Competitive hub location problems : Model and solution approaches

Richa Tiwari
Sachin Jayaswal
Ankur Sinha

W.P. No. WP 2019-12-02
December 2019

**INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD – 380015
INDIA**

Competitive hub location problems : Model and solution approaches

Richa Tiwari
Sachin Jayaswal
Ankur Sinha

1]Richa Tiwari cor1 [cor1]Corresponding Authorrichat@iima.ac.in 1]Sachin Jayaswal 1]Ankur Sinha [1]Indian Institute of Management, Ahmedabad, India

Abstract

In this paper, we study the hub location problem of an entrant airline that tries to maximize its market share, in a market with already existing competing players. The routes open for use can be either of multiple allocation or single allocation type. The entrant's problem is modelled as a non-linear integer program in both the situations, which is intractable for off-the-shelf commercial solvers, like CPLEX and Gurobi, etc. Hence, we propose four alternate approaches to solve the problem. The first is based on a mixed integer second order conic program reformulation, while the second uses lifted polymatroid cuts based approximation of second order cone constraints. The third is the second order conic program within Lagrangian relaxation, while the fourth uses approximated lifted polymatroid cuts within lagrangian relaxation. The four methods performs differently for the single allocation and multiple allocation models, and second approach is the best for single allocation model and for smaller instances in multiple allocation model. As the problem size in multiple allocation model increases, the third method starts to be the better performer in terms of computation time.

`Hub and Spoke Networks, Competition, Non-Linear Program, Exact Solution Methods`

1 Introduction

This paper discusses hub location problems in the presence of competition. A new airline, called the entrant, plans to enter a market where competitor airlines are already meeting the passenger flow demand. The entrant wants to maximize its market share in the said market, wherein the customers' airline choice depends on their utility (instead of just cost), given the competitor's best response of the hub and spoke network design. To the best of our knowledge, there is only one other related study in literature in this area, by Eiselt and Marianov (2009). The shortcoming of Eiselt and Marianov (2009)'s market share model is the assumption that the customer choice depends just on the location of the hubs and not on the path to be chosen to fly by any airline. In such a situation, the airlines will end up flying their aircraft through multiple routes for a particular origin destination (O-D) pair, which doesn't make economic sense. This also suggests that every city is connected to all the open hubs in the network, which defeats the entire purpose of having a hub and spoke network as there are relatively fewer economies of scale due to higher operational and setup costs in the model.

The suggested limitations are addressed in the current paper by designing a more realistic hub and spoke network, which allows for just one route to be operational between every O-D pair by any airline.

The route allowed can be operational in either a single or a multiple allocation setting, resulting in two different formulations. Both these formulations are NP hard and more difficult to solve than the formulation proposed by Eiselt and Marianov (2009), which is even solved using genetic algorithm. The raised difficulty in our formulations arise from the increase in the number of binary routing variables and effect in non-linear integer programs, which are very challenging to solve using off-the-shelf solvers like CPLEX, Gurobi, etc. We address this challenge by providing four alternate approaches to solve the problem. The first approach relies on a mixed integer second order conic program (MISOCP) based reformulation of the problem. The second approach is based on outer approximations of the aforementioned second order cone constraints, generated using lifted polymatroid cuts in an iterative procedure (LP-MISOCP). The third approach employs a Lagrangian relaxation of the mixed integer second order conic program (LR-MISOCP), while the fourth approach uses a Lagrangian relaxation of the lifted polymatroid cuts based method (LRLP-MISOCP). Further, we compare the above four solution approaches based on extensive computational experiments. Our analysis highlights the superiority of the LP-MISOCP approach for all the test networks. With the best approach, we are able to solve problem instances that are otherwise intractable in computational time of less than 1.6 hours. To summarise, the major contributions of this paper are as follows:

1. We propose two different formulations to model a competitive hub location problem in order to rectify the limitations of Eiselt and Marianov (2009)'s model.
2. Also, we present four alternate approaches for each of the formulations to solve the problem that are based on second order cones, lifted polymatroid cuts, lagrangian relaxation of second order cones and lifted polymatroid cuts.
3. Additionally, we compare the mentioned solution approaches based on extensive experiments for their computational performance.

The rest of the paper is organized as follows. In Section 2, we study the problem description, followed by its mathematical formulation. We present our alternate solution approaches in Section 3, followed by extensive computational results in Section 4. Finally, the conclusions and directions for future research are presented in Section 5.

2 Literature Review

We first present in Section 2 the literature on HLP and its variants in the monopolistic setting, focusing mainly on single or multiple allocation and then we review the literature in the competitive setting.

2.1 Hub Location Literature

Designing a hub and spoke network requires solving a hub location problem, for the optimal location of hubs and the routes for any passenger travelling between any O-D pair. O'kelly (1986) studies he first ever hub location problem in literature, with the objective to minimize the transportation cost of the network. However, O'kelly (1987) proposes the first mathematical formulation as an optimisation problem, which is a quadratic p median single allocation problem. Later, Skorin-Kapov et al. (1996) provide a linear reformulation of the model and Ernst and Krishnamoorthy (1996) model the problem as a multi-commodity flow problem. Any formulation is classified as single allocation when the immediate hub visited by any flow originating from (or destined to) a given node is the same irrespective of their destinations (or origins). The single allocation p-hub median problems are proven to be NP-hard in a seminal work by Kara and Tansel (2000). There is an extensive literature on multiple allocation p-hub median problems as well. Campbell (1992) presented the first multiple allocation hub location model which was quadratic in nature. Skorin-Kapov et al. (1996) extended this work and developed a linear model for the problem.

Although hub location problem (HLP) with median objective constitutes the main focus of the literature, other objectives like p center, covering or fixed charge are also investigated by researchers. The objective for a p center HLP is to minimize the maximum transportation cost between any pair of nodes (Campbell, 1994; Kara and Tansel, 2000) for locating p hubs. A covering HLP can either be hub set-covering if the objective is to minimize the number of hubs to cover the entire demand (Campbell, 1994; Kara and Tansel, 2000), or maximal hub-covering if the objective is to maximize the demand covered with a given number of hubs to locate (Campbell, 1994). An interested reader may refer to surveys by Campbell and O'Kelly (2012); Campbell et al. (2002); Alumur and Kara (2008); Kara and Taner (2011) for a detailed discussion of hub location problems.

2.2 Hub Location with competition

Although the literature on competition in location science has been studied in detail, competitive hub location studies in literature are rare, despite the prevalent application in transportation logistics, energy and telecommunication networks. Hub location studies in a competitive setting are either in the form of cooperative games or non-cooperative games. Cooperative games focus on the collective payoffs from coalitions and joint actions amongst firms. Lin and Lee (2010) is one such study, which focuses on cooperative game in freight services in an oligopolistic market.

On the other hand, non-cooperative games focus on payoffs from individual firms' actions. In the hub location literature, these games can be divided into two classes; the one where modelling competition doesn't take into account the response of the competitors while deciding the the hub locations for a player. This kind of situation generally arises when an entering firm has to make a strategic decision, say locating facilities, and it is difficult for incumbent firms to change their already taken strategic decisions as it might be very expensive. These kinds of studies are classified as static in facility location literature Plastria (2001) and we are extending the classification to hub location literature. Marianov et al. (1999) belongs to this category, and to the best of our knowledge, is also the first to introduce competition in the area of hub location. The problem models the decision of an entrant, to locate a set of hubs so as to maximize the demand flow captured from its competitors. They included proportional capture levels in addition to all-or-nothing type in their model. The resulting mixed integer linear program (MILP) was solved using a tabu search heuristic. Wagner (2008) highlights certain shortcomings of the model proposed by Marianov et al. (1999), and describes a new capture set where the follower gets nothing in case of same service levels as the leader . Lüer-Villagra and Marianov (2013) considered hub location as well as pricing decisions of an entrant firm where another firm has already been operating on the market, thereby solving the resulting model using genetic algorithm. Eiselt and Marianov (2009) also studies the problem of an entrant airline that wants to maximize its market share, wherein the customers' choice of an airline depends on their utility (instead of just cost). The problem is formulated as a non-linear integer program, which is again solved using genetic algorithm.

As opposed to the above cited papers, the second class of papers take into account the competitor's response while solving the HLP and are referred to as dynamic. These papers model the problem of

Table 1: Summary of Literature

Reference	Collaboration	Competition	Static	Dynamic	Exact Method
Lin and Lee (2010)	✓				
Sasaki and Fukushima (2001)		✓		✓	
Sasaki (2005)		✓		✓	
Sasaki et al. (2014)		✓		✓	
Mahmutogullari and Kara (2016)		✓		✓	✓
Marianov et al. (1999)		✓	✓		
Wagner (2008)*		✓	✓		✓
Eiselt and Marianov (2009)		✓	✓		
Lüer-Villagra and Marianov (2013)		✓	✓		
This work		✓	✓		✓

* They studied the total traffic captured by the entrant, which being linear is much easier and tractable than our formulations

the leader firm as a Stackelberg game. Sasaki and Fukushima (2001) studies the perspective of a leader who competes with several existing firms to maximize his/her profit. The problem is modelled as a bilevel Stackelberg game, and solved using sequential quadratic programming. Sasaki (2005) extends the problem from a continuous network to a discrete one, which is solved using complete enumeration and greedy heuristics. Sasaki et al. (2014) studies the problem of a leader who tries to locate hub arcs, as opposed to locating hub nodes, to maximize revenue. The resulting bilevel program is solved using implicit enumeration. Mahmutogullari and Kara (2016) studies a duopoly model in a Stackelberg framework, where two competitors sequentially choose their respective hub locations with the aim to maximize their captured flow. The problem is formulated as a bilevel HLP, and solved using implicit enumeration of the leader's problem.

The work in this paper falls under the category of static competitive hub location. As evident from the review above, the literature in this area is scarce. Also the assumption in Eiselt and Marianov (2009)'s market share model is questionable as explained in 1. Further, the extant studies in the area have resorted to heuristic approaches. Specifically, we study the problem of market share maximization by the entrant firm in a competitive airline industry. The market share of the entrant is modeled as a probabilistic function of its routing decisions, which introduces non-linearity in the problem. The resulting mathematical program is a non-linear IP, for which we propose alternate solution approaches. Table 1 summarizes studies in the competitive hub location literature where the last row corresponds to this paper.

3 Problem Description and Model Formulation

Air passenger traffic in a market is served by an existing company (or a set of companies, collectively), called the incumbent (or incumbents), that utilizes a hub and spoke network. Like all fundamental hub location models, our model also assumes reduced transportation costs due to economies of scale in inter-hub, collection and distribution traffic, and that the discount factors are constant. We also assume that the incumbent's hubs are fully connected and are located optimally for cost minimization while serving all the demand, though the incumbent may end up serving less than that after a new company, referred to as the entrant arrives. The entrant intends to enter the same market, using its own hub and spoke network and tries to maximise its market share, giving up some part of profit in the beginning. We consider this objective based on the argument given by Rumelt and Wensley (1981), that it is necessary to look at long term profits, even though there is a cost of capturing market share. Both the incumbent and the entrant may offer the same route between an O-D pair, one among which is patronized by the customers on the basis of price as well as other attributes as travel time and attractiveness measure of the competing companies. Customers' decision based on the utility derived, is modeled using a gravity model. The gravity model is used very often to assist in the forecasting of future consumer behavioral patterns based on hypothesised parameters, which is well validated in the transportation literature (de Dios Ortuzar and Willumsen, 2011). All competitors in the market have same information of the demand structure. Airfares are proportional to the costs incurred by the airlines.

The problem is described over a hub and spoke network, represented by a complete graph $G = (N, A)$, in which incumbent airlines represented by the superscript c are competing to capture maximum demand. Suppose that there exists inelastic demand (f_{ij}) between every origin node $i \in N$ and destination node $j \in N$, which is routed through one or at most two of the hubs from the set of hubs H. χ, α and δ are the discount factors due to flow consolidation in collection (origin to hub), transfer (between hubs)and distribution (hub to destination), respectively. The demand is served by either the incumbent c or the entrant e. The utility of a single passenger represents preferences concerning attractiveness, cost and time based on the gravity model. Model parameters are drawn from Huff (1964, 1966), and the utility (u^e_{ijkl}) that the customers derive from the entrant airline e is defined as:

$$u^e_{ijkl} = A^e_{kl}/\Big(\gamma\big(T^e_{ijkl}\big)^{\beta} + (1-\gamma)\big(B^e_{ijkl}\big)^{\delta}\Big) \tag{1}$$

where A^e_{kl} is the basic attractiveness index of a pair of hubs (k,l) used for the trip, B^e_{ijkl} and T^e_{ijkl} denotes the the cost and the total time required by the flight respectively for traveling along the route $(i \to k \to l \to j)$, parameters β and δ denote the attraction decay of travel time and cost, respectively. The customers' utility from choosing a competing airline c, u^c_{ijkl}, can be similarly computed. In the model, higher values of γ means that customers are very sensitive to travel time and they will mostly choose less time consuming routes. Smaller values of γ means that the customers are more sensitive to price differences, and there will be a higher customers' spread among the different routes. For further details on this models, please refer to de Dios Ortuzar and Willumsen (2011); Huff (1964, 1966). Finally, H^c is the set of nodes where the incumbent's hubs are located.

