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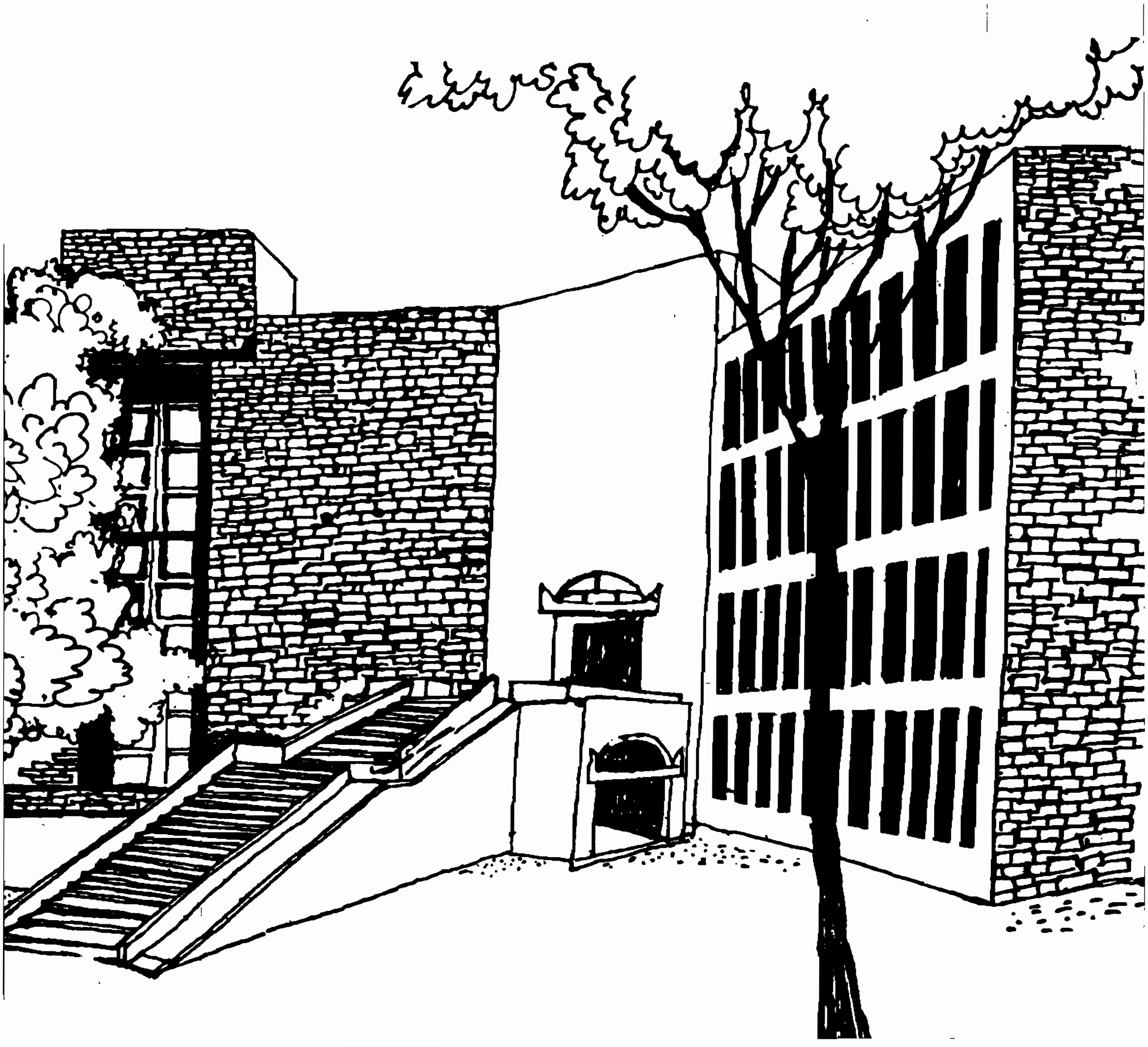

A NOTE ON AXIOMATIC CHARACTERIZATIONS OF THE NASH BARGAINING SOLUTION

By

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Abstract

In this paper, to begin with we present a generalization of the independence of irrelevant expansions assumption to the situation with an arbitrary yet finite number of players, and with the help of a comparatively simpler proof than the one suggested by Thomson (1981), we uniquely characterize the Nash bargaining solution.

