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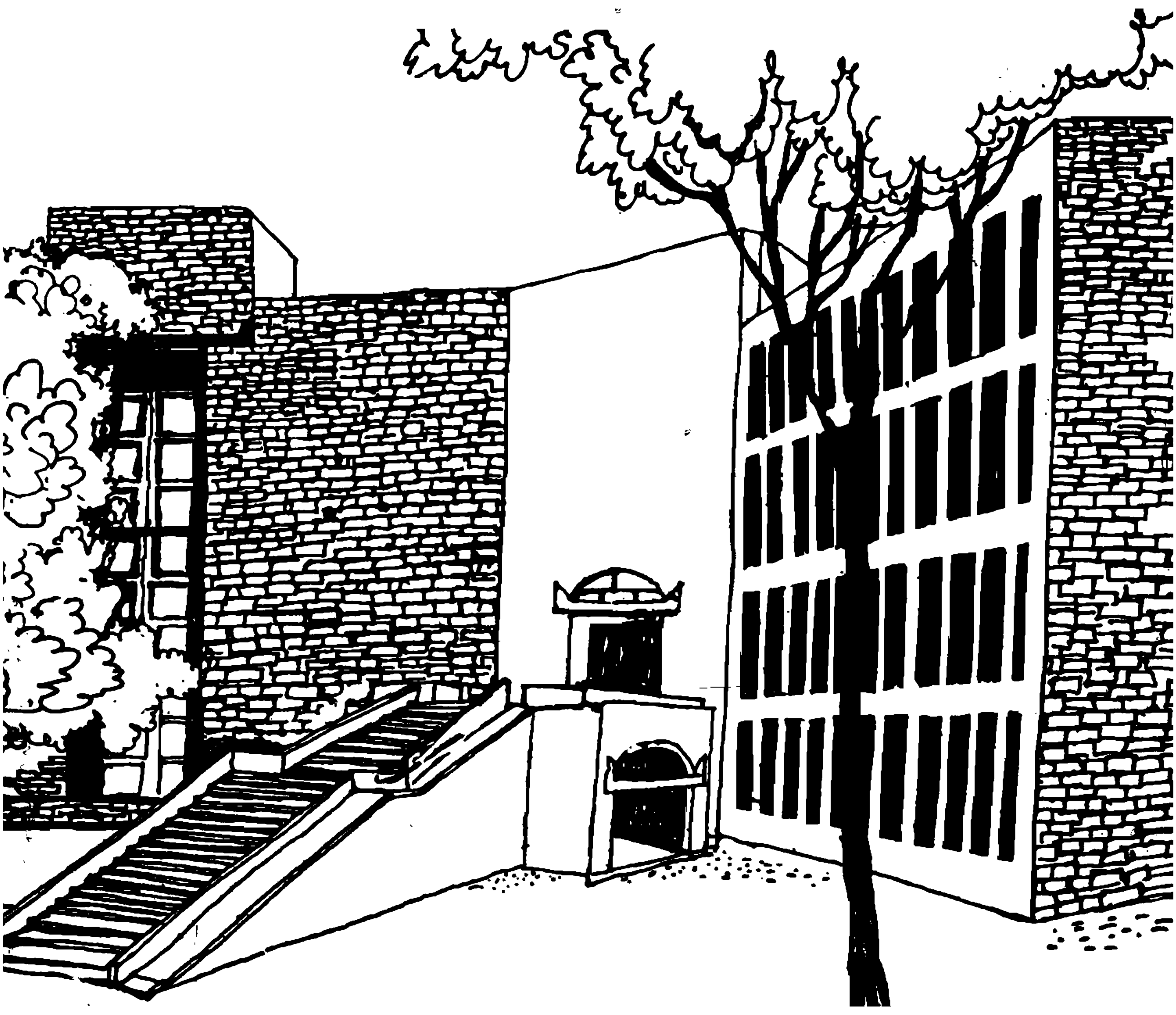

QUASITRANSITIVE RATIONAL CHOICE

By

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1. Introduction

We consider a finite universal set of alternatives and the set of all feasible sets are simply the set of all non-empty subsets of this universal set. A choice function assigns to each feasible set a non-empty subset of it.

An interesting problem in such a context is to explore the possibility of the choice function coinciding with the best elements with respect to a binary relation. This is precisely the problem of rational choice theory. There is a large literature today on this topic.