This gravity based utility function can then be introduced into the probabilistic customer choice model to calculate entrant e's market share for a particular O-D pair as :

$$\rho^e_{ij} = \sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl}/\Big(\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl} + \sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}\Big). \tag{2}$$

The total capture of passengers using the route $(i \to k \to l \to j)$ offered by the entrant is denoted by $f_{ij}\rho^e_{ij}$. The objective of the airline is to capture as large a market share as possible, by locating a fixed p number of hubs. In Eiselt and Marianov (2009)'s model, there is a exists a possibility of one city being connected to multiple or all hubs in the optimal network. Hence, there can be multiple operational paths between any $i - j$ pair for a single airline as well in the network. This defeats the entire purpose of setting up a hub spoke network as it doesn't make economic sense for an airline to operate multiple routes for an O-D pair. To allow for a more realistic model, we assume that each competitor can have only a single route between any $i - j$ pair, passing through different hubs or pairs of hubs.

In order to allow only one operational path, the entrant airline's problem can be modelled in two different ways; one based on single allocation and the other based on multiple allocation. In single allocation model; a non-hub node can be connected to a single hub node. However, in multiple allocation hub network, a non-hub node can be connected to multiple hub nodes. Both single and multiple allocation hub location problems are proved to be NP-hard (Kara and Tansel, 2000). The problem at hand is even more complex owing to the non-linearity arising in the objective function based on utility on top of either single allocation or multiple allocation. Next, we present the mathematical model describing the entrant airline's problem with single allocation in (3.1) followed by multiple allocation in (3.2).

3.1 Single Allocation Model

To define the mathematical model, we define the following decision variables:

$$z_{ik} = \begin{cases} 1 & \text{if non-hub node located at } i \text{ is allocated to hub at node } k \text{ for the entrant,} \\ 0 & \text{otherwise .} \end{cases} \tag{3}$$

$$x_{ijkl} = \begin{cases} 1 & \text{if path } i \to k \to l \to j \text{ exists for the entrant ,} \\ 0 & \text{otherwise ,} \end{cases} \tag{4}$$

Considering the decision variables and the given parameters, the integer non-linear programming for $SACOHLP$, based on (Skorin-Kapov et al., 1996), can be formulated as follows:

$[SACOHLP]$:

$$\theta(x,z) = \max \sum_i \sum_j f_{ij} \left\{ \frac{\sum_{k \in H^e} \sum_{l \in H^e} u^e_{ijkl} x_{ijkl}}{\sum_{k \in H^e} \sum_{l \in H^e} u^e_{ijkl} x_{ijkl} + \sum_{k \in H^c} \sum_{l \in H^c} u^c_{ijkl} x^c_{ijkl}} \right\} \tag{5}$$

$$\text{s.t.} \quad \sum_{k \in H^e} z_{ik} = 1 \qquad \forall\, i \in N \tag{6}$$

$$z_{ik} \leqslant z_{kk} \qquad \forall\, i \in N, k \in H^e \tag{7}$$

$$\sum_k z_{kk} = p \tag{8}$$

$$\sum_{l \in H^e} x_{ijkl} = z_{ik} \qquad \forall\, i, j, k \in H^e \tag{9}$$

$$\sum_{l \in H^e} x_{ijlk} = z_{jk} \qquad \forall\, i, j, k \in H^e \tag{10}$$

$$z_{ik} \in \{0,1\} \qquad \forall\, i, k \tag{11}$$

$$x_{ijkl} \in \{0,1\} \qquad \forall\, i, j, k, l \tag{12}$$

In the above model, x^c_{ijkl} is a parameter derived by solving the p-median hub location problems for the competitors. It is 1, if path $i \to k \to l \to j$ exists for the competitor, 0 otherwise. The objective function (5) maximizes the total demand captured by the entering airline, given the competitors' hub locations and operational paths. Constraint set (6) assures that all nodes are assigned to a hub, while (7) requires that if a node i is assigned to a hub at k, then a hub must be opened there. Constraint sets (9) and (10) are the only linking constraints between x and z variables. Constraint sets (9) and (10) guarantees that if any travel from i to j goes through hub $k[l]$, then node $i[j]$ must be assigned to a hub $k[l]$. Constraint (8) enforces p hubs to be open. Constraint sets (12) and (11) are the binary constraints on x and z variables, respectively. SACOHLP is a non-linear IP, which off-the-shelf solvers like CPLEX and Gurobi cannot handle (since they cannot handle non-linear problems that are non-quadratic).

Please note that SACOHLP can be transformed into a non-linear mixed integer program by relaxing the binary variables x_{ijkl} to take continuous values in $[0,1]$ using the following argument: whenever both z_{ik} and z_{jl}, which are binary, take a value 1, x_{ijkl} will always assume a value of 1 (as objective function (5) is an increasing function in x_{ijkl}). On the other hand, when one of z_{ik} or z_{jl} is 0, x_{ijkl} takes a value 0. Therefore, (12) can be relaxed as:

$$x_{ijkl} \in [0,1] \tag{13}$$

3.2 Multiple Allocation Model

To define the mathematical model, we define the following decision variables:

$$z_k = \begin{cases} 1 & \text{if } k \text{ node is set up as hub for the entrant,} \\ 0 & \text{otherwise .} \end{cases} \tag{14}$$

$$x_{ijkl} = \begin{cases} 1 & \text{if path } i \to k \to l \to j \text{ exists for the entrant ,} \\ 0 & \text{otherwise ,} \end{cases} \tag{15}$$

Considering various variables and parameters, the integer non-linear programming for $MACOHLP$, based on (Hamacher et al., 2004) can be formulated as follows :

$[MACOHLP]$:

$$\theta(x,z) = \max \sum_i \sum_j f_{ij} \left\{ \frac{\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl} x_{ijkl}}{\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl} x_{ijkl} + \sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl}} \right\} \tag{16}$$

$$\text{s.t.} \quad \sum_k z_k = p \tag{17}$$

$$\sum_{k\in H^e}\sum_{l\in H^e} x_{ijkl} = 1 \qquad \forall\, i,j \in N \tag{18}$$

$$\sum_{l\in H^e} x_{ijkl} + \sum_{l\in H^e - k} x_{ijlk} \leqslant z_k \qquad \forall\, i,j,l \in H^e \tag{19}$$

$$z_k \in \{0,1\} \qquad \forall\, i,k \tag{20}$$

$$x_{ijkl} \in \{0,1\} \qquad \forall\, i,j,k,l \tag{21}$$

In the above model, $z_k = 1$, if hub is located at node k, 0 otherwise. The objective function (16) maximizes the total demand captured by the entering airline, given the competitors' hub locations and operational paths. Constraint (17) enforces p hubs to be open. Constraint set (18) states that every i to j pair of demand has to be routed via one or two hub nodes k and l. Constraint set (19) is the only linking constraints between x and z variables and ensures that flow is only sent via open hub. Constraint sets (20) and (21) are the binary constraints on x and z variables, respectively. MACOHLP, like SACOHLP is a non-linear IP, which off-the-shelf solvers like CPLEX and Gurobi cannot handle. Also, similar to SACOHLP, (21) can be relaxed as:

$$x_{ijkl} \in [0,1] \tag{22}$$

However, the resulting non-linear mixed integer programs are still difficult for off-the-shelf solvers. Next, we discuss alternate solution methods to solve the problem efficiently, exploiting (22) wherever possible.

4 Solution Methods

In this section, we propose four novel solution approaches, which solve both $SACOHLP$ and $MACOHLP$. The first approach is based on the reformulation into a mixed integer second order conic program (MISOCP), which can be solved efficiently using an off-the-shelf solver (Alizadeh and Goldfarb, 2003). The second approach is based on approximating the second order cone constraints using lifted polymatroid cuts in an iterative procedure (Sen et al., 2017). The third approach is based on Lagrangian relaxation, which separates the resulting problem into an linear integer program and a non-linear program (NLP) with continuous variables. The fourth approach is used for solving the resulting non-linear program (NLP) with continuous variables using lifted polymatroid cuts. We discuss these four approaches in the upcoming subsections.

4.1 Mixed Integer Second Order Conic Program (MISOCP)

Second order conic programs (SOCPs) are of particular interest as they can be solved efficiently with widely available commercial solvers, like CPLEX and Gurobi. Hence, SOCPs have recently been employed to a variety of problems, like portfolio optimization, value-at-risk minimization, machine scheduling, supply chain network design and airline rescheduling with speed control (Vielma et al. (2008); Aktürk et al. (2014); Antoniou and Lu (2007)). Mixed integer second order conic programs (MISOCPs) have also been studied in a variety of problems like hub location problems with congestion and assortment problems (Sen et al., 2017) among many others. SACOHLP, as discussed in Section 3.1, is a non-linear IP, which cannot be solved using off-the-self solvers. Therefore, in this section, we reformulate SACOHLP as a mixed integer second order conic program (MISOCP). To convert the SACOHLP into an MISOCP, we introduce the following sets of variables:

$$a_{ij} = 1/\Big(\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl}x_{ijkl} + \sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}x^c_{ijkl}\Big) \tag{23}$$

$$g_{ij} = \sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl}x_{ijkl} + \sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}x^c_{ijkl} \tag{24}$$

$$o_{ij} = a_{ij}\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl}x_{ijkl} \tag{25}$$

The above transformations result in the following MISOCP based reformulation of COHLP:

$$[SACOHLP_{MISOCP}]: \tag{26}$$

$$\max \quad \sum_i\sum_j \hat{f}_i - \sum_i\Big\{\sum_j\sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}x^c_{ijkl}\hat{f}_i a_{ij} + \sum_j(\hat{f}_i - f_{ij})o_{ij}\Big\}$$
$$\text{s.t. } (6)-(11),(13),(24),(53)-(54)$$

Likewise, for the multi allocation version the program can be formulated as follows:

$$[MACOHLP_{MISOCP}]: \tag{27}$$

$$\max \quad \sum_i\sum_j \hat{f}_i - \sum_i\Big\{\sum_j\sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}x^c_{ijkl}\hat{f}_i a_{ij} + \sum_j(\hat{f}_i - f_{ij})o_{ij}\Big\}$$
$$\text{s.t. } (17)-(19),(22),(24),(53)-(54)$$

Both $SACOHLP_{MISOCP}$ and $MACOHLP_{MISOCP}$ have $3N^2$ additional variables and $5N^2$ additional constraints, out of which $2N^2$ are SOC constraints. The programs can be solved directly by CPLEX using either of the two parameter settings; *miqcpstrat 1* and *miqcpstrat 2*. In *miqcpstrat 1*, it uses an SOCP based branch-and-bound algorithm, wherein at each node, the continuous relaxation is solved using an interior point algorithm specifically designed for SOCPs. In *miqcpstrat 2*, CPLEX uses outer approximation of the MISOCP, which produces an LP at each node of the branch-and-bound tree. In our numerical experiments, reported in Section 5, we use the default setting of CPLEX, which is *miqcpstrat 0* to allow CPLEX to choose the best strategy, depending on the problem structure.

4.2 MISOCP with lifted polymatroid cuts

Commercial software packages utilize a branch-and-bound algorithm for solving MISOCPs, and their performance can be significantly improved by strengthening the formulations using structural cutting planes. Atamtürk and Narayanan (2008) have pioneered the polymatroid cuts to strengthen the convex relaxation of sub-modular cone constraints. Although there are exponentially many extremal polymatroid inequalities, only a small subset of them are needed in the branch-and-bound search tree. It turns out that, given a solution, finding a violated polymatroid cut can be done easily using a separation problem for the extended polymatroid inequalities. Thus, the separation problem (formulated from problem structure) is an optimization over an extended polymatroid, which is solved by the greedy algorithm of Edmonds and Fulkerson (1970) for sub-modular functions. Solving the separation problem usually is not easy and at times is heuristically done, which is the case for second order cone constraints with only continuous variables. Also, it is important to note that in our formulation we have two different kinds of SOC constraints, Class (I) being equation (53) and the Class (II) being (54).

4.2.1 Single Allocation

We consider an equivalent optimization problem, with very small values of ϵ_1 and ϵ_2 to begin with:

$[RSACOHLP]:$

$$\max \sum_i \sum_j \quad \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k \in H^c} \sum_{l \in H^c} u^c_{ijkl} x^c_{ijkl} \hat{f}_i a_{ij} + \sum_j (\hat{f}_i - f_{ij}) o_{ij} \right\} - \sum_i \sum_j \left\{ \zeta_1 o'_{ij} + \zeta_2 g'_{ij} \right\}$$

$$\text{s.t.} \quad (6) - (11), (13), (24), (52), (53) - (54)$$

$$o'_{ij} - \frac{o_{ij}}{\sum_{k,l \in H^e} u^e_{ijkl}} \geqslant 0 \tag{28}$$

$$g'_{ij} - \frac{g_{ij}}{\sum_{k,l \in H^e} u^e_{ijkl} + \sum_{k,l \in H^c} u^c_{ijkl}} \geqslant 0 \tag{29}$$

$$o'_{ij}, g'_{ij} \in \{0, 1\} \tag{30}$$

Since $a_{ij} \leqslant 1$, hence $o_{ij} \leqslant \sum_{k,l \in H^e} u^e_{ijkl}$. This implies that $o_{ij} / \sum_{k,l \in H^e} u^e_{ijkl} \leqslant 1$. Also $g_{ij} \leqslant \sum_{k,l \in H^e} u^e_{ijkl} + \sum_{k,l \in H^c} u^c_{ijkl}$, since $x_{ijkl} \leqslant 1$ and hence $g_{ij} / \sum_{k,l \in H^e} u^e_{ijkl} + \sum_{k,l \in H^c} u^c_{ijkl} \leqslant 1$.

