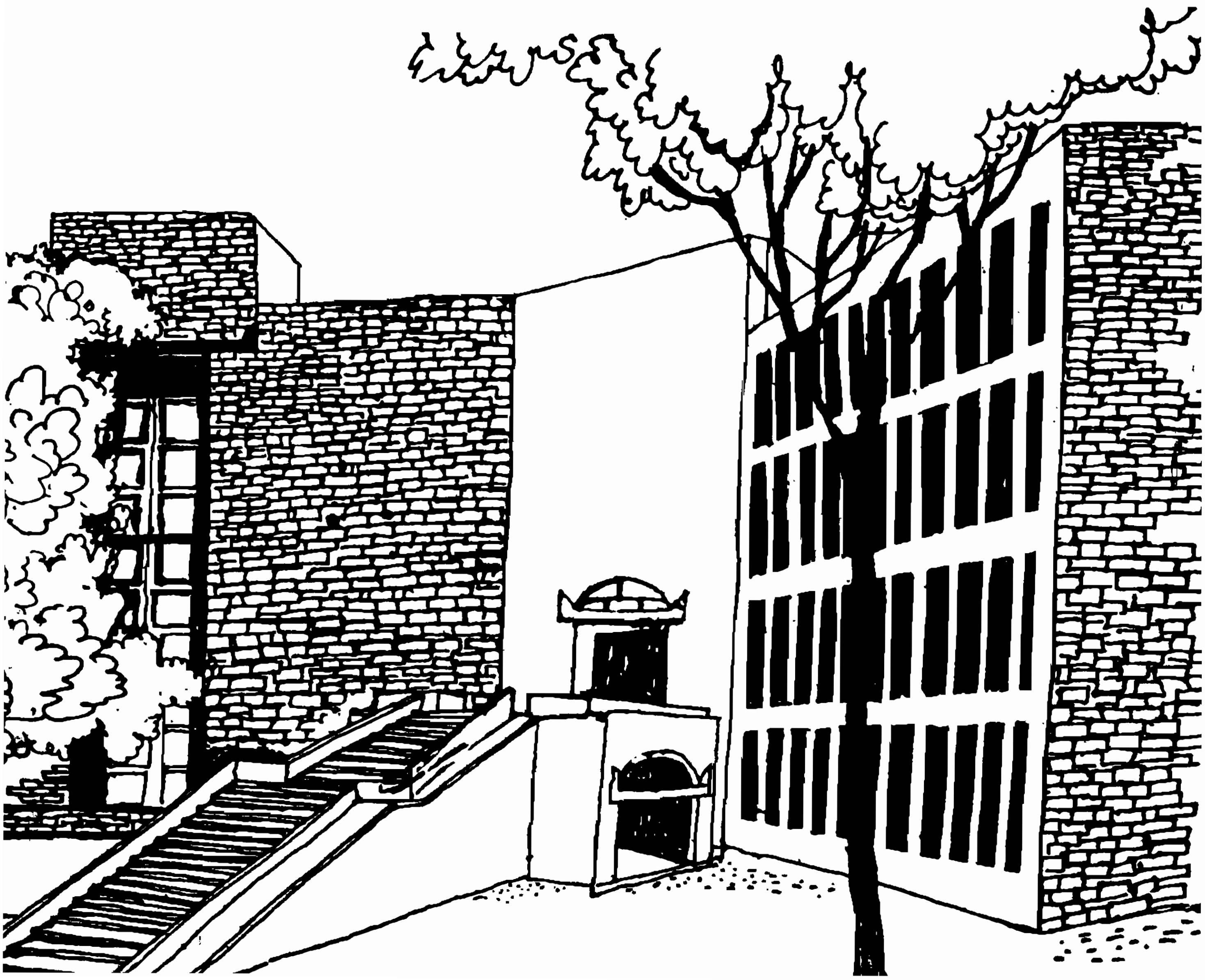
Working Paper



A CONCEPT OF CONSTRAINED EGALITARIANISM
IN FAIR DIVISION PROBLEMS

By

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Abstract

This paper is really a modification of a paper by Diamantaras and Thomson (1990). In that paper, the no-envy concept due to Foley (1967) was refined to accommodate some kind of a radial no-envy comparison, inspired by Chaudhuri (1986). Simply put, each person compares his/her own consumption bundle with all possible radial expansions and contractions of every other person's consumption bundle. A Weakly Pareto Optimal allocation which is envy free against such a maximal expansion is the one selected by Diamantaras and Thomson (1990).

