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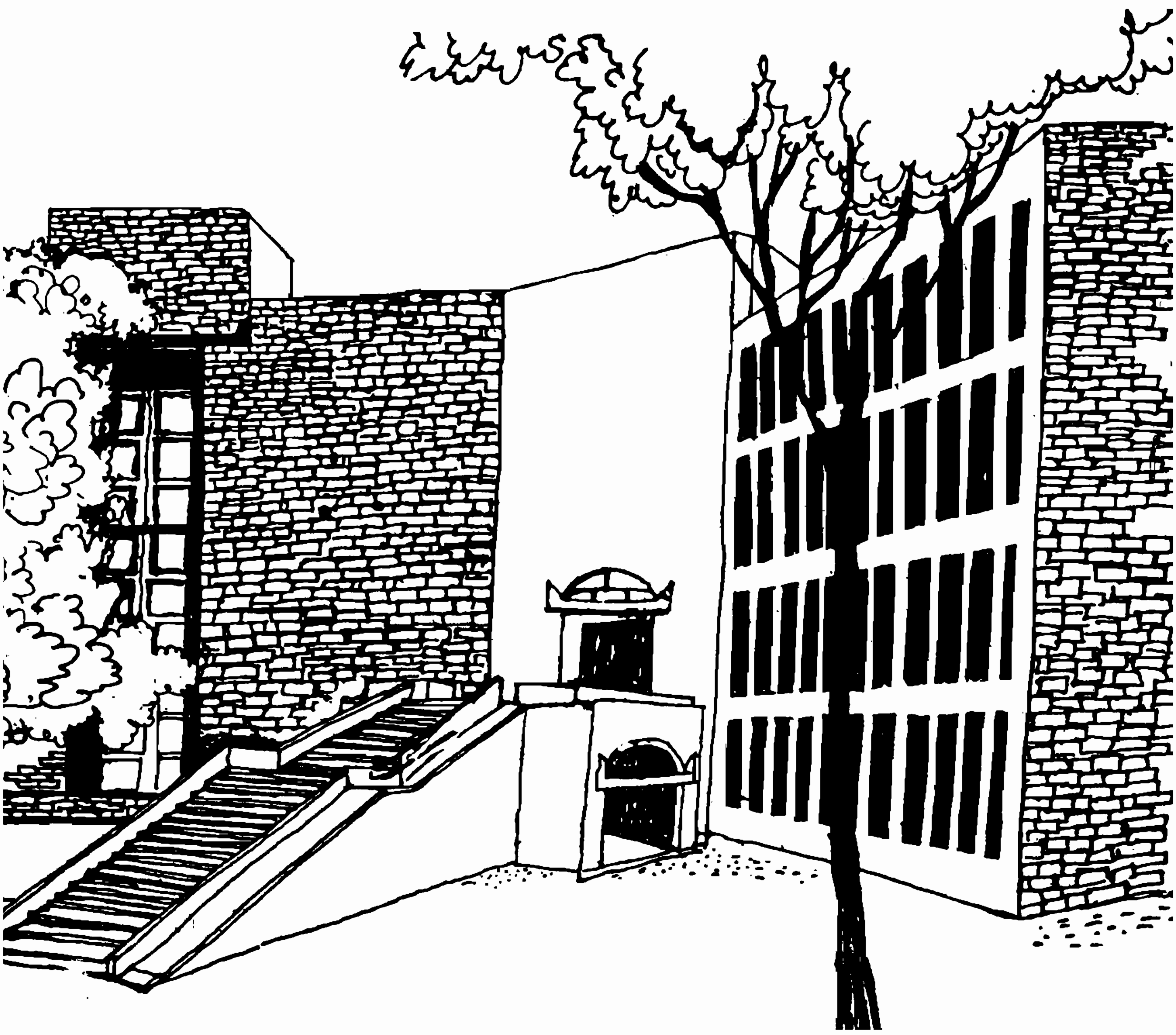

QUASI-UTILITARIAN CHOICE FUNCTION FOR MULTIATTRIBUTE CHOICE PROBLEMS

By

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Abstract

In this paper we propose a generalization of the choice function due to Cao (1981) and axiomatically characterize it. For obvious reasons, we refer to our choice function as the "Quasi-Utilitarian Choice Function".

Introduction :- The theory of multiattribute choice which has its genesis in Keeney and Raiffa (1976), concerns itself with the following type of problem: given a compact, convex subset of a finite dimensional Euclidean space, satisfying some additional regularity conditions, choose a point belonging to the subset which satisfies a given set of properties. The theory of axiomatic bargaining, which began with the path finding work of Nash (1950), is concerned with uniquely characterizing solution procedures to the above type of multiattribute choice problems.

