that the properties (P.1)-(P.4) characterize the value function $V : D \to \mathbb{R}^2$:

<u>Theorem 2</u> : Let $f : D \to \mathbb{R}^2$ be such that $\forall (A,B) \in D, f(A,B) \in \{(K(p,q), L(p,q)) / p \in \Delta_m, q \in \Delta_n\}$. In addition suppose that

(Q.1) If $m = 1$, $n = 1$ then $f(A,B) = (a_{11}, b_{11})$.

(Q.2) For each $(A,B) \in D, (A',B') \in D$ with $a'_{ij} = \min\{a_{ij}, t_1\}, b'_{ij} = \min\{b_{ij}, t_2\}$, $t_k \leq V_k(A,B), k = 1,2, i \in \{1,\dots,m\}$, $j \in \{1,\dots,n\}$), $f(A,B) \geq f(A',B')$.

(Q.3) For each $(A,B) \in D$, $f_1(A,B) = f_2(B^t, A^t)$, $f_2(A,B) = f_1(B^t, A^t)$, where $f(A,B) = (f_1(A,B), f_2(A,B))$ and $f(B^t, A^t) = (f_1(B^t, A^t), f_2(B^t, A^t))$.

(Q.4) For each $(A,B) \in D$ and $(A_S, B_S) \in D$, where $S \subseteq \{1,\dots,m\}$, $S \neq \emptyset$, Let (A_S, B_S) be the bi-matrix game obtained by deleting the rows corresponding to indices not in S. If S is weakly sufficient for player I in the game (A,B), we have $f(A,B) = f(A_S, B_S)$.

<u>Proof</u> : First we note that (Q.3) and (Q.4) imply :

(Q.5) For each $(A,B) \in D$ and $(A_T, B_T) \in D$, where $T \subseteq \{1,\dots,n\}$, $T \neq \emptyset$, and (A_T, B_T) is the bi-matrix game obtained by deleting the columns corresponding to indices not in T, if T is weakly sufficient for player II in the game (A,B), we have $f(A_T, B_T) = f(A,B)$.

Now take an $(A,B) \in D$ with $V(A,B) \in \mathbb{R}^2$ and take real numbers t_1 and t_2 such that $V_k(A,B) = t_k$. We shall show that $f_k(A,B) \geq t_k$. For this reason we introduce the following five two person games :

(1) $\left(\dfrac{A}{a_{m+1}}, \dfrac{B}{b_{m+1}}\right)$ where $\left(\dfrac{A}{a_{m+1}}\right)$ is the (m+1)xn matrix and

$a_{m+1} = (a_{m+1,1}, \dots, a_{m+1,n})$ with $a_{m+1,j} = t_1$ for each $j \in \{1,\dots,n\}$.

A corresponding definition is valid for the (m+1)xn matrix $\left(\dfrac{B}{b_{m+1}}\right)$ where $b_{m+1} = (b_{m+1,1}, \dots, b_{m+1,n})$ and $b_{m+1,j} = t_2$ for each $j \in \{1,\dots,n.\}$

(2) (A',B') where $B' = ((b'_{ij}))_{(m+1)\times n}$ and $A' = ((a'_{ij}))_{(m+1)\times n}$ where $a'_{ij} = \min\{a_{ij}, t_1\}$ for each $i \in \{1,\dots,m+1\}$, $j \in \{1,\dots,n\}$ and $b'_{ij} = \min\{b_{ij}, t_2\}$ for each $i \in \{1,\dots,m+1\}$, $j \in \{1,\dots,n\}$.

(3) (A'',B'') where A'' is an $(m+1)\times(n+1)$ matrix and B'' is an $(m+1)\times(n+1)$ matrix $(a''_{ij}, b''_{ij}) = (a'_{ij}, b'_{ij})\ \forall i \in \{1,\dots,m+1\}$, $j=1,\dots,n$ and $(a''_{i,n+1}\ b''_{i,n+1}) = (t_1,t_2)$ for each $i \in \{1,\dots,m+1\}$.