Our framework differs from the Diamantaras and Thomson (1990) framework in that we consider only the pure exchange situation, with the possibility of quantity constraints on consumption. Thus, since such technical issues with regard to existence of envy free allocation in the sense of Foley (1967) are somewhat secondary (though present) in our framework, we view this no-envy concept as a new equity criterion. In this framework, we prove the Diamantaras and Thomson (1990) result regarding the existence of an envy free allocation on a somewhat larger domain of preferences. We also feel that our existence proof is much simpler than the one due to the two authors, although it is difficult to say whether our proof would extend to the economies with production as studied by them.

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1. Introduction:

This paper is really a modification of a paper by Diamantaras and Thomson (1990). In that paper, the no-envy concept due to Foley (1967) was refined to accommodate some kind of a radial no-envy comparison, inspired by Chaudhuri (1986). Simply put, each person compares his/her own consumption bundle with all possible radial expansions and contractions of every other person's consumption bundle. A Weakly Pareto Optimal allocation which is envy free against such a maximal expansion is the one selected by Diamantaras and Thomson (1990).

Our framework differs from the Diamantaras and Thomson (1990) framework in that we consider only the pure exchange situation, with the possibility of quantity constraints on consumption. Thus, since such technical issues with regard to existence of envy free allocation in the sense of Foley (1967) are somewhat secondary (though present) in our framework, we view this no-envy concept as a new equity criterion. In this framework, we prove the Diamantaras and Thomson (1990) result regarding the existence of an envy free allocation on a somewhat larger domain of preferences. We also feel that our existence proof is much simpler than the one due to the two authors, although it is difficult to say whether our proof would extend to the economies with production as studied by them.

2. Notations, Definitions, Results:

We adopt the framework in Diamantaras and Thomson (1990) for our purposes. There are $k \geq 2$ private goods, $n \geq 1$ agents. We denote the set of agents by N.

Each agent $i \in N$, has a utility function $u_i : \mathbb{R}^k_+ \to \mathbb{R}$ which is assumed to be continuous and weakly increasing ie $x, y \in \mathbb{R}^k_+$, $x \gg y \to u_i(x) > u_i(y)$. Let $u = (u_1, \dots u_n)$.

We make the following assumption about u:

Either (a) $\forall\, i \in N$, u_i is strictly increasing (i.e. $x, y \in \mathbb{R}^k_+$, $x > y \to u_i(x) > u_i(y)$);

or (b) $\forall\, i \in N$, $x \in \mathbb{R}^k_{++}, u_i(x) = u_i(y) \to y \in \mathbb{R}^k_{++}$.

As observed in Lahiri (1997), if u satisfies (b), then $x \in \mathbb{R}^k_{++}, y \in \mathbb{R}^k_+ \setminus \mathbb{R}^k_{++} \to u_i(x) > u_i(y)$. Further it is easy to see that continuity along with (b) implies $u_i(x) = u_i(0)\ \forall\, x \in \mathbb{R}^k_+ \setminus \mathbb{R}^k_{++}$.

Let $\omega \in \mathbb{R}^k_{++}$. This is the aggregate social endowment of the economy.

Let L belonging to $\mathbb{R}^k_{++}$ denote quantity constraints.

Let $A = \left\{ (x_i)_{i=1}^n \in (\mathbb{R}^k_+)^n \,/\, \sum_{i=1}^n x_i \le \omega, x_i \le L \forall i \in N \right\}$ denote the set of all feasible allocations.

An allocation $(x_i)_{i=1}^n \in A$ is said to be weakly Pareto

exist $(y_i)_{i=1}^n \in A$ such that $u_i(y_i) > u_i(x_i)\ \forall i = 1, \ldots, n$.