In a recent paper, Lahiri (1993) introduces the concept of a shift for bargaining problems. A shift for a bargaining problem amounts to a displacement of the origin to a point in the nonnegative orthant of a finite dimensional Euclidean space (in which the bargaining problem is defined) so as to reduce the original problem to a new one consisting only of those points that weakly Pareto dominate the new origin. A characterization of Nash bargaining solution is also obtained in this paper using a convexity assumption. A related version of this convexity assumption and a similar characterization theorem can be found in Chun and Thomson (1990) and Peters (1992). An intermediate property used in the latter characterization called localization, which can be found in Peters (1992) is similar in spirit to the independence of irrelevant alternatives assumption. We also obtain a characterization of the Nash solution, by relaxing this localization property and invoking Pareto continuity.

1. Introduction :- In Thomson (1981) can be found a characterization of the Nash (1950) bargaining solution, which relies on an axiom called independence of irrelevant expansions. This axiom replaces Nash's original axiom called independence of irrelevant alternatives assumption.

The independence of irrelevant alternatives assumption states that if while contracting a bargaining problem, we retain the solution to the original problem, then the solution to the new problem should coincide with the solution to the old one. Independence of irrelevant expansions, proceeds in the reverse direction: if a bargaining problem is expanded to a new bargaining problem by adding points below a specific supporting line at the original solution point, then the new solution outcome should weakly Pareto dominate the old one.

Thomson (1981), stated the axiom in the context of two person bargaining problems, and thus was able to uniquely characterize the two person Nash bargaining solution.

In this paper, to begin with we present a generalization of the independence of irrelevant expansions assumption to the situation with an arbitrary yet finite number of players, and with the help of a comparatively simpler proof than the one suggested by Thomson, we uniquely characterize the Nash bargaining solution.

In a recent paper, Lahiri (1993) introduces the concept of a shift for bargaining problems. A shift for a bargaining problem amounts to a displacement of the origin to a point in the nonnegative orthant of a finite dimensional Euclidean space (in which the bargaining problem is defined) so as to reduce the original problem to a new one consisting only of those points that weakly Pareto dominate the new origin. A characterization of Nash bargaining solution is also obtained in this paper using a convexity assumption. A related version of this convexity assumption and a similar characterization theorem can be found in Chun and Thomson (1990) and Peters (1992). An intermediate property used in the latter characterization called localization, which can be found in Peters (1992) is similar in spirit to the independence of irrelevant alternatives assumption. We also obtain a characterization of the Nash solution, by relaxing this localization property and invoking Pareto continuity.

2. The Model :- Following Moulin (1988), we adopt the following framework of analysis: an n-person bargaining problem is a non-empty subset S of $\mathbb{R}^n_+$. We consider the following class Σ of n-person bargaining problems:

$S \in \Sigma$ if and only if

(i) $0 \in S$

(ii) S is compact and convex

(iii) S is comprehensive i.e. $y \leqq x, x \in S \Rightarrow y \in S$, where $x, y \in \mathbb{R}^n_+$

(iv) S satisfies minimal transferability i.e. $\forall x \in S$, $\forall i \in \{1,\dots,n\}$, $x_i > 0$ implies $\exists\ y \in S$ with $y_i < x_i$ and $y_j > x_j$ $\forall j \in \{1,\dots,n\}, j \neq i$.

Properties (i), (ii) and (iii) are standard in the literature. Property (iv) helps to equate weak Pareto optimality with Pareto optimality.

A <u>bargaining solution</u> on D is a function $F: D \to \mathbb{R}^n_+$ such that $F(S) \in S\ \forall S \in D$, where $D \subseteq \Sigma$. D is called a domain.