The complete algorithm to solve the problem, begins by removing all the SOC constraints from the formulation and iteratively adds violated cuts to the algorithm. The UB to the problem is found using a separation algorithm (1). The LB to the problem is found using proposition 31. The process is repeated upto an optimality gap of 1%.

Proposition 1. *For any given x^q_{ijkl}, (31) provides a lower bound to the optimal objective function value of $SACOHLP$*

$$LB^q = \theta(x^q_{ijkl}) = \sum_{i,j\in N} f_{ij} \frac{\sum_{k,l\in H^e} x_{ijkl} u^c_{ijkl}}{\sum_{k,l\in H^e} x_{ijkl} u_{ijkl} + \sum_{k,l\in H^c} x^c_{ijkl} u^c_{ijkl}} \tag{31}$$

Proof. If (x^q_{ijkl}) is a solution to the relaxed MISOCP: $RSACOHLP_{MISOCP}$, then it is a feasible solution to $SACOHLP$ (since it satisfies constraints (6)-(10). Hence, the objective function of $SACOHLP$ evaluated at (x^q_{ijkl}), given by (31), provides a lower bound on the optimal objective of COHLP. □

4.2.2 Multiple Allocation

Likewise for multiple allocation (since the SOC constraints are of similar format), the complete algorithm to solve the problem, begins by removing all the SOC constraints from the formulation and iteratively adds violated cuts to the algorithm. The UB to the problem is found using a separation algorithm (1). The LB to the problem is found using proposition 37. The process is repeated up to an optimality gap of 1%.

Algorithm 1 Separation Problem

if $\epsilon > .01$ **then**
  Given X^*, X'^*, o^*, g^*, $S^d_m = \varnothing$; sort X_m in non-increasing order such that $X_{(1)} \geqslant X_{(2)} \geqslant \ldots$.
  For $m = 1, 2, 3 \ldots\ldots\ldots |M|$, let $S^d_m = \{(1), (2), (3) \ldots. (m)\}$ and $\pi_m = \sqrt{\sum_{k\in S^d_m} a_k - \sum_{k\in S^d_{m-1}} a_k}$
  if $\pi X' \geqslant o + g + \alpha' \Big\{X' - X\Big\}$ **then**
    Add the polymatroid cut (55)
  else
    For $m = 1, 2, 3 \ldots. |M|$, $S^a_m = N$ let $S^a_m = \{(m), (m-1) \ldots. (2), (1)\}$
    if $\Big(\pi\hat{X}' - \alpha_S(X' - X)\Big)^2 + \sum_{m\in N\setminus S^a_m} a_m X^2_m > (o + g)^2$ **then**
      Add the polymatroid cut (56)
    else
      For given S^a removing largest m from S^a
      if $\pi_S \hat{X}' + \sqrt{\sum_{m\in T} a_m \hat{X}^2} > \alpha_S(X'_S - X_S)$ **then**
        Add polymatroid cut (60)
      end if
    end if
  end if
  Similarly run iterations for $a_1/X_{(1)}, a_2/X_{(2)} \ldots$ permutation.
  Similarly run iterations for $a_1 X_{(1)}, a_2 X_{(2)} \ldots$ permutation.
end if

4.3 Lagrangian Relaxation with SOCP

Lagrangian relaxation (LR) is a popular technique that has been used to solve a wide variety of integer/ mixed integer linear and non-linear programs (NLP) (Narula et al., 1977; Mirchandani et al., 1985; Aykin, 1994; Pirkul and Schilling, 1998, 1991).

4.3.1 Single Allocation

In this section, we apply LR to SACOHLP. The challenge with LR method is to correctly identify the constraints to be relaxed. For this particular work, we relax (6), (9) and (10) using ϵ_i, λ_{ijk} and μ_{ijk} as their respective Lagrange multipliers, which produces the following Lagrangian sub-problem:

$[SACOHLP_{SUB}]$:

$$\theta_{SUB}(\lambda,\mu) = \max \quad \sum_i\sum_j f_{ij}\frac{\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl}x_{ijkl}}{\sum_{k\in H^e}\sum_{l\in H^e} u^e_{ijkl}x_{ijkl} + \sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}x^c_{ijkl}} + \sum_i\sum_j\sum_{k\in H^e}\lambda_{ijk}(z_{ik} - \sum_{l\in H^e} x_{ijkl})$$

$$+ \sum_i\sum_j\sum_{l\in H^e}\mu_{ijk}(z_{jk} - \sum_{l\in H^e} x_{ijlk}) + \sum_i \epsilon_i(1 - \sum_k z_{ik}) \tag{32}$$

$$\text{s.t. } (7)-(8),(11)-(12)$$

$$\lambda_{ijk},\mu_{ijk},\epsilon_i \in [-\infty,\infty] \tag{33}$$

For a given set of (λ,μ,ϵ), the Lagrangian sub-problem provides an upper bound (UB) to $SACOHLP$. The tightest (smallest) UB is obtained by solving the following Lagrangian dual problem:

$$\min_{\lambda,\ \mu,\ \epsilon} \theta_{SUB}(\lambda,\mu) \tag{34}$$

(34) is non-linear optimization problem, which is popularly solved using the sub-gradient algorithm (Held et al., 1974; Fisher, 1981) as elaborated in Algorithm (4). A feasible solution to SACOHLP can be obtained by turning the (infeasible) solution obtained from Lagrangian sub-problem into a feasible solution, which provides a lower bound (LB) to SACOHLP. The best feasible solution (maximum of the known lower bounds) is reported, and the relative optimality gap is calculated at every iteration as 1-(LB/UB), which is used as a termination criterion for the sub-gradient algorithm.

For a given set of Lagrange multipliers (λ,μ,ϵ), the sub-problem (34) decomposes into the following two independent sub-problems, with one involving only z variables, while the other involving x variables.

$[SUB_1]$:

$$\theta_{SUB1} = \max \quad \sum_i\sum_k\sum_j(\lambda_{ijk} + \mu_{jik} - \epsilon_i)z_{ik} + \sum_i \epsilon_i \tag{35}$$

$$\text{s.t.} \quad \sum_{k\in H^e} z_{kk} = p$$

$$z_{ik} \leqslant z_{kk} \qquad \forall\, i,k$$

$$z_{ik} \in \{0,1\} \qquad \forall\, k\in H^e$$

$[SUB_2]$:

$$\theta_{SUB2} = \max \quad \sum_i\sum_j f_{ij}\frac{\sum_{k\in H^e}\sum_{l\in H^e} x_{ijkl}u^e_{ijkl}}{\sum_{k\in H^e}\sum_{l\in H^e} x_{ijkl}u^e_{ijkl} + \sum_{k\in H^c}\sum_{l\in H^c} u^c_{ijkl}x^c_{ijkl}} - \sum_i\sum_j\sum_k\sum_l\left\{\lambda_{ijk}x_{ijkl} + \nu_{ijl}x_{ijkl}\right\} \tag{36}$$

$$\text{s.t. } x_{ijkl} \in [0,1]$$

SUB_1 can be solved optimally for a given set of (λ,μ,ϵ) by locating p hubs with the maximum contribution to objective function (41). The steps of the method to solve SUB_1 are summarized in Algorithm (2)

Algorithm 2 Optimal Solution for SUB_1

Set $z_{kk} = 1 \; \forall \; k$
if $(\sum_j (\lambda_{ijk} + \mu_{ijk} - \epsilon_i) > 0 \; \forall \; i, k)$ **then**
    Set $z_{ik} = 1$
end if
Set $S_k = \sum_i \sum_j (\lambda_{ijk} + \mu_{ijk} - \epsilon_i) \; \forall \; k$
Define an ordered set $\S = \{k : S_k \geqslant S_{k+1}\}$. Let $\S_m$ denote an element in $\S$, $m \in \{1, \ldots, |\S|\}$.
Set $z_{mm} = 1 \; \forall \; m = \S_1, \ldots, \S_p$; $z_{mm} = 0 \; \forall \; m = \S_{p+1}, \ldots, \S_{|\S|}$
if $(\sum_j (\lambda_{ijm} + \mu_{ijm} - \epsilon_i) > 0 \; \forall \; i, m = \S_1, \ldots, \S_p)$ **then**,
    Set $z_{im} = 1$
else
    Set $z_{im} = 0$
end if

SUB_2 is an unconstrained non-linear binary program, which can be solved by reformulating it as a MISOCP, as explained in Section (4.1). For this, we exploit the fact that SUB_2, given by (36), can be reformulated as the following MISCOP, using the transformations (23), (24) and (25) from Section 4.1.

We also introduce the following modifications in SUB_2 on top of the above transformations, to solve it more efficiently. First, (10) can be replaced by (13), although this may not guarantee binary values for x_{ijkl} variables, and therefore, may result in a relatively weaker Lagrangian upper bound. Nonetheless, we still prefer to use (13) since it converts SUB_2 into a Second Order Conic Program (SOCP) named as SUB_2', which have polynomial time complexity. Second, we add $\sum_{k \in H^e} \sum_{l \in H^e} x_{ijkl} = 1$ to SUB_2', which is redundant to $SACOHLP$, but helps to strengthen the Lagrangian upper bound forming a connect between SUB_1 and SUB_2'. SUB_2 with the reformulation and modifications is written as follows:

$$[SUB_2'] :$$

$$\begin{aligned}
\theta_{SUB_2'} = \max \quad & \sum_i \sum_j \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k \in H^c} \sum_{l \in H^c} u^c_{ijkl} x^c_{ijkl} \hat{f}_i a_{ij} + \sum_j (\hat{f}_i - f_{ij}) o_{ij} \right\} - \sum_i \sum_j \sum_k \sum_l \left\{ \lambda_{ijk} + \nu_{ijl} \right\} x_{ijkl} \\
\text{s.t.} \quad & g_{ij} = \sum_{k \in H^e} \sum_{l \in H^e} u^e_{ijkl} x_{ijkl} + \sum_{k \in H^c} \sum_{l \in H^c} u^c_{ijkl} x^c_{ijkl} && \forall \; i, j \in N \\
& (o_{ij} + g_{ij})^2 \geqslant 2 \sum_{k \in H^e} \sum_{l \in H^e} u^e_{ijkl} x^2_{ijkl} + o^2_{ij} + g^2_{ij} && \forall i, j \in N \\
& (a_{ij} + g_{ij})^2 \geqslant 2 + a^2_{ij} + g^2_{ij} && \forall \; i, j \in N \\
& \sum_{k \in H^e} \sum_{l \in H^e} x_{ijkl} = 1 && \forall \; i, j \in N \\
& x_{ijkl} \in [0, 1] && \forall \; i, j, k, l \\
& a_{ij}, g_{ij}, o_{ij} \geqslant 0 && \forall \; i, j \in N
\end{aligned}$$

We now state the following two propositions, which are used in the development of the complete sub-gradient algorithm.

Proposition 2. *For any given set of* $(\lambda^q, \nu^q, \epsilon^q)$, (37) *provides an upper bound on the optimal objective function value of SACOHLP, where* (z^q_{ik}), $(x^q_{ijkl}, a^q_{ij}, o^q_{ij})$ *are the optimal solutions to* SUB_1 *and* SUB_2'

with the objective function values $\theta^q_{SUB_1}$ and $\theta^q_{SUB'_2}$, respectively.

$$UB^q = \theta^q_{SUB_1} + \theta^q_{SUB'_2} = \sum_i \sum_k \sum_j (\lambda^q_{ijk} + \mu^q_{jik} - \epsilon^q_i) z^q_{ik} +$$

$$\sum_i \sum_j \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k \in H^c} \sum_{l \in H^c} u^c_{ijkl} x^c_{ijkl} \hat{f}_i a^q_{ij} + \sum_j (\hat{f}_i - f_{ij}) o^q_{ij} \right\} - \sum_i \sum_j \sum_k \sum_l \left\{ \lambda^q_{ijk} + \nu^q_{ijl} \right\} x^q_{ijkl} \quad (37)$$

Proof. Since $SACOHLP_{SUB}$ is a Lagrangian relaxation of the full problem COHLP, the objective function value of $SACOHLP_{SUB}$, given by θ_{SUB}, provides an upper bound on the optimal objective function value of COHLP. □

Proposition 3. *For any given set of $(\lambda^q, \nu^q, \epsilon^q)$, (38) provides a lower bound to the optimal objective function value of SACOHLP, where (z^q_{ik}) is an optimal solution to SUB_1 and x^q_{ijkl} is calculated using Algorithm 3.*

$$LB^q = \theta(z^q, x^q_{ijkl}) = \sum_{i,j \in N} f_{ij} \frac{\sum_{k,l \in H^e} x^q_{ijkl} u^c_{ijkl}}{\sum_{k,l \in H^e} x^q_{ijkl} u_{ijkl} + \sum_{k,l \in H^c} u^c_{ijkl} x^c_{ijkl}} \quad (38)$$

Proof. If (z^q_{ik}) is an optimal solution to SUB_1 and x^q_{ijkl} is derived from Algorithm 3, then the solution set (z^q_{ik}, x^q_{ijkl}) is feasible to the original problem since it satisfies constraints (6), (9)-(10). These constraints were relaxed in the Langrangian relaxation. Hence, the objective function of $SACOHLP$ evaluated at (z^q_{ik}, x^q_{ijkl}), given by (38), provides a lower bound on the optimal objective of SACOHLP. □

A good, but not necessarily optimal, set of Lagrange multipliers to SACOHLP is obtained using the standard sub-gradient optimization, as summarized in Algorithm 4. At each iteration, the algorithm also produces a feasible solution, the best among which is reported as the final solution once the algorithm terminates, which happens either on reaching the specified maximum time limit (M^*) or upon reaching a particular optimality gap.