An allocation $(x_i)_{i=1}^n \in A$ is said to be Pareto optimal if there does not exist $(y_i)_{i=1}^n \in A$ such that $u_i(y_i) \geq u_i(x_i)\ \forall i = 1, \ldots, n$ with at least one strict inequality.

Clearly any Pareto optimal allocation is weakly Pareto optimal.

Lemma 1: If $L \geq \omega$, then under our assumptions any weakly Pareto Optimal allocation is Pareto Optimal.

Proof:

Let $(x_i)_{i=1}^n$ be weakly Pareto Optimal. Suppose it is not Pareto optimal. Then there exists $(y_i)_{i=1}^n \in A$ such that $u_i(y_i) \geq u_i(x_i)\ \forall_i = 1, \ldots, n$ with at least one strict inequality. Without loss of generality suppose $u_1(y_1) > u_1(x_1)$.

Case 1: u satisfies (a).

Then there exists $a \in \mathbf{R}_+^k \setminus \{0\}$ such that $u_1(y_1 - a) > u_1(x_1)$ (by continuity).

Consider,

$z_1 = y_1 - a$

$z_i = y_i + \frac{a}{n-1}$, $i \geq 2$.

$\therefore u_i(z_i) > u_i(x_i) \; \forall \; i = 1, \ldots, n,$

since, $u_1(y_1 - a) > u_1(x_1)$ and $u_i\left(y_i + \frac{a}{n-1}\right) > u_i(y_i) \; \forall \; i \geq 2$ by su...

monotonicity. This contradicts that $(x_i)_{i=1}^n$ is weakly Pareto optimal.

Case 2: u satisfies (b)

Clearly $y_1 \in \mathbf{R}_{++}^k$. Thus by continuity, there exists $a \in \mathbf{R}_{++}^k$ such that $u_1(y_1 - a) > u_1(x_1)$. Since $u_i\left(y_i + \frac{a}{n-1}\right) > u_i(x_i) \; \forall \; i \geq 2,$ a similar construction such as above leads to a contradiction.

Q.E.D

Let P denote the set of Weakly Pareto optimal allocations.

Lemma 2: P is a closed set.

Proof: Let $\left\{(x_i^m)_{i=1}^n\right\}_{m \in \mathbf{N}}$ be a sequence in P converging to $(x_i)_{i=1}^n$. If $(x_i)_{i=1}^n \notin P$, then these exists (by Lemma 1) $(y_i)_{i=1}^n \in A$ such that $u_i(y_i) > u_i(x_i) \; \forall i \in N$. By continuity of u_i, there exists m large such that $u_i(y_i) > u_i(x_i^m) \; \forall i \in N$. This contradicts $(x_i^m)_{i=1}^n \in P$, and proves the lemma.

Q.E.D

We say that $(x_i)_{i=1}^n \in A$ is λ - fair if

i) $(x_i)_{i=1}^{n} \in P$

ii) $u_i(x_i) \geq u_i(\lambda x_j)\ \forall\ i \neq j,\ i, j \in N$

We only consider λ - fair allocations for $\lambda \geq 0$.

Let $B = \{\lambda \geq / \exists \quad a \quad \lambda\text{-}fair\ allocation\}$.

Lemma 3: B is a closed set.

Proof: Let $\{\lambda^m\}_{m \in \mathbb{N}} \subset B\ with \lim_{m \to \infty} \lambda^m = \lambda \in \mathbb{R}$.

Hence $\forall\ m \in \mathbb{N},\ \exists (x_i^m)_{i=1}^{n} \in A$ such that

i) $(x_i^m)_{i=1}^{n} \in P$

ii) $u_i(x_i^m) \geq u_i(\lambda^m x_j^m)\ \forall\ i \neq j,\ i, j \in \mathbb{N}$.

Since P is a closed subset of a compact set A, P is compact. Hence $\left\{(x_i^m)_{i=1}^{n}\right\}_{m \in \mathbb{N}}$ has a convergent subsequence. Without loss of generality assume that the original sequence is convergent to $(x_i)_{i=1}^{n} \in P$.