Multiattribute choice problems have received various interpretations over the years. Nash (1950), intended it to be a model of how two bargainers (possibly a buyer and a seller) arrive at a decision over the division of a surplus. This interpretation has been the cornerstone of many related developments and extensions as surveyed for instance in Peters (1993). Keeney and Raiffa (1976) and Yu (1985), have interpreted each attribute to be a measurable criterion and have used multiattribute choice theory as a vehicle for resolving multicriteria decision making problems. Moulin (1988) and Thomson (forthcoming) have found proper expression of a society of individual economic agents, making a concensual choice of a feasible outcome (:in their case a vector of utility levels, one for each individual) in multiattribute choice theory. Abad and Lahiri (1993) use multiattribute choice theory to explain output choices by a regulated firm. These are only some interpretations, among many others, like Chun and Thomson (1990) (:wage negotiation between a trade union and management).

A variety of solutions for multiattribute choice problems have been offered. One notable solution is the utilitarian solution which picks the point in the feasible set of attribute vectors where the sum of the magnitudes is maximized. This solution has been axiomatized by Myerson (1981). However, the utilitarian solution has one shortcoming: it is not scale invariant. To take care of this shortcoming Cao (1981) suggested a modification which has been axiomatically characterized by

Lahiri (1993a). However, the domain considered in Lahiri (1993a) is a small subset of the domain considered in modern multiattribute choice theory as discussed for instance in Lahiri (forthcoming, 1993b).

In this paper we propose a generalization of the choice function due to Cao (1981) and axiomatically characterize it. For obvious reasons, we refer to our choice function as the "Quasi-Utilitarian Choice Function".

The Framework :- Following earlier work done by the author, we consider the following framework:

A multiattribute choice problem is an ordered pair (S,c) where $0 \in S \subseteq \mathbb{R}^n_+$ and $c \in \mathbb{R}^n_+$ for some $n \in \mathbb{N}$ (the set of natural numbers). S is called the feasible set of attribute vectors and c is variously referred to as the target point or claims point. We shall consider the following class $\mathcal{L}$ of admissible choice problems: $(S,c) \in \mathcal{L}$ if and only if

(i) S is nonempty, compact and convex;

(ii) S satisfies <u>minimal transferability</u>: $\forall x \in S$, $\forall i \in \{1,\dots,n\}$, if $x_i > 0$, then there exists $y \in S$ with $y_i < x_i$ and $y_j > x_j \; \forall j \neq i$.

(iii) S is <u>comprehensive</u>: $0 \leqq y \leqq x \in S \Rightarrow y \in S$

(iv) $c \gg 0 \; \forall (S,c)$ with $S \neq \{0\}$.

Properties (i), (ii) and (iii) above resemble those of Moulin (1988).

A domain D is any subset of $\mathcal{L}$.

A multiattribute choice function on D is a function $F: D \to \mathbb{R}^n_+$ such that $F(S,c) \in S \; \forall (S,c) \in D$.

An important domain which forms much of the basis of axiomatic bargaining is $\mathcal{L}_u$:

$$\mathcal{L}_u \equiv \{(S,c) \in \mathcal{L} / c = u(S),\; u_i(S) = \max\{x_i / x \in S\},\; i=1,\dots,n\}.$$

For $(S,c) \in \mathcal{L}$, define $P(S) \equiv \{x \in S / y \in S,\; y \geqq x \Rightarrow y = x\}$. P(S) is called the efficient set of S.

Let $\mathcal{L}^0 \equiv \{(S,c) \in \mathcal{L} / x, y \in P(S) \Rightarrow tx + (1-t)y \notin P(S) \text{ if } t \in (0,1)\}$ i.e. $\mathcal{L}^0$ is the class of those choice problems for which the efficient set of feasible alternatives do not contain a line segment.

On $\mathcal{L}^0$ we define the quasi-utilitarian choice function as follows:

$$Q(S,c) = \underset{x \in S}{\operatorname{argmax}} \left(\Sigma^n_{i=1} x_i / c_i \right) \text{ if } S \neq \{0\}$$

$$= 0 \text{ if } S = \{0\}.$$

The following axioms are essentially all we require for characterizing the quasi-utilitarian choice function:

Let $F: \mathcal{L}^0 \to \mathbb{R}^n_+$ be any choice function.

Axiom 1 :- (Efficiency): $F(S,c) \in P(S)$ $\forall$ $(S,c) \in \mathcal{L}^0$.

Axiom 2 :- (Symmetry): Let $\sigma: \{1,\ldots,n\} \to \{1,\ldots,n\}$ be a permutation (i.e. a one-to-one function). For $x \in \mathbb{R}^n$, let $\sigma(x) = (x_{\sigma(1)},\ldots,x_{\sigma(n)})$ and for $S \subseteq \mathbb{R}^n$ let $\sigma(S) \equiv \{\sigma(x)/x \in S\}$.

If $S = \sigma(S)$, $c = \sigma(c)$ for all permutations σ, then $F_i(S,c) = F_j(S,c)$ $\forall$ $i, j \in \{1,\ldots,n\}$, where $(S,c) \in \mathcal{L}^0$.