We explore the possibility of improving this method further. LR-SOCP requires solving SUB_1 and SUB'_2 at each iteration of the sub-gradient algorithm. Solving SUB_1 is computationally much cheaper than solving SUB'_2. Further, an UB is useful only to the extent of identifying when to terminate the algorithm. Hence, computing an UB is not necessarily required at every iteration of the sub-gradient algorithm. We exploit this idea to avoid solving SUB'_2 frequently. We repeat the experiments using LR-SOCP for all the test instances, with the modification that SUB''_2 is solved only after every ψ iterations. From our limited experiments, $\psi = 75$ turned out to be a good choice and we use it for all our further experiments based on lagrangian based relaxation.

Algorithm 3 Feasible

1: Set $z_{ik} = 0$, ∀ i, k.
2: For each i, find k that $\max\{(\lambda_{ijk} + \mu_{jik} - \epsilon_i) : z_{kk} = 1\}$. Set that k $z_{ik} = 1$.
3: Set $x_{ijkl} = z_{ik} * z_{jl} \quad \forall\, i, j, k, l$.

Algorithm 4 Sub-Gradient Optimisation Algorithm

$q \leftarrow 1$, $\lambda^q \leftarrow 0, \mu^q \leftarrow 0$, $\epsilon^q \leftarrow 0$, $UB^q \leftarrow \infty$, $LB^q \leftarrow -\infty$, $UB \leftarrow UB^q$ and $LB \leftarrow LB^q$, $t^q \leftarrow 0$.
Initialise Δ (the step-size multiplier), N_I (maximum iterations with no improvement in UB), ϵ (optimality gap), M (maximum CPU time limit).
do
    Solve SUB_1, SUB_2, and obtain UB^q, LB^q from (37), (38), respectively.
    $UB \leftarrow UB^q$, $LB \leftarrow max\{LB, LB^q\}$.
    $(z^*, x^*) \leftarrow \arg\max_{z,x}\{LB\}$ and $(\lambda^*, \mu^*, \epsilon^*)$ be the corresponding lagrange multipliers.
    Find a feasible solution using Algorithm (3).
    Adjust the multipliers as :

$$\lambda_{ijk}^{q+1} \leftarrow \lambda_{ijk}^{q} + t^q(z_{jk}^q - \sum_{l\in H^e} x_{ijlk}^q) \quad \forall\, i,j,k$$

$$\mu_{ijl}^{q+1} \leftarrow \mu_{ijl}^{q} + t^q(z_{jk}^q - \sum_{l\in H^e} x_{ijlk}^q) \quad \forall\, i,j,l$$

$$\epsilon_{i}^{q+1} \leftarrow \epsilon_{i}^{q} + t^q(1 - \sum_{k\in H^e} z_{ik}^q) \quad \forall\, i$$

$$\text{where, } t^{q+1} \leftarrow \Delta\left\{\frac{UB - LB}{\sum_i\sum_j\sum_k(z_{jk}^q - \sum_{l\in H^e} x_{ijlk}^q)^2 + \sum_i\sum_j\sum_l(z_{jk}^q - \sum_{l\in H^e} x_{ijlk}^q)^2 + \sum_i(1-\sum_{k\in H^e} z_{ik}^q)^2}\right\}.$$

    If no improvement in UB in N_I consecutive iterations, $\Delta \leftarrow \Delta/2$ and $(\lambda, \mu, \epsilon) \leftarrow (\lambda^*, \mu^*, \epsilon^*)$
    $q \leftarrow q + 1$.
while $(1 - LB^q/UB^q \leqslant \epsilon \vee \text{CPU Time} \leqslant M)$

4.3.2 Multiple Allocation

Lagrangian relaxation has been effectively used by Ishfaq and Sox (2011) to solve multi allocation model. They have used Skorin-Kapov et al. (1996)'s model for their multiple allocation problem. Since then, a formulation has been proposed in literature by Hamacher et al. (2004) which provide comparatively tighter LP relaxation. Hence, for computational benefits, we have used this formulation and have relaxed the facet inducing constraint (19) to decompose the problem. For $MACOHLP$, the Lagrangian sub-problem can be written as follows:

$$[MACOHLP_{SUB}]: \quad \theta_{SUB}(\nu) =$$

$$\max \sum_i\sum_j f_{ij}\frac{\sum_{k\in H^e}\sum_{l\in H^e} x_{ijkl}u_{ijkl}^e}{\sum_{k\in H^e}\sum_{l\in H^e} x_{ijkl}u_{ijkl}^e + \sum_{k\in H^c}\sum_{l\in H^c} u_{ijkl}^c} + \sum_i\sum_j\sum_{k\in H^e}\nu_{ijk}(z_k - \sum_{l\in H^e} x_{ijkl} + \sum_{l\in H^e - k} x_{ijlk}) \tag{39}$$

$$\text{s.t. } (16)-(18),(20)-(21)$$

$$\nu_{ijk} \in [-\infty, \infty] \tag{40}$$

For a given set of (ν), , the Lagrangian sub-problem provides an upper bound (UB) to $MACOHLP$. For a given set of Lagrange multipliers (,), the sub-problem (50) decomposes into the following two independent sub-problems, with one involving only z variables, while the other involving x variables

$$[SUB_3]:$$

$$\theta_{SUB3} = \max \sum_i \sum_k \sum_j \nu_{ijk} z_k \tag{41}$$

$$\text{s.t.} \sum_{k \in H^e} z_{kk} = p$$

$$z_k \in \{0,1\}$$

$$[SUB_4]:$$

$$\theta_{SUB4} = \max \quad \sum_i \sum_j f_{ij} \frac{\sum_{k\in H^e} \sum_{l\in H^e} x_{ijkl} u^e_{ijkl}}{\sum_{k\in H^e} \sum_{l\in H^e} x_{ijkl} u^e_{ijkl} + \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl}} - \sum_i \sum_j \sum_{k\in H^e} \nu_{ijk} (\sum_{l\in H^e} x_{ijkl} + \sum_{l\in H^e - k} x_{ijlk}) \tag{42}$$

$$\text{s.t.} \sum_k \sum_l x_{ijkl} = 1 \quad \forall i,j \in N$$

$$x_{ijkl} \in [0,1]$$

SUB_3 can be solved optimally for a given set of (ν) by locating p hubs with the maximum contribution to objective function (41). The steps of the method to solve SUB_3 are summarized in Algorithm 5

Algorithm 5 Optimal Solution for SUB_3

1: Define an ordered set S=$\{i : \zeta_i \geqslant \zeta_{i+1}, \zeta_i \in \{\nu_{ijkl} : i,j \in N \; k,l \in H^e\}\}$. Let S_j denote an element in S, $j \in 1,\ldots,|S|$.
2: Set $z_j = 1 \; \forall \; j = S_1,\ldots,S_p$; $z_j = 0 \; \forall \; j = S_{p+1},\ldots,S_{|S|}$

SUB_4 is an unconstrained non-linear binary program, which can be solved by reformulating it as a MISOCP, as explained in Section (4.1). For this, we exploit the fact that SUB_4, given by (42), can be reformulated as the following MISCOP, using the transformations (23), (24) and (25) from Section 4.1. We also introduce the same modifications that we did for SUB_2. SUB_4 with the reformulation and modifications is written as follows:

$$[SUB_4'] : \quad \theta_{SUB_4'} =$$

$$\max \quad \sum_i \sum_j \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl} \hat{f}_i a_{ij} + \sum_j (\hat{f}_i - f_{ij}) o_{ij} \right\} - \quad \sum_i \sum_j \sum_{k\in H^e} \nu_{ijk} (\sum_{l\in H^e} x_{ijkl} + \sum_{l\in H^e-k} x_{ijlk})$$

$$\text{s.t.} \; g_{ij} = \sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x_{ijkl} + \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl} \qquad \forall \; i,j \in N$$

$$(o_{ij} + g_{ij})^2 \geqslant 2 \sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x^2_{ijkl} + o^2_{ij} + g^2_{ij} \qquad \forall \; i,j \in N$$

$$(a_{ij} + g_{ij})^2 \geqslant 2 + a^2_{ij} + g^2_{ij} \qquad \forall \; i,j \in N$$

$$\sum_{k\in H^e} \sum_{l\in H^e} x_{ijkl} = 1 \qquad \forall \; i,j \in N$$

$$x_{ijkl} \in [0,1] \qquad \forall \; i,j,k,l$$

$$a_{ij}, g_{ij}, o_{ij} \geqslant 0 \qquad \forall \; i,j \in N$$

The above model (SUB_4) is an SOCP, which has a polynomial time complexity, and hence can be solved efficiently using off-the-shelf solvers. It is worth noting that in MACOHLP((17)-(21)), x_{ijkl} is binary. One could have chosen to retain x_{ijkl} as binary variables in SUB'_4, which would have produced a tighter Lagrangian UB. However, this would have resulted in an MISOCP model for SUB'_4, which is not polynomial time solvable.

Algorithm 6 Feasible for MACOHLP

1: For a given ν, Set z_k $\forall k$ using Algorithm 5. Let H^r denote the set of opened hubs, i.e., $z_k = 1$.
2: For each i, chose $m \in H^r$ which $\min_m\{(d_{ik}) : z_m = 1\}$ and set $z_{im} = 1$.
3: Set $x^r_{ijmn} = z_{im} * z_{jn} \quad \forall\, i, j, m, n$.

With the above transformation, MACOHLP can be solved using a sub-gradient algorithm, similar to Algorithm 4. The lower and upper bounds (LB^q and UB^q) are updated according to the following propositions:

Proposition 4. *For any given set of* (ν^q), (43) *provides an upper bound on the optimal objective function value of SACOHLP, where* (z^q_k), $(x^q_{ijkl}, a^q_{ij}, o^q_{ij})$ *are the optimal solutions to* SUB_3 *and* SUB'_4 *with the objective function values* $\theta^q_{SUB_3}$ *and* $\theta^q_{SUB'_4}$*, respectively.*

$$UB^q = \theta^q_{SUB_3} + \theta^q_{SUB'_4} = \sum_i \sum_k \sum_j \nu_{ijk} z^q_k +$$

$$\sum_i \sum_j \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl} \hat{f}_i a^q_{ij} + \sum_j (\hat{f}_i - f_{ij}) o^q_{ij} \right\} - \sum_i \sum_j \sum_{k\in H^e} \nu^q_{ijk} \Big(\sum_{l\in H^e} x^q_{ijkl} + \sum_{l\in H^e - k} x^q_{ijlk} \Big) \tag{43}$$

Proof. Since $MACOHLP_{SUB}$ is a Lagrangian relaxation of the full problem MACOHLP, the objective function value of $MACOHLP_{SUB}$, given by θ_{SUB}, provides an upper bound on the optimal objective function value of MACOHLP. □

Proposition 5. *For any given set of* (ν^q), (44) *provides a lower bound to the optimal objective function value of MACOHLP, where* (z^q_{ik}) *is an optimal solution to* SUB_1 *and* x^{rq}_{ijkl} *is calculated using Algorithm 3.*

$$LB^q = \theta(z^q, x^{rq}_{ijkl}) = \sum_{i,j\in N} f_{ij} \frac{\sum_{k,l\in H^e} x^{rq}_{ijkl} u^c_{ijkl}}{\sum_{k,l\in H^e} x^{rq}_{ijkl} u_{ijkl} + \sum_{k,l\in H^c} u^c_{ijkl} x^c_{ijkl}} \tag{44}$$

Proof. If (z^q_{ik}) is an optimal solution to SUB_3 and x^{rq}_{ijkl} is derived from Algorithm 6, then the solution set $(z^q_{ik}, x^{rq}_{ijkl})$ is feasible to the original problem since it satisfies constraints (17)-(19). These constraints were relaxed in the Langrangian relaxation. Hence, the objective function of $MACOHLP$ evaluated at $(z^q_{ik}, x^{rq}_{ijkl})$, given by (38), provides a lower bound on the optimal objective of MACOHLP. □

With the above transformation, $MACOHLP$ can be solved using a sub-gradient algorithm, similar to Algorithm 4, with the exception that SUB_3 is solved as an SOCP (SUB'_4). The lower and upper bounds (UB^q and LB^q) are updated according to proposition 43, 44 respectively.