In this paper, we propose three new axioms which are used to fully characterize all choice functions which are rationalized by quasi-transitive, semi-transitive and a third kind of "almost" transitive (: the property is called intervality in the literature) binary relations. These "almost" transitive (: but not exactly so!) binary relations, which are now quite popular in the literature (: see Yu [1985]), have the rather interesting feature of revealing intransitive indifference for single valued choice functions. This phenomena has been dealt with rather elegantly by Kim [1987]. Our purpose, is to shed new light on the problem in the absence of the single-valuedness assumption. We, propose axiomatic characterizations which are minimal. Several examples are provided, to show that the assumptions we use are logically independent.

While characterizing choice functions which coincide with the best elements with respect to a binary relation satisfying intervality, we invoke a property due to Fishburn [1971], which we refer to in the paper as Fishburn's Intervality Axiom. In Aizerman and Aleskerov [1995], can be found an axiom called Functional Acyclicity, which generalizes Fishburn's Intervality Axiom. It is correctly claimed in Aizerman and Aleskerov [1995], that satisfaction of Functional Acyclicity is equivalent to the existence of two real valued functions, one with domain being the finite universal set and the other with domain being the set of all finite subsets of the universal set, such that given a feasible set, only those alternatives are chosen whose value corresponding to the first function is at-least as much as the value assigned to the feasible set by the second function. Such choice functions are called threshold rationalizable. In a final section to the paper, we provide a correct proof of this result, in view of obvious logical discrepencies in the proof available in Aizerman and Aleskerov [1995].

2. Model

Let X be a finite, non-empty universal set. If S is any non-empty subset of X, let [S] denote the set of all non-empty subsets of S. A choice function on X is a function C : [X] → [X] such that

$C(S) \subset S \; \forall S \in [X]$. Given a binary relation R on X and S ∈ [X], let G (S, R) = {x ∈ S / {x, y) ∈ R ∀ y ∈ S}. This set is called the set of best elements in S with respect to R. Given a choice function C on X, let

$$R^c = \{(x,y) \in X \times X \,/\, x \in C(\{x,y\})\} \text{ and } R_c = \bigcup_{S \in [X]} [C(S) \times S]$$

The following result is well known in the literature on rational choice.

Proposition 1: Given a choice function C on X if there exists a binary relation R on X such that C(S) = G(S,R) ∀ S ∈ [X], then $R = R^c$.

A binary Relation R on X is said to be

i. Reflexive if (x, x) ∈ R ∀ x ∈ S;

ii. Complete if ∀ x, y ∈ X, x ≠ y implies (x, y) ∈ R or (y, x) ∈ R.

iii. Quasitransitive if ∀ x,y,z ∈ X, (x,y) ∈ P(R), (y,z) ∈ P(R) implies (x,z) ∈ P(R), where P(R) = {(x,y) ∈ R / (y, x) ∉ R}.

iv) A Quasi-ordering if it satisfies (i), (ii) and (iii).

v) Transitive if ∀ x, y, z ∈ X, (x, y) ∈ R & (y, z) ∈ R implies (x, z) ∈ R;

vi) An Ordering if it satisfies (i), (ii) and (v).

A choice function is said to satisfy:

a) Chernoff's Axiom (CA) if $\forall S, T \in [X], [S \subset T] \rightarrow [C(T) \cap S \subset C(S)]$

b) Generalized Condorcet (GC) if ∀ S ∈ [X], $G(S, R^c) \subset C(S)$;

c) Bandopadhyay - Sengupta Acyclicity Axiom (BSAA) if $\forall S \in [X], [x \in S \setminus C(S) \rightarrow \exists y \in S: (x, y) \notin R_c]$.

Proposition 2: Given a choice function $C : [X] \rightarrow [X]$ there exists a binary relation R on X such that $C(S) = G(S,R)\ \forall S \in [X]$ if and only if at least one of the following two conditions hold:

1) C satisfies CA and GC;

2) C satisfies BSAA

The above results are available in Suzumura [1983] and Bandopadhyay and Sengupta [1991].

The reason why we refer to one of the axioms above as an acyclicity axiom is that if $\forall S \in [X]$, $C(S) = G(S,R)$ where R is a binary relation on X, then R must be acyclic in the following sense;

there does not exist $t \in N$ and $\{x^i\}_{i=1}^t$ all in X such that

$(x^i, x^{i+1}) \in P(R)\ \forall i \in \{1, \ldots, t-1\}$ with $(x^t, x^1) \in P(R)$.

