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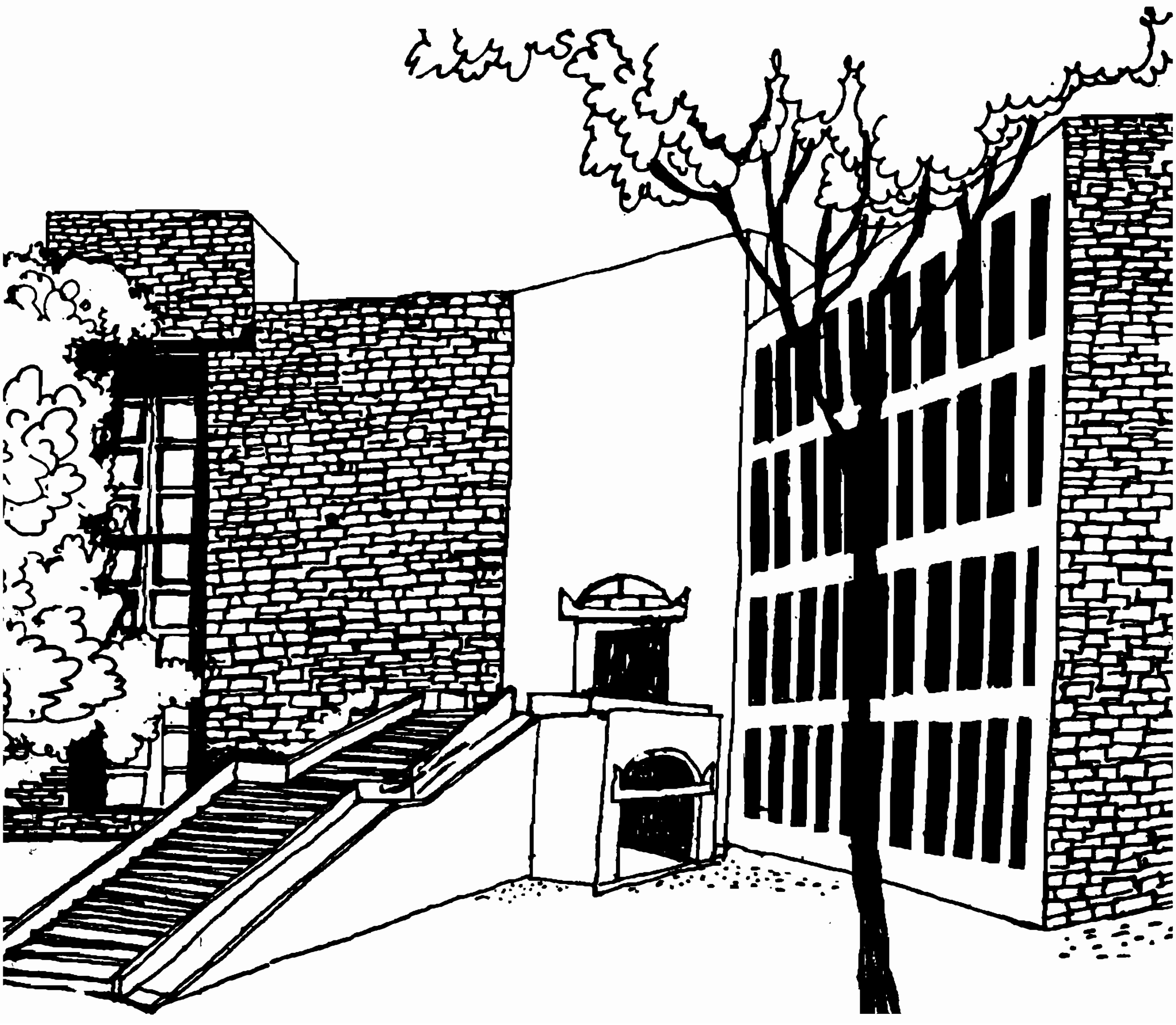

REDUCING A MULTI STAGE VECTOR OPTIMIZATION PROBLEM TO A SINGLE STAGE VECTOR OPTIMIZATION PROBLEM

By

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A B S T R A C T

In this paper we study the problem of universally reducing a multi-stage vector optimization problem to a one-stage vector optimization problem. This paper draws heavily and modifies on results obtained in Aizerman and Malishevski [1986]. Given the importance of such problems as mathematical representations of real world phenomena, particularly in economics and the management sciences, the results reported here have great interest.