4.4 Lagrangian relaxation with Lifted Polymatroid Cuts

In this section, like in Section 4.3, we use the sub-gradient algorithm (refer to Algorithm 4) to solve $SACOHLP$ and $MACOHLP$. However, the UB to them is computed by solving SUB_2 and SUB_4 in a different way, as opposed to using MISOCP based reformulation. The UB is calculated for both the problems by exploiting the fact that SUB_2 and SUB_4 can be solved by approximating the conic constraints with the help of lifted polymatroid cuts, as discussed in section 4.2. The relaxed problem is then solved iteratively to give a solution within a set optimality gap.

5 Numerical Experiments

We first describe in Section 5.1 the relevant data used in our experiments, followed by a discussion of our results in Section 5.2.

5.1 Data Generation

All our computational experiments are based on the Australian Post (AP) data-set from HLP literature. This data-set, introduced by Ernst and Krishnamoorthy (1996), consists of nodes representing district postcodes, along with their coordinates and mail flow volumes. In the context of competitive HLP, this data-set has been used by Eiselt and Marianov (2009); Marianov et al. (1999), which are the closest to our work. However, there is no information available on computation of customers' utility in either of the data-sets, and hence we provide below the scheme used in this paper to generate them.

In all our experiments, we assume there is only one incumbent player, denoted by c, operating with a set of hubs in the network, and the entrant, denoted by e, intends to capture a part of its market by locating its own set of p hubs. As discussed in Section 3, the share of the market captured by the entrant is given by (**??**), wherein the customer utility appearing in (**??**) are given by (1). The parameters used in (1) are set as follows: $\beta = 1, \delta = 1, \gamma = 0.75$, and $A^c_{kl} = 1.25 \ \forall \ k, l : l = k; 1$ otherwise. The attractiveness index A^c_{kl} is set 25% higher when $l = k$ to capture the fact that single hub routes are more attractive to customers than multiple hub routes. We assume that the the travel time for the same route ($i \rightarrow k \rightarrow l \rightarrow j$) is the same for both the airlines, which is computed as $T^a_{ijkl} = T^a_{ik} + T^a_{kl} + T^a_{lj} \ \forall \ a \in \{c, e\}$. The travel time (in minutes) between any two cities i and k is given by $30 + 0.12d_{ik}$, where d_{ik} is the distance (in miles) between the cities i and k, and 30 is an approximation for the layover time (in minutes) at the two cities (Grove and O'Kelly, 1986). The operational routes and hubs for the competitor airline is found by solving a p-median hub location problem. The transportation cost per unit volume is also assumed to be the same for a given route ($i \rightarrow k \rightarrow l \rightarrow j$), which is computed as $B^a_{ijkl} = \chi T^a_{ik} + \alpha T^a_{kl} + \eta T^a_{lj} \ \forall \ a \in \{c, e\}$, where χ and η are the transportation costs per unit volume on the collection and distribution legs, respectively. All the parameter values used in our experiments are summarized in Table 2. All the experiments are run on a personal computer with 2.20 GHz Intel(R) Xenon(R) E5-2630 CPU and 64 GB RAM. The four solution methods, as described in Section 4, are coded in C++, and ILOG CPLEX 12.7.1 is used as the default solver. The optimality gap $(UB - LB)/LB$ for all the iterative algorithms is set at .01, and the maximum CPU time limit (M) of 2 hours is used for each instance. For MISOCP in Section 4.1, we use the default value of 0 for the CPLEX parameter *miqcpstrat*, as discussed earlier in Section 4.1. For the Lagrangian based methods (LR-MISOCP and LRLP-MISOCP), the parameters in the sub-gradient algorithm are initialized as: $\Delta = 6$ and $N_I = 50$.

Table 2: Parameters

	Common Parameters					Entrant's Parameters			
Allocation	N	p	χ	ν	α	γ	β	δ	A
Single	$\{10,15,20,25\}$	$\{2,3,4\}$	1	1	$\{.1,.2,.3,.4,.5,.6,.7,.8,.9,1\}$	.75	1	1	1
Multiple	$\{10,15,20,25,50\}$	$\{2,3,4\}$	1	1	$\{.1,.2,.3,.4,.5,.6,.7,.8,.9,1\}$	.75	1	1	1

5.2 Results and Analysis

Tables 3 and 4 present the computational performances (in terms of CPU time) of the four alternate solution approaches corresponding to different combinations of $|N|, p, \alpha$ for the AP dataset. In each of these tables, the column "CPU Time" reports the time required to reach the final solution, while the column "Gap%" reports the optimality gap (in %) at algorithm termination. For the iterative methods (MISOCP, LP-MISOCP, LR-MISOCP, LRLP-MISOCP), the termination criterion used is: 1% optimality gap or 2 hours of CPU time. For MIS, which is directly solved using the CPLEX solver, we use the default termination criteria of CPLEX or 2 hours of CPU time.

As obvious from Table 3, for all the instances that MIS could solve, it was consistently the worst in terms of CPU time. There are other instances, particularly for $|N| \geqslant 15$, for which MIS runs out of memory, which are indicated using a "**". On the other hand, for 97% of the instances, LP-MIS outperforms all the other methods.

As obvious from Table 4, 5, for all the instances, MISOCP was consistently the worst in terms of CPU time. There are no test instances for upto $|N| = 50$, for which MISOCP runs out of memory. This is because of the presence of integer variables of size $|N|$, rather than $|N|^2$ in single allocation. For 54% of the instances, LP-MISOCP outperforms all the other methods and the for the rest of the 46% of the instances, LR-MIS outperforms all the other methods. Also in general, for larger instances (¿ $|N| = 20$), LR-MISOCP turns out to be the best, and for smaller instances, LP-MISOCP is the best performing method (LP-MISOCP).

To further tease out the difference in the performances of the 4 different methods, we show their performance profiles (Dolan and Moré, 2002) through Figures 1a-1b. For this, let $t_{p,s}$ be the CPU time taken to solve the instance $p \in P$ using method $s \in S$. Then, performance ratio $r_{p,s}$ is calculated as:

$$r_{p.s} = \frac{t_{p,s}}{\min_{s \in S} t_{p,s}}$$

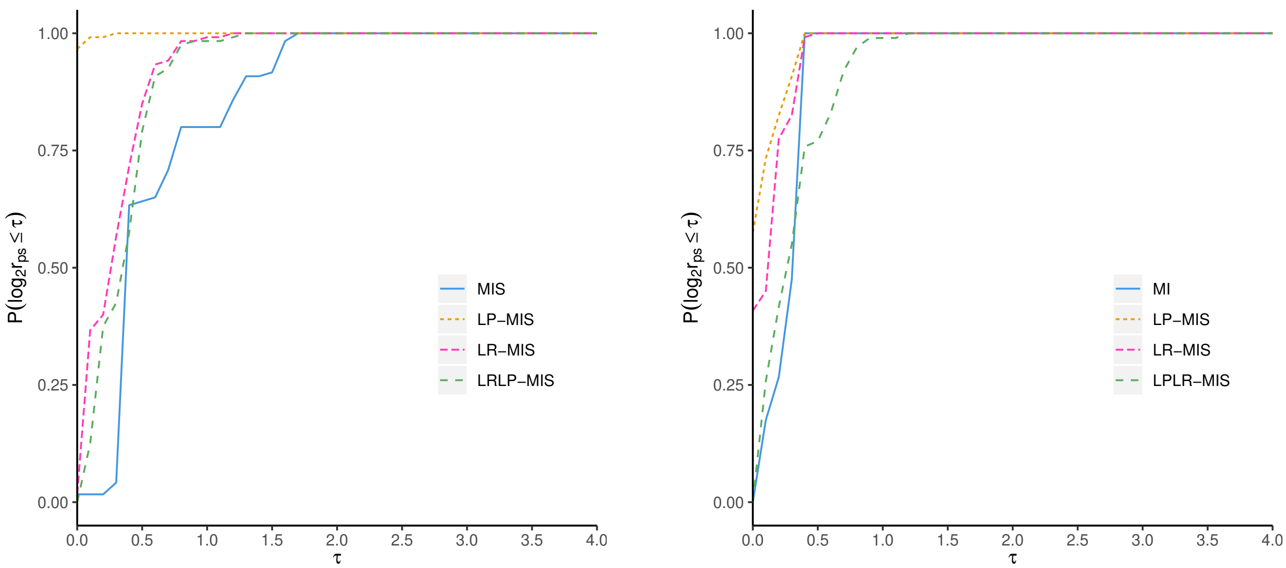
Treating $r_{p,s}$ as a random variable, a performance profile $\rho_s(\tau) = P(r_{p,s} \leqslant 2^\tau)$ gives its cumulative probability distribution. In other words, it gives the probability with which the CPU time taken by a given approach s does not exceed 2^τ times the CPU time required by the best among all the approaches under study. Specifically, the intercept on the y-axis for a given approach s gives the proportion of instances for which it is the best. As clear from Figure 1a, LR-MISOCP is the best for 97% of the 30×4 instances for single allocation model. Further, LR-MISOCP is able to solve 100% of the instances at $\tau = .2$, whereas MISOCP is unable to solve all of them at $\tau = 1.8$. For multiple allocation model, there is a close competition in terms of computation time between LP-MISOCP and LR-MISOCP, which is also clear in the table 4-5. The methods MISOCP, LR-MISOCP and LP-MISOCP are able to solve all the instances given about 1.44 time the time required by the best method. LPLR-MISOCP is performing particularly poorly as is also the case for single allocation model.