Thus, $u_i(x_i) = \lim_{m \to \infty} u_i(x_i^m) \geq \lim_{m \to \infty} u_j(\lambda^m x_j^m) = u_j(\lambda\ x_j)$

$\forall\ i \neq j,\ i, j \in \mathbb{N}$.

Thus $\lambda \in B$. <u>Q.E.D.</u>

Lemma 4: We make the following assumption about u:

Either (a) $\forall\, i \in N$, u_i is strictly

increasing (i.e. $x, y \in \mathbb{R}_+^k$, $x > y \rightarrow u_i(x) > u_i(y)$); further

$\forall\, x \in \mathbb{R}_+^k$, $\forall\, i \in N$, $\{y \in \mathbb{R}_+^k / u_i(x) \geq u_i(y)\}$ is bounded;

or (b) $\forall\, i \in N$, $x \in \mathbb{R}_{++}^k, u_i(x) = u_i(y) \rightarrow y \in \mathbb{R}_{++}^k$.

Then B is bounded above.

Proof:

Case 1: u satisfies (a).

Let $\{\lambda^m\}_{m \in \mathbf{N}}$ be a sequence in $\mathbb{R}_+$ with $\lim_{m \to \infty} \lambda^m = +\infty$,

Let $\{(x_i^m)\}_{m \in \mathbf{N}}$ be the sequence of allocations associated with $\{(\lambda^m)\}_{m \in \mathbf{N}}$. Without loss of generality and since P is compact assume $\{(x_i^m)\}_{m \in \mathbf{N}}$ converges to $(x_i) \in P$. There exists at least one i such that $x_i > 0$. Thus for large m, $j \neq i$, $u_j(\lambda^m x_i^m) > u_j(x_j^m)$ since the lower contour sets of u_j are compact and $\|\lambda^m x_i^m\| \rightarrow +\infty$. This contradiction establishes the non-existence of a diverging sequence.

Case 2: u satisfies (b)

Since $u_i\left(\frac{\omega}{n}\right) > 0 \ \forall\, i$, and equal division is feasible, in the above construction

of a divergent sequence $x_i \in \mathbb{R}^k_{++} \ \forall i$. But then $u_j\left(\lambda^m x_i^m\right) > u_j\left(x_j^m\right)$ for

sufficiently large m, leading to a contradiction.

Thus B must be bounded. Q.E.D.

We are now equipped to prove the following theorem:

Theorem 1: We make the following assumption about u:

Either (a) $\forall\, i \in N$, u_i is strictly

increasing (i.e. $x, y \in \mathbb{R}^k_+$, $x > y \to u_i\,(x) > u_i\,(y)$); further

$\forall\, x \in \mathbb{R}^k_+$, $\forall\, i \in N$, $\left\{y \in \mathbb{R}^k_+ / \ u_i(x) \geq u_i(y)\right\}$ is bounded;

or (b) $\forall\, i \in N$, $x \in \mathbb{R}^k_{++}, u_i(x) = u_i(y) \to y \in \mathbb{R}^k_{++}$.

Under our assumptions max (B) exists.

Proof: Since $0 \in B, B$ is nonempty.

Since it is closed and bounded above max (B) exists and belongs to $\mathbb{R}_+$.

Q.E.D.

Let $\bar{\lambda}$ = max (B) and $(x_i)_{i=1}^n$ be $\bar{\lambda}$ - fair. This allocation is the implied

recommendation in Diamantaras and Thomson (1990), as adapted to our framework.

Note: In the above theorem we implicitely use the fact that $P \neq \phi$ (in order to claim that $B \neq \phi$). But this follows easily from the compactness of A and the continuity of the utility functions.

Remark:

It is shown in Ichiishi (1983) that if the utility functions are quasi-concave and continuous then a constrained competitive equilibrium from strictly positive initial endowments exists. Further both the first and second fundamental theorems of welfare economics with quantity constraints on consumption, continue to hold.