Axiom 3:- (Scale Invariance): For $a \in \mathbb{R}^n_{++}$ and $x \in \mathbb{R}^n$, let $ax \equiv (a_1 x_1, \ldots, a_n x_n)$ and for $S \subseteq \mathbb{R}^n$ let $aS \equiv \{ax/x \in S\}$. Then,

$F(aS,ac) = aF(S,c)$ $\forall$ $a \in \mathbb{R}^n_{++}$ and $\forall$ $(S,c) \in \mathcal{L}^0$

Axiom 4:- (Partial Additivity Under Convex Combinations): If $(S,c),(T,c) \in \mathcal{L}^0$ and $\lambda \in [0,1]$ then

$$F(\lambda S + (1-\lambda)T, c) = \lambda F(S,c) + (1-\lambda)F(T,c).$$

Axiom 4 which is a modification of a similar axiom in Myerson (1981) had been tailored in Lahiri (1993a) to apply on $\mathcal{L}_{ur}$. It is easy to check that if $(S,c),(T,c) \in \mathcal{L}^0$, then $(\lambda S + (1-\lambda)T, c) \in \mathcal{L}^0$ $\forall$ $\lambda \in [0,1]$.

3. The Main Theorem:-

Theorem 1:- There is one and only one choice function on $\mathcal{L}^0$

function.

<u>Proof</u>:- That Q satisfies Axioms 1 to 4 is clear. Hence let us prove the converse. Towards that end suppose $F: \mathcal{L}^0 \to \mathbb{R}^n_+$ is a choice function satisfying Axioms 1 to 4 and suppose $F \neq Q$. Let $(S,c) \in \mathcal{L}^0$. If $S=\{0\}$, then $F(S,c)=0=Q(S,c)$. Hence suppose $S \neq \{0\}$. By Axiom 3, we may assume c=e i.e. the vector in $\mathbb{R}^n$ with all co-ordinates equal to one.

$$\text{Let } R(S)=\{p \in \Delta / \max_{x \in S} \Sigma^n_{i=1} p_i x_i > \Sigma^n_{i=1} p_i F_i(S,e)\}$$

where $\Delta = \{p \in \mathbb{R}^n_+ / \Sigma^n_{i=1} p_i = 1\}$. Suppose $\Delta \neq \bigcup_{\substack{(S,c) \in \mathcal{L}^0 \\ S \neq \{0\}}} R(S)$. Then $\exists p^0 \in \Delta$ such

that $\Sigma^n_{i=1} p^0_i F_i(S,e) = \max_{x \in S} \Sigma^n_{i=1} p^0_i x_i \ \forall (S,e) \in \mathcal{L}^0$.

By Axiom 2, we may conclude $p^0_i = \frac{1}{n} \ \forall i \in \{1,...,n\}$ i.e. $F \equiv Q$.

Thus if $F \neq Q$, $\Delta = \bigcup_{\substack{(S,e) \in \mathcal{L}^0 \\ S \neq \{0\}}} R(S)$.

Each R(S) is open in Δ and Δ is compact. Hence $\exists S_1, ..., S_k$ with $(S_k, e) \in \mathcal{L}^0 \ \forall k \in \{1,...,k\}$ such that

$$\Delta = \bigcup^K_{k=1} R(S_k).$$

Let $T = \Sigma^K_{k=1} \frac{1}{k} S_k$ and consider F(T,e). Since by Axiom 1,

$F(T,e) \in P(T)$ by the separating hyperplane theorem $\exists p \in \Delta$, such

that $\Sigma^n_{i=1} p_i F_i(T,e) = \max_{x \in T} \Sigma^n_{i=1} p_i x_i$. But then $\exists k^0 \in \{1,...,k\}$ such that $p \in R(S_{k_0})$ i.e. $\exists x^{k_0} \in S_{k_0}$ such that $\Sigma^n_{i=1} p_i x^{k_0}_i > \Sigma^n_{i=1} p_i F_i(S_{k_0}, e)$.

Consider $y = \frac{1}{k} [x^{k_0} + \Sigma_{k \neq k_0} F(S_k, e)]$ $y \in T$.

$\Sigma^n_{i=1} p_i y_i > \Sigma^n_{i=1} p_i \frac{1}{k} \Sigma^k_{k=1} F_i(S_k, e) = \Sigma^n_{i=1} p_i F_i(T, e)$ by Axiom 4

This contradiction establishes the theorem.

Q.E.D.

4. Conclusion:- Our proposed axiomatic characterization of the quasi-utilitarian choice function is complete.

One possible application of the above choice function is in social choice theory i.e. the weighted utilitarian social choice function. In order to allow for scale invariance we must allow the weights to vary with the choice problem at hand. In particular, the inverse of the weight should change by the same scale by which the utility function of an individual changes. This principle seems to agree with traditional concepts of fairness. It should be noted that Q restricted to $\mathcal{L}_u \cap \mathcal{L}^0$ is the solution suggested in Cao (1981).

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