Table 3: Computational Performance for SA. MISOCP referred as MIS in this table

p	CPU Time				Avg % Gap				CPU Time				Avg % Gap			
	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS
									N=10							
				$\alpha = .1$								$\alpha = .2$				
2	51.64	**41.96**	49.10	56.13	0.00	0.00	0.00	0.00	55.05	**44.48**	67.55	75.11	0.00	0.00	0.00	0.00
3	122.62	**98.34**	120.09	123.04	0.00	0.00	0.00	0.00	121.46	**97.66**	135.83	145.70	0.00	0.00	0.00	0.00
4	182.24	**146.75**	179.77	184.89	0.00	0.00	0.00	0.00	185.13	**148.44**	193.18	196.80	0.00	0.00	0.00	0.00
				$\alpha = .3$								$\alpha = .4$				
2	57.31	**46.10**	54.66	64.36	0.00	0.00	0.00	0.00	60.10	**48.92**	69.44	76.44	0.00	0.00	0.00	0.00
3	124.39	**99.89**	121.08	127.16	0.00	0.00	0.00	0.00	125.23	**101.18**	140.83	144.32	0.00	0.00	0.00	0.00
4	188.97	**151.74**	185.66	193.22	0.00	0.00	0.00	0.00	192.28	**153.90**	195.79	201.24	0.00	0.00	0.00	0.00
				$\alpha = .5$								$\alpha = .6$				
2	60.91	**48.75**	56.43	58.97	0.00	0.00	0.00	0.00	56.39	**45.61**	72.65	78.67	0.00	0.00	0.00	0.00
3	128.26	**103.44**	124.53	131.85	0.00	0.00	0.00	0.00	123.52	**99.06**	141.87	150.65	0.00	0.00	0.00	0.00
4	195.90	**157.37**	194.23	195.42	0.00	0.00	0.00	0.00	192.06	**154.50**	195.39	199.62	0.00	0.00	0.00	0.00
				$\alpha = .7$								$\alpha = .8$				
2	49.02	**39.82**	46.13	52.37	0.00	0.00	0.00	0.00	48.35	**38.98**	73.92	81.64	0.00	0.00	0.00	0.00
3	119.80	**95.87**	116.82	126.97	0.00	0.00	0.00	0.00	119.25	**95.54**	137.66	139.34	0.00	0.00	0.00	0.00
4	190.63	**152.74**	186.01	190.19	0.00	0.00	0.00	0.00	189.65	**151.90**	193.11	193.68	0.00	0.00	0.00	0.00
				$\alpha = .9$								$\alpha = 1$				
2	44.19	**35.98**	41.45	50.75	0.00	0.00	0.00	0.00	38.00	**30.75**	68.49	76.84	0.00	0.00	0.00	0.00
3	118.14	**95.08**	117.68	119.84	0.00	0.00	0.00	0.46	112.50	**90.06**	132.50	141.83	0.00	0.00	0.00	0.00
4	183.87	**147.60**	181.27	189.33	0.00	0.00	0.00	0.90	182.04	**146.46**	189.92	194.40	0.00	0.00	0.00	0.00
									N=15							
				$\alpha = .1$								$\alpha = .2$				
2	623.79	**499.38**	603.98	739.28	0.00	0.00	0.00	0.99	625.89	**501.06**	666.02	669.90	0.00	0.00	0.00	0.90
3	745.62	**597.08**	718.80	843.98	0.00	0.00	0.00	0.95	750.17	**600.53**	861.73	868.76	0.00	0.00	0.00	0.68
4	1260.04	**1008.69**	1212.46	1361.45	0.00	0.00	0.00	0.40	1264.79	**1011.97**	1692.75	1699.66	0.00	0.00	0.00	0.40
				$\alpha = .3$								$\alpha = .4$				
2	624.37	**499.54**	620.73	718.29	0.00	0.00	0.00	0.39	628.55	**502.92**	667.60	668.42	0.00	0.00	0.00	0.33
3	753.82	**603.15**	702.28	745.42	0.00	0.00	0.00	0.37	755.61	**604.97**	872.02	878.76	0.00	0.00	0.00	0.57
4	1270.97	**1017.30**	1251.66	1320.90	0.00	0.00	0.00	0.75	1271.94	**1018.52**	1693.96	1700.26	0.00	0.00	0.00	0.21
				$\alpha = .5$								$\alpha = .6$				
2	632.26	**506.34**	623.06	637.00	0.00	0.00	0.00	0.89	627.06	**502.05**	666.13	667.15	0.00	0.00	0.00	0.23
3	761.34	**609.64**	759.21	788.36	0.00	0.00	0.00	0.10	756.25	**605.02**	875.24	883.30	0.00	0.00	0.00	0.83
4	1278.06	**1022.76**	1259.80	1323.00	0.00	0.00	0.00	0.20	1276.19	**1021.82**	1693.13	1702.15	0.00	0.00	0.00	0.05
				$\alpha = .7$								$\alpha = .8$				
2	627.72	**502.21**	580.04	706.53	0.00	0.00	0.00	0.44	623.83	**499.41**	659.53	665.84	0.00	0.00	0.00	0.91
3	754.43	**604.54**	714.57	745.70	0.00	0.00	0.00	0.58	748.30	**599.32**	868.50	870.74	0.00	0.00	0.00	0.02
4	1272.46	**1018.02**	1263.76	1295.07	0.00	0.00	0.00	0.92	1272.73	**1018.22**	1686.83	1689.58	0.00	0.13	0.13	0.73
				$\alpha = .9$								$\alpha = 1$				
2	624.21	**500.04**	574.75	579.36	0.00	0.00	0.00	0.82	620.85	**496.72**	653.91	659.85	0.00	0.00	0.00	0.31
3	748.96	**600.07**	698.75	838.38	0.00	0.00	0.00	0.66	744.33	**595.75**	866.29	875.83	0.00	0.10	0.10	0.35
4	1267.95	**1014.75**	1265.68	1381.79	0.00	0.56	0.54	0.09	1264.50	**1011.84**	1683.71	1686.62	0.00	0.38	0.38	0.28
									N=20							
				$\alpha = .1$								$\alpha = .2$				
2	2919.05	**2335.57**	2677.28	2846.07	0.00	0.00	0.00	0.37	**2920.83**	3030.17	3189.52	3427.68	0.00	0.00	0.00	0.47
3	3097.78	**2478.28**	2881.70	3087.00	0.00	0.19	0.22	0.36	**3101.38**	3139.34	3303.79	3315.27	0.00	0.52	0.46	0.29
4	3123.16	**2499.07**	2870.39	2894.46	0.00	0.89	0.88	0.99	**	**3271.57**	3443.11	3672.50	0.00	0.91	0.90	0.81
				$\alpha = .3$								$\alpha = .4$				
2	2923.08	**2339.17**	2900.75	2976.76	0.00	0.00	0.00	0.75	**	**3039.21**	3198.31	3306.03	0.00	0.67	0.67	0.88
3	**	**2422.96**	3230.18	3379.75	0.00	0.81	0.74	0.73	**	**3144.29**	3309.25	3383.14	0.00	0.43	0.36	0.14
4	**	**2558.88**	3410.92	3601.85	0.00	0.52	0.45	0.89	**	**3278.02**	3450.11	3604.57	0.00	0.46	0.44	0.54
				alpha=.3								**alpha=.4**				
2	**	**2294.21**	3057.82	3147.07	0.00	0.53	0.54	0.34	**	**3039.74**	3199.11	3425.80	0.00	0.28	0.98	0.46
3	**	**2459.79**	3279.71	3320.59	0.00	0.76	0.74	0.76	**	**3139.49**	3303.80	3477.34	0.00	0.59	0.54	0.63
4	**	**2503.40**	3337.76	3495.90	0.00	0.72	0.93	0.94	**	**3276.10**	3448.27	3451.43	0.00	0.39	0.20	0.36
				alpha=.7								**alpha=.8**				
2	**	**2337.48**	3116.34	3168.05	0.00	0.75	0.75	0.23	**	**3037.94**	3197.33	3445.72	0.00	0.54	0.54	0.86
3	**	**2408.46**	3210.46	3370.30	0.00	0.58	0.42	0.38	**	**3132.39**	3297.21	3336.86	0.00	0.82	0.79	0.79
4	**	**2505.66**	3340.06	3403.32	0.00	0.36	0.06	0.70	**	**3271.77**	3443.78	3525.08	0.00	0.43	0.37	0.29
				alpha=.9								**alpha=1**				
2	**	**2393.05**	3190.06	3310.44	0.00	0.29	0.13	0.66	**	**3035.13**	3194.44	3407.37	0.00	0.46	0.44	0.99
3	**	**2430.40**	3240.32	3284.00	0.00	0.75	0.73	0.89	**	**3128.55**	3292.37	3424.30	0.00	0.42	0.43	0.66
4	**	**2545.58**	3393.18	3469.50	0.00	0.59	0.48	0.17	**	**3264.64**	3436.43	3449.99	0.00	0.29	0.14	0.26
									N=25							
				alpha=.1								**alpha=.2**				
2	6121.66	4897.71	**4596.81**	4731.90	0.00	0.55	0.53	0.48	**	**4422.00**	4654.02	4919.51	0.00	0.13	0.11	0.48
3	7033.14	5626.90	**4669.69**	4692.08	0.00	0.62	0.37	0.75	**	**4470.38**	4705.62	4758.52	0.00	0.67	0.65	0.98
4	**	**5744.03**	6045.47	6081.23	0.00	0.49	0.48	0.05	**	**5831.46**	6137.66	6411.98	0.00	0.96	0.97	0.36
				alpha=.3								**alpha=.4**				
2	**	**4335.21**	4562.58	4622.87	0.00	0.43	0.38	0.64	**	**4425.94**	4658.22	5042.29	0.00	0.92	0.87	0.82
3	**	**4462.55**	4697.18	5018.25	0.00	0.32	0.30	0.95	**	**4476.85**	4711.44	4903.92	0.00	0.88	0.89	0.41
4	**	**5813.06**	6118.54	6426.07	0.00	0.19	0.12	0.84	**	**5840.45**	6147.78	6404.73	0.00	0.62	0.60	0.94
				alpha=.5								**alpha=.6**				
2	**	**4381.88**	4612.41	4881.79	0.00	0.17	0.12	0.43	**	**4424.25**	4656.11	4938.36	0.00	0.61	0.56	0.38
3	**	**4387.73**	4618.30	4704.12	0.00	0.16	0.19	0.90	**	**4476.27**	4711.84	4752.85	0.00	0.32	0.34	0.12
4	**	**5689.14**	5987.85	6207.39	0.00	0.71	0.75	0.02	**	**5842.11**	6148.68	6366.89	0.00	0.35	0.37	0.98
				alpha=.7								**alpha=.8**				
2	**	**4352.66**	4581.68	4663.98	0.00	0.41	0.37	0.20	**	**4422.68**	4655.08	4973.17	0.00	0.34	0.01	0.30
3	**	**4384.37**	4614.76	4891.14	0.00	0.19	0.16	0.95	**	**4470.55**	4705.24	4959.40	0.00	0.72	0.65	0.04
4	**	**5820.60**	6126.68	6281.50	0.00	0.38	0.35	0.58	**	**5837.36**	6143.55	6342.62	0.00	0.63	0.67	0.32
				alpha=.9								**alpha=1**				
2	**	**4327.33**	4554.52	4705.33	0.00	0.52	0.51	0.55	**	**4411.18**	4643.16	4756.31	0.00	0.19	0.25	0.77
3	**	**4457.34**	4690.99	4912.00	0.00	0.68	0.66	0.30	**	**4464.99**	4699.97	4772.27	0.00	0.29	0.17	0.28
4	**	**5827.74**	6134.35	6471.28	0.00	0.35	0.34	0.84	**	**5824.38**	6130.41	6203.42	0.00	0.57	0.56	0.56

Table 4: Computational performance for all solution approaches for MA. MISOCP referred as MIS in this table