Our analysis is restricted to the case of finite sets of alternatives, and thus has independent appeal from the stand point of finite/discrete mathematics as well.

1. <u>Introduction</u>: In this paper we study the problem of universally reducing a multi-stage vector optimization problem to a one-stage vector optimization problem. This paper draws heavily and modifies on results obtained in Aizerman and Malishevski [1986]. Given the importance of such problems as mathematical representations of real world phenomena, particularly in economics and the management sciences, the results reported here have great interest.

Our analysis is restricted to the case of finite sets of alternatives, and thus has independent appeal from the stand point of finite/discrete mathematics as well.

2. <u>Notations and Definitions</u>: Let $\mathbf{N}$ denote the set of natural numbers. Given a, b $\in \mathbf{N}$, let a > b denote "a is greater than b", a = b denote "a is equal to b" and a $\geq$ b denote "either a > b or a = b". Given n $\in \mathbf{N}$, let $\mathbf{N}^n$ denote the set of all functions from $\{1,\ldots,n\}$ to $\mathbf{N}$ i.e., the set of all n-tuples of natural numbers. Given a, b $\in \mathbf{N}^n$, we write:

$a \gg b$ to denote $a_i > b_i \ \forall\ i \in \{1,\ldots,n\}$

$a = b$ to denote $a_i = b_i \ \forall\ i \in \{1,\ldots,n\}$

$a \times b$ to denote "neither $a \gg b$ nor $b \gg a$"

$a \not\gg b$ to denote "not $a \gg b$"

Let U be a given non-empty set consisting of N elements. Thus U is assumed to finite. Let Σ denote the collection of all non-empty subsets of U.

Let $f : U \to \mathbf{N}^n$ be a function for some $n \in \mathbf{N}$. Given $X \in \Sigma$, $\underset{x \in X}{argmax}\,[f(x)]$ is defined to be equal to

$\{y \in X / \exists\, x \in X \text{ with } f(x) > f(y)\}$. This set is denoted by $Y(f, X)$.

A f-triad in U is an ordered triplet (u, v, w), u, v, w $\in$ U such that $f(u) > f(v)$, $f(u) \times f(w)$, $f(v) \times f(w)$.

The following result is essentially Lemma 1 in Aizerman and Malishevski [1986].

Lemma 1: Given $f : U \to \mathbf{N}$ with $n \in \mathbf{N}$, there exists $g : U \to \mathbf{N}$ such that $\forall X \in \Sigma$, $Y(f, X) = \{y \in X / \exists\, x \in X \text{ with } g(x) > g(y)\}$ if and only if there does not exist any f-triad in U.

Let $f : U \to \mathbf{N}^n$, $g : U \to \mathbf{N}^m$, $n, m \in \mathbf{N}$ be given. We let Y (f, g, X) denote the set Y[g, Y(f,x)] where X $\in \Sigma$.

The central problem in Section 4 is to obtain necessary and sufficient conditions for the existence of $p \in \mathbf{N}$ *and a function* $h : U \to \mathbf{N}^p$ such that for all X in Σ, Y (f,g,X) = Y(h,X).

The following results appear in Aizerman and Malishevski [1986].

Theorem 1: If n = 1, m = 1, then there exists $h: U \to \mathbf{N}$ such that Y (f, g, X) = Y (h, X) $\forall X \in \sum$.

<u>Theorem 2:</u> If n = 1, m ≥ 1, then there exists $p \in \mathbf{N}$ *and* $h: U \to \mathbf{N}^p$ such that Y (f, g, X) = Y (h, X) $\forall X \in \Sigma$.

<u>Theorem 3:</u> If n = 1, m ≥ 1 and there does not exist any g-triad (u,v,w) with f(u) = f(v) = f(w), then there exists $h : U \to \mathbf{N}$ such that Y (f, g, X) = Y (h, X) $\forall X \in \Sigma$.

Hence, what really remains to be investigated is the case where n ≥ 1. An answer to that question is available in Theorem 2 of Aizerman and Malishevski [1986]. Here we provide a different characterization using a slightly different approach.

3. <u>Binary Relations:</u> A binary relation on U is any set of ordered pairs of elements in U. Let P be a binary relation on U. P is said to be asymmetric if (u, v) ∈ P implies (v, u) ∉ P. Hence if P is asymmetric (u, u) ∉ P. P is said to be transitive if (u, v) ∈ P, (v, w) ∈ P implies (u, w) ∈ P.