A choice function C is said to satisfy the Bandopadhyay Sengupta Quasi Transitivity Axiom (BSQTA) if $\forall S \in [X]$, $x \in S \setminus C(S)$ implies that there exists $y \in C(S)$ such that $(x, y) \notin R_c$.

The following result has been established in Bandopadhyay and Sengupta [1991].

Proposition 3: Given a choice function $C : [X] \rightarrow [X]$, there exists a quasi-ordering R on X such that $C(S) = G(S,R)\ \forall S \in [X]$ only if C satisfies BSQTA.

3. Quasi-Transitive Rationality: A choice function C on X is said to satisfy

d) Outcasting (O) if $\forall S, T \in [X]$, $C(T) \subset S \subset T$ implies $C(S) = C(T)$;

e) Superset Axiom (SUA) if $\forall S, T \in [X]$, $C(T) \subset C(S) \subset T$ implies $C(S) = C(T)$;

f) Jamison and Lau's Quasi Transitivity Axiom (JLQTA) if $\forall S, T \in [X]$, $S \subset T \setminus C(T)$ implies $C(T \setminus S) = C(T)$;

g) Sen's Quasi Transitivity Axiom (SQTA) if $\forall S,T \in [X]$, $S \subset T$, $x, y \in C(S)$, $x \neq y$ implies $C(T) \neq \{x\}$.

h) Fishburn's Quasi Transitivity Axiom (FQTA) if $\forall S, T \in [X]$, $[S \setminus C(S)] \cap C(T) \neq \emptyset$ implies $C(S) \not\subset T$.

Outcasting is generally attributed to Nash [1950]; the Superset Axiom can be found in Suzumura [1983]; Jamison and Lau's Quasi Transitivity Axiom can be found in Jamison and Lau [1973], Sen's Quasi-Transitivity Axiom can be found in Sen [1971]; Fishburn's Quasi-Transitivity Axiom can be found in Fishburn [1975]. The following result can be found in the above mentioned papers and in Aizerman and Aleskerov [1995].

Theorem 1: Given a choice function C on X such that $C(S) = G(S,R)\ \forall S \in [X]$, where R is a binary relation on X, R is a quasi-transitive if and only if any one of the following holds:

i Outcasting;
ii Superset Axiom;
iii Jamsion and Lau's Quasi-Transitivity Axiom;
iv Sen's Quasi-Transitivity Axiom;
v Fishburn's Quasi-Transitivity Axiom.

We now introduce a new quasi-transitivity axiom, similar in spirit to Sen's Quasi-Transitivity Axiom.

New Quasi Transitivity Axiom (NQTA): A choice function C on X is said to satisfy the New Quasi Transitivity Axiom if

$$\forall S \in [X],\ \forall x, y \in S \setminus C(S),\ [x \neq y \text{ implies } x \notin C(S \setminus \{y\})].$$

We now introduce the following result:

Theorem 2: Let C be a choice function on X such that $C(S) = G(S, R)\ \forall S \in [X]$, where R is a binary relation on X. Then R is quasi-transitive if and only if C satisfies NQTA.

Proof: Suppose $C(S) = G(S, R)\ \forall S \in [X]$, where R is a quasi-ordering on X. Let $x, y \in S \setminus C(S)$. Since S is finite and R is a quasi-ordering, there exists $z \in C(S)$ such that $(z,x) \in P(R)$. Thus, $z \in S \setminus \{y\}$. Hence, $x \notin G(S \setminus \{y\}, R) = C(S \setminus \{y\})$.

Now suppose $C(S) - G(S,R)\ \forall S \in [X]$ and C satisfies NQTA. Let $(x,y) \in P(R)$, $(y,z) \in (P,R)$. Let $S = \{x,y,z\}$. Since $C(S) \neq \emptyset$, we must have $C(S) = \{x\}$. Hence $(z,x) \notin P(R)$. If $(x,z) \notin P(R)$, then $C(\{x,z\}) = \{x,z\}$. However, then $y, z \in S \setminus C(S)$ and $z \in C(S \setminus \{y\})$, contradicting NQTA. Thus, $(x,z) \in P(R)$. This proves the theorem.