p	CPU Time				Avg % Gap				CPU Time				Avg % Gap			
	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS
								N=10								
				$\alpha = .1$								$\alpha = .2$				
2	17.30	**14.02**	16.65	18.11	0.00	0.00	0.00	0.00	18.97	**15.56**	18.25	18.62	0.00	0.00	0.00	0.00
3	47.53	**38.09**	50.75	50.89	0.00	0.00	0.00	0.00	46.11	**37.07**	46.43	48.81	0.00	0.00	0.00	0.00
4	69.15	**55.81**	71.75	72.85	0.00	0.00	0.00	0.00	72.09	**58.32**	73.43	73.95	0.00	0.00	0.00	0.00
				alpha=.3								**alpha=.4**				
2	20.04	**16.79**	18.55	20.47	0.00	0.00	0.00	0.00	21.26	**17.73**	20.12	21.35	0.00	0.00	0.00	0.00
3	48.27	**39.12**	48.91	51.21	0.00	0.00	0.00	0.00	48.04	**39.04**	51.28	52.28	0.00	0.00	0.00	0.00
4	73.25	**59.60**	73.42	75.41	0.00	0.00	0.00	0.00	75.77	**60.92**	77.88	81.40	0.00	0.00	0.00	0.00
				alpha=.5								**alpha=.6**				
2	23.31	**19.46**	23.18	23.47	0.00	0.00	0.00	0.00	19.34	**15.53**	18.14	19.75	0.00	0.00	0.00	0.00
3	49.36	**40.01**	52.57	53.50	0.00	0.00	0.00	0.00	48.52	**39.66**	51.16	54.24	0.00	0.00	0.00	0.00
4	77.31	**62.28**	78.62	82.29	0.00	0.00	0.00	0.00	76.24	**61.15**	77.78	81.77	0.00	0.00	0.00	0.00
				alpha=.7								**alpha=.8**				
2	17.99	**14.81**	16.57	17.36	0.00	0.00	0.00	0.00	16.96	**14.09**	16.22	18.61	0.00	0.00	0.00	0.00
3	45.11	**36.78**	45.12	45.54	0.00	0.00	0.00	0.00	45.78	**36.79**	46.31	48.01	0.00	0.00	0.00	0.00
4	74.30	**59.89**	74.31	75.24	0.00	0.00	0.00	0.00	72.91	**59.32**	73.15	77.11	0.00	0.00	0.00	0.00
				alpha=.9								**alpha=1**				
2	14.33	**11.47**	13.89	17.07	0.00	0.00	0.00	0.00	13.70	**11.32**	12.58	15.99	0.00	0.00	0.00	0.00
3	46.50	**37.32**	46.71	47.82	0.00	0.00	0.00	0.00	41.65	**33.72**	42.98	45.33	0.00	0.00	0.00	0.00
4	69.89	**56.44**	72.55	74.87	0.00	0.00	0.00	0.00	71.58	**57.86**	74.43	77.31	0.00	0.00	0.00	0.00
								N=15								
				alpha=.1								**alpha=.2**				
2	186.34	**149.72**	164.68	203.83	0.00	0.00	0.00	0.00	186.09	**149.07**	164.74	174.56	0.00	0.00	0.00	0.00
3	222.19	**177.78**	197.63	248.61	0.00	0.00	0.00	0.00	222.68	**178.22**	197.82	233.20	0.00	0.00	0.00	0.00
4	376.49	**301.85**	334.74	379.41	0.00	0.00	0.00	0.00	378.49	**303.29**	336.37	393.05	0.00	0.00	0.00	0.00
				alpha=.3								**alpha=.4**				
2	186.60	**149.69**	166.04	199.68	0.00	0.00	0.00	0.00	186.14	**149.24**	164.69	209.62	0.00	0.00	0.00	0.00
3	223.49	**179.51**	197.65	217.80	0.00	0.00	0.00	0.00	224.23	**179.84**	198.89	257.38	0.00	0.00	0.00	0.00
4	378.72	**303.72**	336.30	354.20	0.00	0.00	0.00	0.00	378.64	**302.94**	336.63	358.50	0.00	0.00	0.00	0.00
				alpha=.5								**alpha=.6**				
2	186.77	**149.60**	164.88	179.99	0.00	0.00	0.00	0.00	187.22	**149.89**	165.42	216.57	0.00	0.00	0.00	0.00
3	227.42	**182.31**	201.36	217.98	0.00	0.00	0.00	0.00	224.03	**179.73**	198.26	244.27	0.00	0.00	0.00	0.00
4	382.07	**305.95**	339.55	358.92	0.00	0.00	0.00	0.00	381.80	**306.20**	339.60	359.26	0.00	0.00	0.00	0.00
				alpha=.7								**alpha=.8**				
2	186.45	**149.71**	164.51	205.91	0.00	0.00	0.00	0.00	186.69	**150.14**	165.94	220.54	0.00	0.00	0.00	0.00
3	225.79	**180.68**	200.29	222.53	0.00	0.00	0.00	0.00	222.55	**178.67**	197.64	225.99	0.00	0.00	0.00	0.00
4	381.42	**305.16**	338.15	372.89	0.00	0.00	0.00	0.00	381.00	**305.61**	338.07	359.41	0.00	0.00	0.00	0.00
				alpha=.9								**alpha=1**				
2	186.43	**149.51**	165.02	204.50	0.00	0.00	0.00	0.00	183.28	**146.92**	162.57	216.44	0.00	0.00	0.00	0.00
3	221.17	**177.81**	196.76	213.79	0.00	0.00	0.00	0.00	221.72	**178.10**	197.16	240.91	0.00	0.00	0.00	0.00
4	379.30	**304.21**	336.93	365.36	0.00	0.00	0.00	0.00	378.46	**303.76**	335.53	373.21	0.00	0.00	0.00	0.00
								N=20								
				alpha=.1								**alpha=.2**				
2	1020.02	**816.22**	927.69	1023.28	0.00	0.00	0.00	0.50	1020.92	**817.07**	929.03	1030.61	0.00	0.00	0.00	0.32
3	1240.14	**992.45**	1128.50	1363.73	0.00	0.19	0.22	0.80	1241.97	**994.35**	1129.16	1350.80	0.00	0.52	0.46	0.54
4	1248.74	**999.39**	1135.47	1416.15	0.00	0.89	0.88	0.72	1259.71	**1007.81**	1145.04	1294.12	0.00	0.91	0.90	0.78
				alpha=.3								**alpha=.4**				
2	1170.67	**936.83**	990.53	1058.35	0.00	0.00	0.00	0.72	1399.27	1331.54	**1330.98**	1418.49	0.00	0.67	0.67	0.18
3	1459.62	**1369.32**	1383.62	1617.55	0.00	0.81	0.74	0.58	1460.67	1371.93	**1371.02**	1431.01	0.00	0.43	0.36	0.05
4	1578.21	**1490.74**	1588.20	1697.29	0.00	0.52	0.45	0.90	1584.68	1492.20	**1491.78**	1637.82	0.00	0.46	0.44	0.98
				alpha=.5								**alpha=.6**				
2	1367.00	1331.93	**1331.64**	1515.28	0.00	0.53	0.54	0.99	1424.28	1329.42	**1329.23**	1335.02	0.00	0.28	0.98	0.49
3	1449.86	1370.96	**1370.94**	1416.15	0.00	0.76	0.74	0.78	1459.35	1371.28	**1370.49**	1371.85	0.00	0.59	0.54	0.77
4	1580.89	1493.65	**1492.52**	1574.47	0.00	0.72	0.93	0.66	1506.27	1490.88	**1490.39**	1522.28	0.00	0.39	0.20	0.08
				alpha=.7								**alpha=.8**				
2	1359.72	1328.10	**1327.58**	1338.91	0.00	0.75	0.75	0.51	1345.63	1329.31	**1328.50**	1392.93	0.00	0.54	0.54	0.85
3	1381.13	1371.92	**1370.65**	1488.82	0.00	0.58	0.42	0.25	1377.23	1368.73	**1367.67**	1569.57	0.00	0.82	0.79	0.75
4	1520.97	1489.84	**1489.27**	1493.47	0.00	0.36	0.06	0.53	1565.12	1489.48	**1489.16**	1702.27	0.00	0.43	0.37	0.90
				alpha=.9								**alpha=1**				
2	1329.49	1327.20	**1326.74**	1472.83	0.00	0.29	0.13	0.41	1381.82	1326.84	**1325.49**	1395.82	0.00	0.46	0.44	0.47
3	1452.77	1370.30	**1368.96**	1384.22	0.00	0.75	0.73	0.47	1412.45	1370.31	**1369.08**	1534.22	0.00	0.42	0.43	0.19
4	1587.61	1487.81	**1486.91**	1678.51	0.00	0.59	0.48	0.19	1494.03	1488.12	**1487.46**	1721.73	0.00	0.29	0.14	0.77
								N=25								
				alpha=.1								**alpha=.2**				
2	3049.13	2958.57	**2690.92**	2757.03	0.00	0.55	0.53	0.90	3017.18	2959.36	**2692.94**	2785.11	0.00	0.13	0.11	0.26
3	4069.91	4062.77	**3533.45**	3689.98	0.00	0.62	0.37	0.83	4096.83	4064.41	**3535.94**	3676.98	0.00	0.67	0.65	0.76
4	4391.00	4316.80	**3453.38**	3484.54	0.00	0.49	0.48	0.95	4329.39	4320.06	**3455.29**	3536.72	0.00	0.96	0.97	0.84
				alpha=.3								**alpha=.4**				
2	3010.88	2961.48	**2694.55**	2746.50	0.00	0.64	0.57	0.47	2976.72	2962.15	**2694.58**	2797.69	0.00	0.48	0.87	0.22
3	3424.77	3387.89	**2947.17**	3128.45	0.00	0.22	0.16	0.79	3439.77	3389.20	**2948.16**	2951.41	0.00	0.29	0.89	0.80
4	3660.10	3602.23	**2881.13**	3090.34	0.00	0.19	0.04	0.07	3702.23	3603.46	**2882.75**	2967.02	0.00	0.06	0.60	0.75
				alpha=.5								**alpha=.6**				
2	3012.37	2963.77	**2696.67**	2856.98	0.00	0.17	0.67	0.71	3017.51	2961.59	**2693.67**	2715.65	0.00	0.96	0.56	0.71
3	3393.24	3390.15	**2948.82**	3101.29	0.00	0.76	0.66	0.75	3685.64	3658.32	**3181.71**	3417.90	0.00	0.28	0.34	0.76
4	3644.41	3607.28	**2885.51**	3095.77	0.00	0.80	0.24	0.84	3962.53	3893.85	**3114.07**	3222.78	0.00	0.66	0.37	0.46
				alpha=.7								**alpha=.8**				
2	3005.79	2958.85	**2691.37**	2908.74	0.00	0.38	0.03	0.61	3017.30	2959.13	**2692.65**	2874.77	0.00	0.65	0.61	0.43
3	3672.06	3656.25	**3180.36**	3386.84	0.00	0.77	0.76	0.49	3721.63	3655.51	**3180.17**	3427.75	0.00	0.21	0.55	0.85
4	3916.80	3893.01	**3114.39**	3159.22	0.00	0.92	0.80	0.15	3959.14	3890.31	**3111.41**	3111.84	0.00	0.29	0.66	0.19
				alpha=.9								**alpha=1**				
2	2972.99	2958.29	**2690.76**	2910.83	0.00	0.10	0.42	0.51	3000.03	2956.80	**2690.36**	2700.24	0.00	0.73	0.25	0.93
3	3673.37	3653.72	**3178.52**	3220.24	0.00	0.88	0.95	0.24	3664.57	3653.35	**3177.73**	3385.55	0.00	0.41	0.17	0.03
4	3916.50	3889.03	**3110.24**	3293.75	0.00	0.43	0.69	0.22	3916.11	3886.59	**3108.20**	3229.48	0.00	0.04	0.56	0.36



(a) Single Allocation

(b) Multiple Allocation

Figure 1: Performance Profile of proposed exact solution methods for CAB and AP dataset

Table 5: Computational performance for all solution approaches for MA. MISOCP referred as MIS in this table

p	CPU Time				Avg % Gap				CPU Time				Avg % Gap			
	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS	MIS	LP-MIS	LR-MIS	LRLP-MIS
								N=50								
				$\alpha = .1$								$\alpha = .2$				
2	4684.35	4666.12	**4152.45**	4331.90	0.00	0.55	0.53	0.88	4684.87	4665.47	**4151.44**	4153.67	0.00	0.13	0.11	0.95
3	6517.90	6481.97	**5639.24**	5704.91	0.00	0.62	0.37	0.98	6533.95	6480.95	**5637.68**	5720.61	0.00	0.67	0.65	0.74
4	7081.35	7065.41	**5651.54**	5883.70	0.00	0.49	0.48	0.65	7154.77	7064.66	**5651.03**	5666.27	0.00	0.96	0.97	0.07
				alpha=.3								**alpha=.4**				
2	4965.15	4918.64	**4474.89**	4485.50	0.00	0.30	0.48	0.94	4695.73	4684.94	**4169.51**	4357.43	0.00	0.38	0.87	0.52
3	6889.72	6832.09	**5942.60**	5983.14	0.00	0.91	0.67	0.87	5619.82	5570.56	**4956.56**	5197.23	0.00	0.31	0.89	0.58
4	7175.92	7173.93	**5737.99**	5981.99	0.00	0.19	0.49	0.81	6772.36	6679.81	**5943.65**	5974.40	0.00	0.03	0.60	0.03
				alpha=.5								**alpha=.6**				
2	4951.18	4938.84	**4493.38**	4518.18	0.00	0.17	0.45	0.69	4799.02	4704.07	**4185.69**	4395.35	0.00	0.11	0.56	0.91
3	5941.31	5872.43	**5107.71**	5153.29	0.00	0.65	0.90	0.32	5645.05	5593.27	**4976.66**	5141.80	0.00	0.87	0.34	0.89
4	7077.17	7041.51	**5631.90**	5698.97	0.00	0.42	0.72	0.98	6778.15	6707.14	**5968.73**	6178.66	0.00	0.42	0.37	0.71
				alpha=.7								**alpha=.8**				
2	5048.30	4959.12	**4511.45**	4712.21	0.00	0.76	0.15	0.21	4764.51	4723.59	**4297.24**	4339.46	0.00	0.94	0.59	0.44
3	5933.30	5896.41	**5128.80**	5277.82	0.00	0.54	0.73	0.55	5678.65	5617.15	**4998.65**	5051.51	0.00	0.25	0.58	0.81
4	7142.87	7070.14	**5654.74**	5664.47	0.00	0.16	0.60	0.70	6784.78	6735.50	**6196.55**	6370.84	0.00	0.76	0.95	0.65
				alpha=.9								**alpha=1**				
2	4983.61	4979.86	**4581.05**	4618.89	0.00	0.76	0.09	0.35	4768.22	4742.85	**4361.96**	4463.58	0.00	0.89	0.25	0.64
3	5968.31	5921.60	**5446.68**	5530.97	0.00	0.96	0.08	0.18	5698.21	5640.05	**5188.17**	5237.19	0.00	0.62	0.17	0.90
4	7176.24	7100.25	**6532.17**	6579.85	0.00	0.88	0.57	0.08	6822.20	6762.52	**6287.95**	6495.24	0.00	0.25	0.56	0.21

6 Conclusions and Directions for Future Research

In this paper, we proposed two alternate formulations for competitive hub location problem , wherein an entrant is making a strategic decision of locating its hubs, thereby routing its traffic in a market with already existing competing players. The routes can either be based on single allocation or multiple allocation, which result in the aforementioned two formulations. The entrant has the objective to maximize its market share, which is a function of the utility that customers get from using its services. Both the classes of problems are non-linear IP, which is computationally intractable. Papers that have studied similar problems (Marianov et al., 1999; Eiselt and Marianov, 2009) have resorted to heuristics, even when they are not deciding on the routes of the passenger. In this paper, we proposed four alternate solution approaches, MISOCP, LP-MISOCP, LR-MISOCP, LRLP-MISOCP.

Besides proposing a new formulation for the problem, our other contributions lie in second order conic reformulation, in using lifted polymatroid cuts to approximate second order cone constraints and also in using second order cone programming within Lagrangian relaxation, to solve the problem efficiently for the two allocation classes. For the single allocation type problem, the method LP-MISOCP is the most efficient and is able to solve all the problem instances within 1% optimality gap in less than 1.7 hours of CPU time. For the multiple allocation class problem, the method LR-MISOCP is the more efficient one for larger test instances but LP-MISOCP performs better for smaller test instances.

Prompted by our success in the current study, we foresee the application of SOCP approximation using lifted polymatroid cuts in similar classes of problems, where non-linearity may arise due to competition, congestion, economies of scale, uncertainties, etc. The problem studied in this paper does not take into account the response of the competing firms to the entrant's actions. A possible extension of this work could be to model it as a leader-follower game, wherein the entrant takes into account the follower's response while solving its problem.

References

Aktürk, M. S., Atamtürk, A., and Gürel, S. (2014). Aircraft rescheduling with cruise speed control. *Operations Research*, 62(4):829–845.

Alamdari, F. E. and Black, I. G. (1992). Passengers' choice of airline under competition: the use of the logit model. *Transport Reviews*, 12(2):153–170.

Alizadeh, F. and Goldfarb, D. (2003). Second-order cone programming. *Mathematical programming*, 95(1):3–51.

Alumur, S. and Kara, B. Y. (2008). Network hub location problems: The state of the art. *European journal of operational research*, 190(1):1–21.

Antoniou, A. and Lu, W.-S. (2007). Semidefinite and second-order cone programming. *Practical Optimization: Algorithms and Engineering Applications*, pages 449–499.

Atamtürk, A. and Narayanan, V. (2008). Polymatroids and mean-risk minimization in discrete optimization. *Operations Research Letters*, 36(5):618–622.

Aykin, T. (1994). Lagrangian relaxation based approaches to capacitated hub-and-spoke network design problem. *European Journal of Operational Research*, 79(3):501–523.