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Appendix

<u>Theorem</u>: Suppose $u:\mathbf{R}^k_+ \to \mathbf{R}$ is quasi-concave i.e $\forall x, y \in \mathbf{R}^k_+$ with $\forall\ t \in [0,1], u(tx+(1-t)y) \geq \min\{u(x), u(y)\}$. Then it is semi-strictly quasi-concave i.e.

$\forall x, y \in \mathbf{R}^l_+$ $u(x) \neq u(y)$ $\forall\ t \in (0,1), u(tx+(1-t)y) > \min\{u(x), u(y)\}$.

<u>Proof:</u> Let $x, y \in \mathbf{R}^k_+$ with u (x) > u (y) and towards a contradiction assume that there exists $\alpha \in (0,1)$ *with* $u(\alpha x+(1-\alpha)y) = u(y)$.

Clearly neither x >> y nor y >> x (∵ by weak monotonicity; for in that case $\alpha x+(1-\alpha)y \gg y$, leading to a contradiction).

<u>Case 1</u>:- x >> 0.

Then choose $\bar{x} \in \mathbf{R}^k_{++}$ such that $x \gg \bar{x}$ *and* $u(\bar{x}) > u(y)$. This is clearly possible by continuity. Clearly $\alpha x+(1-\alpha)y \gg \alpha\bar{x}+(1-\alpha)y$

By weak monotonicity

$u(y)=u(\alpha x+(1-\alpha)y)>u(\alpha\bar{x}+(1-\alpha)y)$ contradicting u is quasi-concave.

<u>Case 2</u>:- $x\in\mathbb{R}^{\ell}_{+} \setminus \mathbb{R}^{\ell}_{++}$ but u is strictly monotonic. Clearly $x > 0$.

Then choose $\bar{x}\in\mathbb{R}^{\ell}_{+}$ such that $x>\bar{x}>0$ and $u(\bar{x})>u(y)$. This is clearly possible by continuity. Clearly $\alpha x+(1-\alpha)y>\alpha\bar{x}+(1-\alpha)y$.

By strict monotonicity

$u(y)=u(\alpha x+(1-\alpha)y)>u(\alpha\bar{x}+(1-\alpha)y)$ contradicting u is quasi-concave.

<u>Case 3</u>:- $x\in\mathbb{R}^{\ell}_{+} \setminus \mathbb{R}^{\ell}_{++}$ *but* $a\in\mathbb{R}^{\ell}_{++}$, $b\in\mathbb{R}^{\ell}_{+} \setminus \mathbb{R}^{\ell}_{++}$ implies $u(a) > u(b)$.

In this case clearly $y\in\mathbb{R}^{\ell}_{+} \setminus \mathbb{R}^{\ell}_{++}$. Suppose $u(tx + (1-t)y)>u(y)$ for some $t\in(0,1)$. By weak monotonicity there exists

$y \succ tx+(1-t)y$ with $u(y)>u(tx+(1-t)y)$. By continuity there exists

$\alpha\epsilon(0,1)$ such that $u(\alpha y+(1-\alpha)y) = u(tx+(1-t)y)$. But $\alpha y+(1-\alpha)y\epsilon\mathbb{R}^k_{++}$ and

$tx+(1-t)y\epsilon\mathbb{R}^k_+ \setminus \mathbb{R}^k_{++}$ which leads to a contradiction.

Hence $u(tx+(1-t)y)=u(y)\forall t\epsilon(0,1)$.

Thus $u(x)=u(y)$ by continuity.

Hence Case 3 is not possible.

This proves the theorem.

Q. E. D.

We make the following assumption about u:

Either (a) $\forall\ i \in N$, u_i is strictly

increasing (i.e. $x, y \in \mathbb{R}^k_+$, $x > y \rightarrow u_i(x) > u_i(y)$); further

$\forall x \in \mathbb{R}^k_+$, $\forall\ i \in N$, $\{y \in \mathbb{R}^k_+ / u_i(x) \geq u_i(y)\}$ is bounded;

or (b) $\forall\ i \in N$, $x \in \mathbb{R}^k_{++}, u_i(x) = u_i(y) \rightarrow y \in \mathbb{R}^k_{++}$.