The following result is available in Aizerman and Malishevski [1986], Aizerman and Aleskerov [1995], Donaldson and Weymark [1998].

<u>Theorem 4:</u> Let P be any asymmetric and transitive binary relation on U. Then there exists $p \in \mathbf{N}$ *and a function* $h: U \to \mathbf{N}^p$ such that for all u, v, in U, (u, v) ∈ P if and only if $h(u) > h(v)$.

4. <u>Preliminary Results</u>: Given $f: u \to \mathbf{N}^n$ *and* $g: u \to \mathbf{N}^m$ with n, m ∈

$\mathbb{N}$ an f-g complex in U is an ordered triplet (u, v, w) such that:

a) (u, v, w) is a f-triad in U

b) $g(w) \not> g(v)$ and $g(u) \not> g(w)$

c) either $g(v) > g(w)$ or $g(w) > g(u)$.

Theorem 5: Given $f: U \rightarrow \mathbb{N}^n$ and $g: U \rightarrow \mathbb{N}^m$ with n, $m \in \mathbb{N}$, there exists $p \in \mathbb{N}$ and $h: U \rightarrow \mathbb{N}^p$ such that $Y(f, g, X) = Y(h, X)$ $\forall X \in \Sigma$ if and only if there does not exist any f-g complex in U.

Proof: Let $h: U \rightarrow \mathbb{N}^p$ satisfy $Y(f, g, X) = Y(h, X)$ $\forall X \in \Sigma$.

Towards a contradiction assume that (u, v, w) is a f-g complex in U. Let $X = \{u, v, w\}$. Thus $v \notin Y(f, g, X)$. Thus either $h(u) > h(v)$ or $h(w) > h(v)$.

Suppose $h(w) > h(v)$. Let $X' = \{v, w\}$. Now $g(w) \not> g(v)$ implies $v \in Y(f, g, X')$. However, $h(w) > h(v)$ implies $v \notin Y(h, X')$. Hence $h(w) \not> h(v)$. Thus $h(u) > h(v)$.

Now $g(w) > g(u)$ implies $u \notin Y(f, g, X)$
$\therefore$ $\{w\} = Y(f, g, X) = Y(h, X)$
Since $h(u) > h(v)$, and $u \notin Y(f,g,X)$ we must have $h(w) > h(u)$.

But this implies $h(w) > h(v)$ contradicting the previous step. Thus $g(w) \not> g(u)$. Hence $g(v) > g(w)$. Thus, $(v) = Y(f, g, X')$. But then $h(v) > h(w)$ and hence $h(u) > h(w)$. But this combined $f(u) \times f(w)$ implies $g(u) > g(w)$ which contradicts that (u, v, is a f-g complex in U. Thus, there does not exist any f-g compl in U.

Now suppose there does not exist any f-g complex in U. Define a binary relation P on U as follows :

$(u,v) \in P$ if and only if either (a) $f(u) > f(v)$ or (b) $f(u) \times f(v)$ and $g(u) > g(v)$.

Clearly P is asymmetric. Let us show that P is transitive. Thus, let $(u,v) \in P$, $(v,w) \in P$.

<u>Case 1</u>: $f(u) \times f(v)$ and $f(v) \times f(w)$.

Then $g(u) > g(v)$ and $g(v) > g(w)$. Thus $g(u) > g(w)$. Since $f(w) > f(u)$ would make (w, u, v) a (f,g)- complex, we must have $f(w) \not> f(u)$. Thus, $(u,w) \in P$.

<u>Case 2</u>: $f(u) > f(v)$ and $f(v) > f(w)$. Then $f(u) > f(w)$ and hence $(u,w) \in P$.

<u>Case 3:</u> $f(u) \times f(v)$ and $f(v) > f(w)$.

Then $g(u) > g(v)$. If $f(u) \times f(w)$ then $(u,w) \notin P$ implies $g(u) \not> g(w)$.

Now (v, w, u) is a f-triad.

Further $g(u) \not> g(w)$ and $g(v) \not> g(u)$ and $g(u) > g(v)$. , Hence, (v, w,

u) is a f-g complex contradicting hypothesis. Thus $f(u) \times f(w)$ implies $(u,w) \in P$.

If $f(w) > f(u)$, then $f(v) > f(u)$, contradicting $f(u) \times f(v)$. Thus $(u, w) \in P$.