Campbell, J., Ernst, A., and Krishnamoorthy, M. (2002). Hub location problems.

Campbell, J. F. (1992). Location and allocation for distribution systems with transshipments and transportion economies of scale. *Annals of operations research*, 40(1):77–99.

Campbell, J. F. (1994). Integer programming formulations of discrete hub location problems. *European Journal of Operational Research*, 72(2):387–405.

Campbell, J. F. and O'Kelly, M. E. (2012). Twenty-five years of hub location research. *Transportation Science*, 46(2):153–169.

de Dios Ortuzar, J. and Willumsen, L. G. (2011). *Modelling transport*. John wiley & sons.

Dolan, E. D. and Moré, J. J. (2002). Benchmarking optimization software with performance profiles. *Mathematical programming*, 91(2):201–213.

Edmonds, J. and Fulkerson, D. R. (1970). Bottleneck extrema. *Journal of Combinatorial Theory*, 8(3):299–306.

Eiselt, H. A. and Marianov, V. (2009). A conditional p-hub location problem with attraction functions. *Computers & Operations Research*, 36(12):3128–3135.

Ernst, A. T. and Krishnamoorthy, M. (1996). Efficient algorithms for the uncapacitated single allocation p-hub median problem. *Location science*, 4(3):139–154.

Fisher, M. L. (1981). The lagrangian relaxation method for solving integer programming problems. *Management science*, 27(1):1–18.

Grove, P. G. and O'Kelly, M. E. (1986). Hub networks and simulated schedule delay. *Papers in Regional Science*, 59(1):103–119.

Hamacher, H. W., Labbé, M., Nickel, S., and Sonneborn, T. (2004). Adapting polyhedral properties from facility to hub location problems. *Discrete Applied Mathematics*, 145(1):104–116.

Held, M., Wolfe, P., and Crowder, H. P. (1974). Validation of subgradient optimization. *Mathematical programming*, 6(1):62–88.

Huff, D. L. (1964). Defining and estimating a trading area. *The Journal of Marketing*, pages 34–38.

Huff, D. L. (1966). A programmed solution for approximating an optimum retail location. *Land Economics*, 42(3):293–303.

Ishfaq, R. and Sox, C. R. (2011). Hub location–allocation in intermodal logistic networks. *European Journal of Operational Research*, 210(2):213–230.

Kara, B. Y. and Taner, M. R. (2011). Hub location problems: the location of interacting facilities. In *Foundations of location analysis*, pages 273–288. Springer.

Kara, B. Y. and Tansel, B. C. (2000). On the single-assignment p-hub center problem. *European Journal of Operational Research*, 125(3):648–655.

Lin, C.-C. and Lee, S.-C. (2010). The competition game on hub network design. *Transportation Research Part B: Methodological*, 44(4):618–629.

Lüer-Villagra, A. and Marianov, V. (2013). A competitive hub location and pricing problem. *European Journal of Operational Research*, 231(3):734–744.

Mahmutogullari, A. I. and Kara, B. Y. (2016). Hub location under competition. *European Journal of Operational Research*, 250(1):214–225.

Marianov, V., Serra, D., and ReVelle, C. (1999). Location of hubs in a competitive environment. *European Journal of Operational Research*, 114(2):363–371.

Mirchandani, P. B., Oudjit, A., and Wong, R. T. (1985). 'multidimensional'extensions and a nested dual approach for the m-median problem. *European Journal of Operational Research*, 21(1):121–137.

Narula, S. C., Ogbu, U. I., and Samuelsson, H. M. (1977). An algorithm for the p-median problem. *Operations Research*, 25(4):709–713.

O'kelly, M. E. (1986). The location of interacting hub facilities. *Transportation science*, 20(2):92–106.

O'kelly, M. E. (1987). A quadratic integer program for the location of interacting hub facilities. *European journal of operational research*, 32(3):393–404.

Pirkul, H. and Schilling, D. A. (1991). The maximal covering location problem with capacities on total workload. *Management Science*, 37(2):233–248.

Pirkul, H. and Schilling, D. A. (1998). An efficient procedure for designing single allocation hub and spoke systems. *Management Science*, 44(12-part-2):S235–S242.

Plastria, F. (2001). Static competitive facility location: an overview of optimisation approaches. *European Journal of Operational Research*, 129(3):461–470.

Rumelt, R. P. and Wensley, R. (1981). In search of the market share effect. In *Academy of management proceedings*, volume 1981, pages 2–6. Academy of Management Briarcliff Manor, NY 10510.

Sasaki, M. (2005). Hub network design model in a competitive environment with flow threshold. *Journal of the Operations Research Society of Japan*, 48(2):158–171.

Sasaki, M., Campbell, J. F., Krishnamoorthy, M., and Ernst, A. T. (2014). A stackelberg hub arc location model for a competitive environment. *Computers & operations research*, 47:27–41.

Sasaki, M. and Fukushima, M. (2001). Stackelberg hub location problem. *Journal of the Operations Research Society of Japan*, 44(4):390–402.

Sen, A., Atamturk, A., and Kaminsky, P. (2017). A conic integer programming approach to constrained assortment optimization under the mixed multinomial logit model. *arXiv preprint arXiv:1705.09040*.

Skorin-Kapov, D., Skorin-Kapov, J., and O'Kelly, M. (1996). Tight linear programming relaxations of uncapacitated p-hub median problems. *European journal of operational research*, 94(3):582–593.

Vielma, J. P., Ahmed, S., and Nemhauser, G. L. (2008). A lifted linear programming branch-and-bound algorithm for mixed-integer conic quadratic programs. *INFORMS Journal on Computing*, 20(3):438–450.

Wagner, B. (2008). A note on "location of hubs in a competitive environment". *European Journal of Operational Research*, 184(1):57–62.

7 Appendix

The objective function (5) in $MCOHLP$ can be rewritten as :

$$\theta(x,z) = \sum_i \sum_j \hat{f}_i - \sum_i \sum_j \hat{f}_i + \sum_i \sum_j f_{ij} \left\{ \frac{\sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x_{ijkl}}{\sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x_{ijkl} + \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl}} \right\}, \quad (45)$$

where $\hat{f}_i = \max_j f_{ij}$. Further, using the newly defined sets of variables, (45) can be restated as:

$$\theta(o) = \sum_{i,j} \hat{f}_i - \sum_{i,j} \hat{f}_i + \sum_i \sum_j f_{ij} o_{ij} \quad (46)$$

From (23) & (25), we can conclude that:

$$o_{ij} + \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} a_{ij} = 1 \quad (47)$$

Introducing the relation (47) in (46), we get:

$$\theta(a,o) = \sum_i \sum_j \hat{f}_i - \sum_i \sum_j \hat{f}_i \Big(o_{ij} + \sum_{k\in H^c} \sum_{l\in H^c} u^e_{ijkl} a_{ij} \Big) + \sum_i \sum_j f_{ij} o_{ij} \quad (48)$$

On rearranging the terms, (48), can be restated as:

$$\theta(a,o) = \sum_i \sum_j \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} x^c_{ijkl} \hat{f}_i a_{ij} + \sum_j (\hat{f}_i - f_{ij}) o_{ij} \right\}$$

Using the above form of the objective function, $SACOHLP$ can be restated as:

$$\max \quad \sum_i \sum_j \hat{f}_i - \sum_i \left\{ \sum_j \sum_{k\in H^c} \sum_{l\in H^c} u^c_{ijkl} \hat{f}_i a_{ij} + \sum_j (\hat{f}_i - f_{ij}) o_{ij} \right\} \quad (49)$$

$$\text{s.t.}(6) - (11)$$

$$a_{ij} g_{ij} \geqslant 1 \qquad \forall\, i,j \in N \quad (50)$$

$$o_{ij} g_{ij} \geqslant \sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x_{ijkl} \qquad \forall\, i,j \in N \quad (51)$$

$$a_{ij}, g_{ij}, o_{ij} \geqslant 0 \qquad \forall\, i,j \in N \quad (52)$$

In (49)-(52), the constraint set (50) is derived using (23) and (24). Constraint set (51) is derived using (23) and (25). Also, since $x_{ijkl} \in \{0,1\}$, (51) can be rewritten as $o_{ij} g_{ij} \geqslant \sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x^2_{ijkl}\ \forall\, i,j \in N$, which is a rotated second order conic constraint. The rotated second order cone (SOC) can be transformed to a standard SOC as follows (Alamdari and Black, 1992):

$$(o_{ij} + g_{ij})^2 \geqslant 2 \sum_{k\in H^e} \sum_{l\in H^e} u^e_{ijkl} x^2_{ijkl} + o^2_{ij} + g^2_{ij} \quad \forall\, i,j \in N \quad (53)$$

Similarly, (50), which is also a rotated SOC constraint, can be transformed to a standard SOC as follows:

$$(a_{ij} + g_{ij})^2 \geqslant 2 + a^2_{ij} + g^2_{ij} \quad \forall\, i,j \in N \quad (54)$$

The vector X_{ij} represents N^2+2 elements, with the first N^2 elements being x_{ijkl} and the last two elements being o_{ij}, g_{ij} for a given i, j. The vector $X_{ij}^{'}$ corresponds to the maximum value of every X_{ij}. We add three different types of cuts for each SOC constraint set to the separation problem. The first type of cut is:

$$\pi X_{ij}^{'} \leqslant o_{ij} + g_{ij} + \alpha' \Big\{ X_{ij}' - X_{ij} \Big\} \quad \forall i, j \tag{55}$$

Here $\alpha_{ij}' = \frac{a_{ij}}{\sqrt{a_1 + a_2 + \cdots + a_{ij}}}$, where a_{ij} has N^2+2 elements. The first N^2 elements being $2u_{ijkl}^e$ and the last two being 1 each. The second type of cut is written as follows:

$$\kappa_1 + \frac{1}{f_1(\hat{X}_{ij}, \hat{X}_{ij}^{'})} \left\{ \tau_1(\hat{X}_{ij}, \hat{X}_{ij}^{'}) \left\{ \pi_{S^a} X_{S^a}^{'} - \alpha_{S^a}' \Big\{ X_{S^a}^{'} - X_{S^a} \Big\} \right\} + \sum_{m \in N^2+2 \setminus S^a} a_{ijm} X_{ijm} \hat{X}_{ijm} \right\} \leqslant o_{ij} + g_{ij} \tag{56}$$

$$f_1(X_{ij}, X_{ij}^{'}) = \sqrt{\tau_1(X_{S^a}, X_{S^a}^{'})^2 + \sum_{m \in N^2+2 \setminus S^a} a_{ijm} X_{ijm}^2} \tag{57}$$

$$\tau_1(X_{S^a}, X_{S^a}^{'}) = \pi_{s^a} X_{S^a}^{'} - \alpha_{S^a}' \Big\{ X_{S^a}^{'} - X_{S^a} \Big\} \tag{58}$$

$$\kappa_1 = f_1(\hat{X}_{ij}, \hat{X}_{ij}^{'}) - \frac{\tau_1^2(\hat{X}_{ij}, \hat{X}_{ij}^{'}) + \sum_{m \in N^2+2 \setminus S^a} a_{ijm} \hat{X}_{ijm}^2}{f_1(\hat{X}_{ij}, \hat{X}_{ij}^{'})} \tag{59}$$

Likewise, the third type of cut is written as follows:

$$\kappa_2 + \frac{1}{f_2(\hat{X}_{ij}, \hat{X}_{ij}^{'})} \left\{ \tau_2(\hat{X}, \hat{X}^{'}) \left\{ \pi_{S^a} X_{S^a}^{'} - \alpha_{S^a}(X^{'} - X) + \sum_{m \in T} \frac{a_{ijm} \hat{X}_{ijm}}{\nu \hat{X}_{ij}} X_{ijm} \right\} + \sum_{m \in N^2+2 \setminus (S^a \cup T)} a_{ijm} \hat{X}_{ijm} X_{ijm} \right\} \leqslant o_{ij} + g_{ij} \tag{60}$$

$$f_2(X, X^{'}) = \sqrt{\tau_2(X_{S^a}^{'}, X_{S^a \cup T})^2 + \sum_{m \in N^2+2 \setminus (S \cup T)} a_{ijm} X_{ijm}^2} \tag{61}$$

$$\tau_2(X_{S^a}^{'}, X_{S \cup T}) = \pi_{S^a} X^{'} + \nu(X_T) - \alpha_{S^a}(X_{S^a}^{'} - X_{S^a}) \tag{62}$$

$$\nu(X_T) = \sqrt{\sum_{m \in T} a_{ijm} X_{ijm}^2} \tag{63}$$

$$\kappa_2 = f_2(\hat{X}, \hat{X}^{'}) - \frac{\tau_2(\hat{X}_S, \hat{X}_{S \cup T}^{'}) \Big\{ \pi_S \hat{X}_S^{'} - \alpha_S(\hat{X}_{S^a}^{'} - \hat{X}_{S^a}) + \sum_{m \in T} \frac{a_m}{\nu \hat{X}_m} \hat{X}_m^2 \Big\} + \sum_{m \in N^2+2 \setminus (S \cup T)} a_m \hat{X}_m^2}{f_2(\hat{X}, \hat{X}^{'})} \tag{64}$$