We require the following axioms to be satisfied by a bargaining solution F:

<u>Axiom 1</u> :- (<u>Pareto Optimality</u>): $F(S) \in P(S) \equiv \{x \in S / y \geqq x, y \in S \Rightarrow x = y\}$ $\forall\ S \in \Sigma$.

<u>Axiom 2</u> :- (<u>Scale Invariance</u>): $\forall x \in \mathbb{R}^n, \forall a \in \mathbb{R}^n_{++}$, $x = (x_1, \dots, x_n)$, $a = (a_1, \dots, a_n)$ define $ax = (a_1 x_1, \dots, a_n x_n)$. If $S \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n_{++}$, define $aS \equiv \{ax / x \in S\}$. We require, $\forall\ a \in \mathbb{R}^n_{++}$, $\forall\ S \in \Sigma$, $F(aS) = aF(S)$.

<u>Axiom 3</u> :- (<u>Symmetry</u>): Given a one-to-one function $\sigma : \{1,\dots,n\} \to \{1,\dots,n\}$ and $x \in \mathbb{R}^n$, define $\sigma(x) \in \mathbb{R}^n$ as follows: $\forall i \in \{1,\dots,n\}$, $\sigma_i(x) = x_{\sigma(i)}$. We require the following: if $S = \sigma(S)\ \forall \sigma$ as above then $F_i(S) = F_j(S)\ \forall i, j \in \{1,\dots,n\}$.

Axiom 4 :- (Independence of Irrelevant Expansions) :- $\forall S \in \Sigma$, $\exists p \in \mathbb{R}^n_+$ with $\Sigma^n_{i=1} p_i = 1$ such that

(i) $p.x = p.F(S)$ is the equation of a supporting line of S at F(S)

(ii) $\forall$ $T \in \Sigma$ with $S \subseteq T$ and $p.x \leq p.F(S)$ $\forall x \in T$, we have $F(T) \geq F(S)$.

(Here $p.x = \Sigma^n_{i=1} p_i x_i$).

Axiom 5 :- (Strict Individual Rationality) :- $F(S) >> 0$ $\forall$ $S \in \Sigma$ with $S \neq \{0\}$.

Nash (1950) defined the following solution: $N: \Sigma \to \mathbb{R}^n_+$ is defined as $N(S) = \underset{x \in S}{\text{argmax}} \, \pi^n_{i=1} x_i$.

3. A First Theorem :-

Theorem 1 :- The only solution to satisfy Axioms 1 to 5 on Σ is N.

Proof :- Nash (1950) showed that N satisfied Axiom 1, 2 and 3 and 5. Axiom 4 is valid with $p_i = \dfrac{1/N_i(S)}{\Sigma^n_{i=1} 1/N_i(S)}$, $i=1,\ldots,n$,

whenever $S \in \Sigma$, $S \neq \{0\}$.

To prove the converse we need the following Lemma.

Lemma 1 :- Let $D = \{S \in \Sigma / x \in P(S),\ x >> 0 \Rightarrow \exists$ a unique $p \in \mathbb{R}^n_+$ with $\Sigma^n_{i=1} p_i = 1$, such that $p.y = p.x$ is the equation of a supporting line of S at x$\}$.

If F satisfies Axioms 1 to 4 on D then $F(S) = N(S)$ $\forall$ $S \in D$.

Proof :- Let $S \in D$. If $S = \{0\}$, there is nothing to prove. Hence assume $S \neq \{0\}$. Thus $N(S) >> 0$, By Axiom 2, we may assume $N(S) = e$ (the vector in $\mathbb{R}^n$ with all co-ordinates equal to one).

Let $\Delta^{n-1} = \{x \in \mathbb{R}^n_+ / \Sigma^n_{i=1} x_i \leq 1\}$.

P (Δ^{n-1}) supports S at e.

By Axioms 1 and 3, $F(\Delta^{n-1}) = e$

Let $T = \{x \in S / \sigma(x) \in S$ $\forall$ $\sigma : \{1,\ldots,n\} \to \{1,\ldots n\}$ which are one to one$\}$.