(4) (A''', B''') where each of A''' and B''' is a $1 \times (n+1)$ matrix and $(a'''_{1,j}\ \ b'''_{1,j}) = (a'_{m+1,j}\ \ b'_{m+1,j})\ \forall j \in \{1,\dots,n+1\}$.

(5) $(a'_{m+1,n+1}, b'_{m+1,n+1})$.

Since $V_k(A,B)\ \ t_k$, there exists $p^* \in \Delta_m$ and $q^* \in \Delta_n$ such that
$K(p^*,j) \geq t_1 = a_{m+1,j}\ \forall j \in \{1,\dots,n\}$ such that $q^*_j > 0$
$L(p^*,q^*) \geq t_2 = b_{m+1,j}\ \forall j \in \{1,\dots,n\}$.

Hence, by the definition of weak sufficiency $\{1,\dots,m\}$ is weakly sufficient for player I in the game $\left(\frac{A}{a_{m+1}}, \frac{B}{b_{m+1}}\right)$. By (P.4), (J.4) and by the fact that as per our definition $\left(\frac{A}{a_{m+1}}, \frac{B}{b_{m+1}}\right)$ satisfies the efficiency property, we may conclude that

(J.6) $\left(\frac{A}{a_{m+1}}, \frac{B}{b_{m+1}}\right) \in D$ and $f\left(\frac{A}{a_{m+1}}, \frac{B}{b_{m+1}}\right) = f(A,B)$.

It follows from (J.1) that

(J.7) $(a'_{m+1,n+1}, b'_{m+1,n+1}) = (t_1,t_2)$

In the game (A''', B''') the set $\{n+1\}$ is weakly sufficient for player II. Hence in view of (P.3) and (P.4) $(A''', B''') \in D$. By (J.5)

(J.8) $f(A''', B''') = (a'_{m+1,n+1}, b'_{m+1,n+1}) = (t_1,t_2)$

In the game (A'',B'') the set $\{m+1\}$ is weakly sufficient for player I because for each $i \in \{1,\dots,m\}$ and $q \in \Delta_{n+1}$

$$a''_{m+1,j} \geqslant a''_{ij} \quad \forall\, j \in \{1,\dots,n+1\}$$

$$b''_{m+1,j} \geqslant b''_{m+1,j} \quad \forall\, j \in \{1,\dots,n+1\}$$

By (P.4) and (Q.4) we obtain : $(A'',B'') \in D$ and

(Q.9) $f(A'',B'') = f(A''', B''')$

It is eash to see that $\{1,\dots,n\}$ is weakly sufficient for player II in the game (A''', B'''). Hence by (P.3) and (P.4) : $(A',B') \in D$; and then by (Q.5) :

(Q.10) $f(A''', B''') = f(A',B')$

Now by (P.2)

(Q.11) $f\left(\dfrac{A}{a_{m+1}}, \dfrac{B}{b_{n+1}}\right) \geqslant f(A',B')$

Combining (Q.6) to (Q.11) we get that $f(A,B) \geqslant (t_1,t_2)$. Thus we have proved that $f(A,B) \geqslant (t_1,t_2)$ for each $(A,B) \in D$ with $t_k = V_k(A,B)$, $k = 1,2$. But then

(Q.12) $f(A,B) \geqslant V(A,B)$ for each $(A,B) \in D$.

By the efficiency property satisfied by all $(A,B) \in D$,

(Q.13) $f(A,B) = V(A,B)$

which concludes our theorem.

5. <u>Conclusion</u> : In this paper we have obtained an axiomatic characterization of the value function for bimatrix games satisfying the efficiency and equivalency property. Matrix games, constant sum games and indeed all two person games where by a transition from one situation to another, the payoffs of the players move in opposite directions, all satisfy these two properties. An eash and direct extension of our results is valid for the set of all two person games satisfying the efficiency and equivalency property, where the relevant mixed extension is the c-mixed extension as defined in Tijs (1981).

Many games, including threat bargaining games (Lahiri,(1989), Lahiri(1990)) satisfy the above properties. Thus apart from the analytical niceties of our extension what we have achieved is a characterization of the value function in many game theoretic contexts.

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