Case 4: $f(u) > f(v)$ and $f(v) \times f(w)$. Then $g(v) > g(w)$.
If $f(w) > f(u)$, then $f(w) > f(v)$ contradicting $f(v) \not> f(w)$. Thus $f(w) \not> f(u)$. Suppose $f(w) \times f(u)$. Now $(u,w) \notin P$ implies $g(u) \not> g(w)$. Now (u,v,w) is a f-triad $g(v) > g(w)$ implies $g(w) \not> g(v)$. We also have $g(u) \not> g(w)$. In addition $g(v) > g(w)$.
Thus, (u, v, w) is a f-g complex contradicting our hypothesis. Thus $(u,w) \in P$. Also $f(u) > f(w)$ implies $(u,w) \in P$. Thus, $(u,w) \in P$ in this case too.

Hence, P is transitive. By Theorem 4, there exists $p \in \mathbf{N}$ and $h : U \to \mathbf{N}^p$ such that for all u, v in U, $(u, v) \in P$ if and only if $h(u) > h(v)$.

Since $Y(f,g,X) = \{ y \in X / \nexists\, x \in X \text{ with } (x,y) \in P\}$, we get

$Y(f,g,X) = Y(h,X)\ \forall X \in \Sigma$. Q.E.D.

Corollary 1: If in Theorem 5, m = 1, then $Y(f,g,X) = Y(h,X)\ \forall X \in \Sigma$ with $h : U \to \mathbf{N}^p$ for some $p \in \mathbf{N}$ if and only if there is no f-triad in U with $g(u) \leq g(w) \leq g(v)$ and with one of the inequalities being strict.

Note: Theorem 3, now follows as an easy Corollary of Theorem 5, since if n=1, we cannot have any f-triad and hence no (f,g)-complex.

5. Multi-Stage Vector Optimization and The Main Results:

Let $K \in \mathbf{N}$, $K \geq 2$ *and let* $m_i \in \mathbf{N}$ for i = 1,...,K. For each i ϵ {1,...,K}, let $f_i : U \to \mathbf{N}^{m_i}$ be given . Given $X \in \Sigma$, let

$$Y_1 (X) = \{ x \in X / \nexists\, y \in X \text{ with } f_1 (y) > f_1 (x)\} \cdot$$

Having defined $Y_k (X)$, $1 \leq k < K$, $k \in \mathbf{N}$,

define, $Y_{k+1} (X) = \{x \in Y_k (x) \;/\; \nexists\, y \in Y_k(X) \text{ with } f_{k+1} (y) > f_{k+1} (x)\} \cdot$

We are interested in knowing a set of necessary and sufficient conditions which will ensure the existence of p ϵ $\mathbf{N}$ and $h : U \to \mathbf{N}^p$ such that $Y_K (X) = Y(h, X \; \forall \; X \in \Sigma$ ·

Let u, v, w ϵ U. The ordered triple (u, v, w) will be called a $\{f_i\}_{i=1}^K$ - complex in U if there exists k, r, s ϵ $\mathbf{N}$, K $\geq$ s $\geq$ r > k such that

$f_i(u) \times f_i(v)$, $f_i(u) \times f_k(w)$, $f_i(v) \times f_i(w)$ for $i=1,\ldots,k-1$

$f_k(u) > f_k(v)$, $f_k(u) \times f_k(w)$, $f_k(v) \times f_k(w)$

$f_i(u) \times f_i(v)$, $f_i(u) \times f_i(w)$, $f_i(v) \times f_i(w)$ for $i = k+1,\ldots,r-1$

and either

i) $f_r(w) > f_r(u)$, $f_i(w) \overline{>} f_i(v)$ for $i = r,\ldots,K$

or

ii) $f_r(w) > f_r(u)$, $f_i(w) \overline{>} f_i(v)$ for $i = r,\ldots,s$ and $f_s(v) > f_s(w)$

or

iii) $f_i(u) \overline{>} f_i(w)$, and $f_r(v) > f_r(w)$ for $i = r,\ldots,K$

or

iv) $f_i(u) \overline{>} f_i(w)$, $f_r(v) > f_r(w)$ for $i = r,\ldots,s$ and $f_s(w) > f_s(u)$.

where in the above, i = t, ..., u with u < t implies that the step is automatically satisfied.

Note: If a $\{f_i\}_{i=1}^{K}$ complex does not exist in U, then a $\{f_i\}_{i=1}^{n}$ complex does not exist in U, for $K \geq n \geq 2$.