$e \in T$ (the largest symmetric set in D, which is contained in S).

By Axioms 1 and 3, $F(T) = e$.

Further $P(\Delta^{n-1})$ supports T uniquely at e.

$\therefore$ By Axiom 4, $F(S) \geqq F(T) = e$.

But e is Pareto optimal in S.

$\therefore$ $F(S) = e$.

To prove the main theorem, let us take any $S \neq \{0\}$, $S \in \Sigma$. By Axioms 2 and 5, we may assume $F(S) = e$. We have to show that the vector p in Axiom 4 is $(1/n)e$. Suppose not i.e. suppose $p \neq (1/n)e$. Then we can always find a $T \in D$ such that

(i) $S \subseteq T$; (ii) $p.x \leqq p.e \; \forall \; x \in T$; (iii) $e \in T$

By Lemma 1, since $T \in D$, $F(T) = N(T)$

By Axiom 4, $F(T) \geqq F(S) = e$

Since e is Pareto optimal in T, $e = F(T) = N(T)$

But since p uniquely supports T at e, $p = (1/n)e$ (since $F(T) = N(T)$)

This contradiction establishes the theorem.

Q.E.D.

4. Shifts in Bargaining Problems :- Given $S \in \Sigma$, $c \in \mathbb{R}^n_+$, let $S(c) = (S - \{c\}) \cap \mathbb{R}^n_+$.

Let $D \subseteq \Sigma$ be a given domain and $F: D \to \mathbb{R}^n_+$ be a bargaining solution.

F is said to satisfy <u>strong convexity</u> if $\forall$ S, $S(c) \in D$, $F(S(c)) = \alpha c$ for some $\alpha \geqq 0$, then $F(S) = \mu c + F(S(\mu c))$ for $0 \leqq \mu \leqq 1$ whenever $S(\mu c) \in D$.

F is said to satisfy <u>weak convexity</u> if $\forall$ $S \in D$ and $0 \leqq \mu \leqq 1$, $F(S) = \mu F(S) + F(S(\mu F(S)))$ whenever $S(\mu F(S)) \in D$.

Taking $c = F(S)$ and $\alpha = 0$, it is easy to see that strong convexity implies weak convexity on Σ.

The definitions are a minor adaptation of similar definitions in Peters (1992). Convexity refers to a possibility of considering the variable origin to lie anywhere on the straightline connecting the old origin to the origin of the shifted bargaining problem.

5. Towards a second characterization theorem :- We begin with a lemma.

Lemma 2 :- If $F:\Sigma \to \mathbb{R}^n_+$ satisfies Axiom 5 and weak convexity, then it must satisfy Axiom 1.

Proof :- Let $S \in \Sigma$, $S \neq \{0\}$ be given. Suppose $F(S) \notin P(S)$. By SIR, since $S(F(S)) \neq \{0\}$, $F(S(F(S))) \gg 0$.

By weak convexity, $F(S)=F(S)+F(S(F(S))) \gg F(S)$ which is a contradiction.

Hence $F(S) \in P(S)$.

Q.E.D.

Let us invoke the following property for $F:D \to \mathbb{R}^n_+$:

Localization :- $\forall$ $S,T \in D$ if $U \cap S = U \cap T$ for an open neighbourhood U of $F(S)$, then $F(S)=F(T)$.

Lemma 3 :- Let $F:\Sigma \to \mathbb{R}^n_+$ satisfy strong convexity and Axiom 5. Then F satisfies localization.

Proof :- Let S, $T \in \Sigma$ with $S \neq \{0\}$, $T \neq \{0\}$ and suppose there exists a neighbourhood of $F(S)$ such that $U \cap S = U \cap T$

Now there exists $0 < \mu^* < 1$ such that $c = \mu^* F(S) \in U \cap S$.

By strong convexity (which implies weak convexity)

$c+F(S(c))=F(S)$

Now $S(c)=T(c)$

$\therefore$ $F(S(c))=F(T(c))$

By strong convexity, $c+F(T(c))=F(T)$

$\therefore$ $F(S)=F(T)$.