Note: If a $\{f_i\}_{i=1}^{K}$ complex in U exists, then there does not exist any $p \in \mathbb{N}$ and $h : U \to \mathbb{N}^p$ such that $Y_k(X) = Y(h,X)\ \forall X \in \Sigma$. For if otherwise, then by taking X = {u, v, w} we get Y_k (X) = {u, w} and $w \in Y_k$ (X) and by taking X' = {v,w} we get Y_k (X') = {v, w} and v

$\epsilon\ Y_K\ (X')$. Now if in the above (i) or (ii) holds, then $u \notin Y_K\ (X)$, so that $Y_K\ (X) = \{w\}$ and thus either $h(w) > h(v)$ or $h(u) > h(v)$. If $h(u) > h(v)$ then we must have $h(w) > h(u)$ since $u \notin Y_K\ (X)$. Thus, $h(w) > h(v)$. Further if (i) or (ii) holds $v \in Y_K\ (X')$ and hence $h(w) \not> h(v)$ which is a contradictions. If (iii) or (iv) holds, then $w \notin Y_K\ (X')$ and hence $h(v) > h(w)$. However, $w \in Y_K\ (X)$ and so $h(v) \not> h(w)$ which is a contradiction.

<u>Theorem 6</u>: There exists $p \in \mathbf{N}$ and $h : U \to \mathbf{N}^p$ such that $Y_K\ (X) = Y(h, X)\ \forall\ X \in \Sigma$ if and only if there does not exist any $\{f_i\}_{i=1}^K$ complex in U.

<u>Proof:</u> Necessity has been shown above and sufficiency for k = 2, has been proved in Theorem 5.

Let us assume sufficiency for $K = n \in \mathbf{N}$ and consider now the case where $K = n + 1$. By the induction hypothesis and since no $\{f_i\}_{i=1}^K$ - complex exists in U there exists $q \in \mathbf{N}$ and $g : U \to \mathbf{N}^q$ such that $Y_n(X) = Y(g, X)\ \forall\ X \in \Sigma$. Hence

$Y_{n+1}(X) = Y(f_{n+1},\ Y(g, X)) = Y(g, f_{n+1},\ X)$. . Now a necessary and sufficient condition for $p \in \mathbf{N}$ to exist along with a $h : U \to \mathbf{N}^p$ such that $Y(g, f_{n+1},\ X) = Y(h, X)\ \forall\ X \in \Sigma$ is the non existence of a

(g, f_{n+1}) - complex in U. However, (u, v, w) is a (g, f_{n+1}) complex if and only if

$g(u) > g(v)$, $g(u) \times g(w)$, $g(v) \times g(w)$

$f_{n+1}(u) \bar{>} f_{n+1}(w)$, $f_{n+1}(w) \bar{>} f_{n+1}(v)$ and either

a) $f_{n+1}(w) \times f_{n+1}(u)$

or b) $f_{n+1}(v) \times f_{n+1}(w)$

or c) both (a) and (b).

Now $g(u) > g(v)$ implies that there exists $k \in \mathbf{N}$, $n \geq k$, such that $f_i(u) \times f_i(v)$ for $i = 1,\ldots,k-1$ and $f_k(u) \times f_k(v)$; $g(u) \times g(w)$ implies $f_i(u) \times f_i(w)$ for $i = 1,\ldots,n$; $g(v) \times g(w)$ implies $f_i(v) \times f_i(w)$ for $i = 1,\ldots,n$. But then (u,v,w) is a $\{f_i\}_{i=1}^{n+1}$ - complex in U, which is ruled out by hypothesis. Hence there exists $h : U \to \mathbf{N}^p$ such that $Y(g, f_{n+1}, X) = Y(h, X)\ \forall X \in \Sigma$. Since sufficiency is true for K = 2 and it has been shown to be true for K = n+1 if it is assumed true for K = n, it follows by induction that it is true for all $K \in \mathbf{N}$. Q.E.D.

The situation above lends itself to a direct result if $m_i = 1$ for $i=1,\ldots K$.

<u>Theorem 7:</u> If $m_i = 1$ for $i=1,\ldots K$, then there exists $h : U \to \mathbf{N}$

such that $Y_k(X) = Y(h,X) \ \forall X \in \Sigma$.

Proof: Once again we apply induction and appeal to Theorem 1.