Q.E.D.

Lemma 4 :- The only solution on Σ to satisfy Axioms 1, 2, 3 and localization is N.

Proof :- That N satisfies the above properties is clear. So, let us establish the converse. Let $S \in \Sigma$, $S \neq \{0\}$ and suppose towards a contradiction that for an F satisfying the above properties $F(S) \neq N(S)=z$. By Strong Individual Rationality and Scale Invariance of F, we may assume $F(S)=e$. Let $T=\{x \in \mathbb{R}^n_+ / \Sigma^n_{i=1} x_i/z_i < n\}$. Observe

$z \gg 0$ by Strong Individual Rationality of N.

By Axioms 1, 2, 3, $F(T)=z=N(T)$ since $F(\Delta)=e=N(\Delta)$ where $\Delta=\{x\in \mathbb{R}^n_+ / \Sigma^n_{i=1} x_i \leq n\}$.

Suppose $\Sigma^n_{i=1} 1/z_i < n$. Hence there exists $0<\alpha 1$ such that $\Sigma^n_{i=1} 1/z_i < (1-\alpha)n$.

Thus $e\in S \cap K$ where $K=\{x\in \mathbb{R}^n_+ / \Sigma^n_{i=1} x_i / z_i \leq (1-\alpha)n\}$. Further, there exists a neighbourhood U of e such that $U \cap S=U \cap (S \cap K)$. By localization $F(S)=F(S \cap K)$. Thus $F(S \cap K)=e$.

On the other hand, $F(K)=(1-\alpha)F(T)=(1-\alpha)z$.

Further $(1-\alpha)z$ belongs to the interior of S.

Thus there exists a neighbourhood V of $(1-\alpha)z$ such that $V \cap (S \cap K)=V \cap K$.

By localization $F(S \cap K)=F(K)$

$\therefore (1-\alpha)z=e$.

But $(1-\alpha)z$ is not Pareto optimal in S whereas e is. This contradiction establishes $\Sigma^n_{i=1} 1/z_i = n$.

Now suppose $z \neq e$.

Since S is convex $y=1/2(z+e)\in S$

Note that the function $\mathbb{R}_{++} \to \mathbb{R}_{++} :: \beta \mapsto 1/\beta$ is strictly convex. Thus $\Sigma^n_{i=1} 1/y_i < 1/2\ \Sigma^n_{i=1} 1/z_i + (1/2)n=n$.

Let $W=\{x\in \mathbb{R}^n_+ / \Sigma^n_{i=1} x_i/y_i \leq n\}$.

$N(W)=y=F(W)$. Thus there exists $0<\alpha<1$, such that $\Sigma^n_{i=1} 1/y_i < (1-\alpha)n$. By the same argument as above we get now $\Sigma^n_{i=1} 1/y_i = n$ which is a contradiction.

Hence $z=e$.

Q.E.D.

As an immediate consequence of the above lemma we have:

<u>Theorem 2</u> :- The only solution on Σ to satisfy Axioms 1, 2, 3, 5 and strong convexity is N.

<u>Proof</u> :- By lemmas 3 and 4 the proof is immediate.

Q.E.D.

By invoking a continuity assumption, we can obtain a characterization of the Nash bargaining solution using a property slightly weaker than localization.

Weak Localization :- $\forall S, T \in D$ if $U \cap P(S) = U \cap P(T)$ for an open neighbourhood U of $F(S)$ and if the set $U \cap P(S)$ is a nonempty, nondegenerate, convex set, then $F(S)=F(T)$

The proposed continuity property is derived as follows:

Let $S, T \subseteq \mathbb{R}^n$ be nonempty closed sets. Then the Hausdorff distance between S and T denoted $d(S,T)$ is defined by

$$d(S,T) = \sup_{x \in S, y \in T} \{ \inf_{x \in T} \| x-z \|, \inf_{z \in S} \| y-z \| \}$$

where the norms in the definition are Euclidean norms.