Q.E.D.

Theorem 8: Given a multi-stage vector optimization problem there exists $h : U \to \mathbf{N}$ such that $Y_k(X) = Y(h,X) \ \forall X \in \Sigma$ if and only

i) there does not exist any $\{f_i\}_{i=1}^K$ - complex in U

ii) there does not exist any u, v, w ϵ U and k ϵ $\mathbf{N}$, k $\le$ K such that $f_i(u) \times f_i(w) \times f_i(v)$ for i = 1, . . . ,K, f_i (u) $\times$ f_i (v) for i = 1, . . . ,k-1 and f_k (u) $>$ f_k (v).

Proof: A necessary and sufficient condition for a function $g : U \to \mathbf{N}^p, p \in \mathbf{N}$ to exist so that $Y_k(X) = Y(g,X) \ \forall X \in \Sigma$ is that there does not exist away $\{f_i\}_{i=1}^K$ - complex in U. A necessary and sufficient for a function h : U $\to$ $\mathbf{N}$ to exist such that $Y(g,X) = Y(h,X) \ \forall X \in \Sigma$ is that there is no g-triad in u, which is equivalent to (ii).

Appendix

In this appendix we prove the following result which appears in the main text of the paper.

Theorem:- Let n = 1, m ≥ 1. Then there exists $h : U \to \mathbf{N}$ such that

$Y(f, g X) = Y(h, X) \ \forall X \in \Sigma$ if and only if there does not exist any g-triad (u, v, w) in u with f(u) = f(v) = f(w).

Proof: Suppose (u, v, w) is a g-triad in u with f(u) = f(v) = f(w). Suppose towards a contradiction $Y(f, g X) = Y(h, X) \ \forall X \in \Sigma$ where

$h : U \to \mathbf{N}$.

Let X = {u, v, w}. Thus Y(f, g, X) = {u.w). Hence h(u) = h(w) > h(v). Now let X' = {v, w}. Then Y(f, g, X') = {v, w} and so h(v) = h(w) which contradicts the previous result.

Now suppose there exists no g-triad (u, v, w) in U with f(u) = f(v) = f(w). Given v, ∈ U, say that (u, v) ∈ P ↔ either f(u) > f(v) or f(u) = f(v) and g(u) > g(v). It is easy to check that P is transitive.

Let $U_1 = \{ u \in U \ / \ \nexists \ v \in u \ with \ (v, u) \in P \}$

and having defined U_s, s ∈ $\mathbf{N}$, s ≥ 1, let

$$U_{s+1} = \{u \in U \setminus \bigcup_{i=1}^{s} U_s \ / \ \nexists \ v \in U \setminus \bigcup_{i=1}^{s} U_s \ with \ (v, u) \in P\}$$

Since U is finite, there exists k ∈ $\mathbf{N}$ such that $U = \bigcup_{i=1}^{k} U_i$. Let h(u) = k - s + 1, if $u \in U_s$. If $(u, v) \in P$, then $u \in U_r$ and $v \in U_s$ implies $r < s$. $\therefore$ h(u) > h(v).

Now Suppose, h(u) > h(v).

If $(v,u) \in P$, then h(v) > h(u) which is not possible. Thus, suppose $(v, u) \notin P$ and $(u,v) \notin P$. Let $u \in U_r$ and $v \in U_s$.

Thus $r < s$. By transitivity of P, we may assume that there exists $w \in U_r$ such that $(w,u) \in P$. Clearly, $(w,u) \notin P$ and $(u,w) \notin P$. Thus f(u) = f(v) = f(w), g(w) $\succ$ g(v), g(v) $\times$ g(u), g(w) $\times$ g(u). Thus, (w, v, u) is a g-triad with f(w) = f(v) = f(u), contradicting our hypothesis. Thus, $(u, v) \in P$.

Thus h(u) > h(v) $\leftrightarrow$ either f(u) > f(v)
or f(u) = f(v) and g(u) $\succ$ g(v).

Thus, $Y(f, g\ X) = Y(h, X)\ \forall X \in \Sigma$ where $h : U \to N$ is defined above. Q.E.D.

Note: Since $Y(g, X) = Y(f, g, X)\ \forall X \in \Sigma$, whenever, $f : U \to N$ is a constant function, it follows from above that $Y(g, X) = Y(h, X)\ \forall X \in \Sigma$ where $h : U \to N$ is some function if and only if there is no g-triad in U.

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