A sequence of closed sets $S^1, S^2, \ldots$ is said to converge to a closed set S if $\lim_{k \to \infty} d(S^k, S) = 0$.

Axiom 6 :- (Pareto Continuity) :- For every sequence $S^1, S^2, \ldots, S^k, \ldots \in D$ and $S \in D$, if $\lim_{k \to \infty} d(P(S^k), P(S)) = 0$,

then $\lim_{k \to \infty} F(S^k) = F(S)$.

Theorem 3 :- The only solution on Σ to satisfy Axioms 1, 2, 3, 6 and Weak Localization is N.

Proof :- That N satisfies the above assumptions is obvious. Thus let F be any solution on Σ and let $S \in \Sigma$, $S \neq \{0\}$. We may assume by Axioms 1, 2 and 5 that $N(S)=e$. To show $F(S)=e$. Let $\blacktriangle = \{x \in \mathbb{R}^n / \Sigma^n_{i=1} x_i \leq n$. By Axioms 1 and 3, $F(\blacktriangle) = e$ and by Axiom 2, $F(\alpha \blacktriangle) = \alpha e$ $\forall$ $\alpha \geq 0$ where $\alpha \blacktriangle = \{\alpha x / x \in \blacktriangle\}$.

Let $T(\alpha) = (\alpha \blacktriangle) \cap S$.

By, Weak Localization $F(T(\alpha)) = F(\alpha \blacktriangle) = \alpha e$ $\forall$ $0 < \alpha < 1$.

Let $\{\alpha_k\}^\infty_{k=1}$ be a sequence in the open interval $(0,1)$ such that $\lim_{k \to \infty} \alpha_k = 1$

Then $\lim_{k \to \infty} d(P(T(\alpha_k)), P(S)) = 0$

By Axiom 6, $\lim_{k \to \infty} F(T(\alpha_k)) = F(S)$

i.e. $F(S) = e$.

Q.E.D.

An example below illustrates the necessity of Axiom 6 in Theorem 3 above.

<u>Example</u> :- Let $D_1 = \{S \in \Sigma / \exists\ x, y \in P(S)$ such that $\forall\ t \in [0,1], tx+(1-t)y \in P(S)\}\ x \neq y$

$D_2 = \{S \in \Sigma / \exists$ a neighbourhood of U of $N(S)$ such that $U \cap P(S)$ is a convex set containing $N(S)$ in its relative interior$\}$.

$\forall\ S \in D_1\ (D_2)\ \forall\ a \in \mathbb{R}^n_{++},\ aS \in D_1\ (D_2)$

$\forall S \in \Sigma \setminus D_1\ (\Sigma \setminus D_2)\ \forall\ a \in \mathbb{R}^n_{++},\ aS \in \Sigma \setminus D_1\ (\Sigma \setminus D_2)$.

Let $u_i\ (S) = \max\{x_i / x \in S\}$ and $u(S) = (u_1\ (S), \ldots, u_n(S))$

Define $K(S) = \bar{\alpha}(S)u(S)$ where $\bar{\alpha}(S) = \max\{\alpha \in \mathbb{R}_+ / \alpha u(S) \in S\}\ \forall\ S \in \Sigma$.

$$\text{Let } F^1\ (S) = \begin{cases} N(S) & \text{if } S \in D_1 \\ K(S) & \text{if } S \in \Sigma \setminus D_1 \end{cases}$$

$$F^2\ (S) = \begin{cases} N(S) & \text{if } S \in D_2 \\ K(S) & \text{if } S \in \Sigma \setminus D_2 \end{cases}$$

Both F_1 and F_2 satisfy Axioms 1, 2, 3 and Weak Localization. Neither satisfies Axiom 6, however.

6. Conclusion :- In the concluding section we clarify some vector notation used in the paper: for $x, y \in \mathbb{R}^n$

(i) $x \geq y$ means $x_i \geq y_i\ \forall\ i \in \{1, \ldots, n\}$

(ii) $x >> y$ means $x_i > y_i\ \forall\ i \in \{1, \ldots, n\}$.

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