4. Complete Logical Independence of CA, GC and NQTA :

Example 1:

A choice function which does not satisfy either CA or GC or NQTA: Let $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(\{x,y\}) = \{x,y\}$, $C(\{y,z\}) = \{y,z\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. Since $x \notin C(\{x,z\})$, C does not satisfy CA. Since $z \notin C(X)$, C does not satisfy GC; since $y, z \in X \setminus C(X)$ and $z \in C(X \setminus \{y\})$, C does not satisfy NQTA. We have here a choice function which does not satisfy BSAA either : $x \notin C(\{x,z\})$ and BSAA implies $(x,z) \notin R_c$. However $z \in X$ and $x \in C(X)$, contradicting BSAA.

Example 2:

A choice function which does not satisfy either CA or GC but satisfies NQTA : $X = \{x,y,z\}$, $C(X) = \{x,y\}$, $C(\{x,y\}) = \{x,y\}$, $C(\{y,z\}) = \{y,z\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C does not satisfy CA since $x \notin C(\{x,z\})$; C does not satisfy GC since $z \notin C(X)$. However, C satisfies NQTA. Note C does not satisfy BSAA : $x \notin C(\{x,z\})$ implies by BSAA, $(x,z) \notin R_c$. However $z \in X$ and $x \in C(X)$, contradicting BSAA.

Example 3:

A choice function which does not satisfy either CA or NQTA, but satisfies GC : $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C does not satisfy CA, since, $x \notin C(\{x,z\})$; C does not satisfy NQTA, since $y, z \notin C(X)$, but $z \in C(X \setminus \{y\})$. However, C satisfies GC vacuously. Note that C does not satisfy BSAA: $x \notin C(\{x,z\})$ and BSAA imply $(x,z) \notin R_c$ contradicting $z \in X$ and $x \in C(X)$.

Example 4:

A choice function which does not satisfy either GC or NQTA but satisfies CA : $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(S) = S\ \forall S \in [X], S \neq X$. C does not satisfy GC since $y \notin C(X)$. C does not satisfy NQTA, since $y, z \in X \setminus C(X)$ but $z \in C(X \setminus \{y\})$. However, C satisfies CA. Note that C does not satisfy BSAA : $y \in X \setminus C(X)$ implies either $(y,x) \notin R_c$ or $(y,z) \notin R_c$ contradicting $y \in C(\{x,y\})$ and $y \in C(\{y,z\})$.

Example 5:

A choice function which does not satisfy CA, but satisfies GC and NQTA : $X = \{x,y,z\}$, $C(X) = X$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{z\}$, $C(\{a\}) = \{a\}\ \forall a \in X$. C does not satisfy CA, since, $y \notin C(\{x,y\})$. However it satisfies GC and NQTA vacuously. Note C does not satisfy BSAA: $y \in \{x,y\} \setminus C(\{x,y\})$ implies by $(x,y) \notin R_c$ contradicting $y \in X$ and $x \in C(X)$.

Example 6:

A choice function which does not satisfy GC, but satisfies CA and NQTA : $X = \{x,y,z\}$, $C(X) = \{x,y\}$, $C(S) = S\ \forall S \in [X], S \neq X$. C does not satisfy GC, since $z \notin C(X)$. C satisfies CA. C satisfies NQTA vacuously. Note C does not satisfy BSAA : $z \in X \setminus C(X)$ implies by BSAA *either $(z,x) \notin R_c$ or $(z,y) \notin R_c$ contradicting $z \in C(\{x,z\})$ and $z \in C(\{y,z\})$.*

Example 7:

A choice function which does not satisfy NQTA, but satisfies CA and GC : $X = \{x,y,z\}$, $C(X) = \{x\}$, $C(\{x,y\}) = \{x\}$, $C(\{y,z\}) = \{y\}$, $C(\{x,z\}) = \{x,z\}$. C satisfies CA and GC. But C does not satisfy NQTA: $y, z \in X \setminus C(X)$ and yet $z \in C(X \setminus \{y\})$. Note C satisfies BSAA.

Example 8:

A choice function which satisfies CA, GC and NQTA : X = {x,y,z,w}, C(X) = {x,w}, C ({x,y}) = {x}, C({y,z}) = {y}, C({x,z}) = {x}, C({x,w}) = {x,w}, C({z,w}) = {z,w}, C({y,w}) = {y,w}, C ({x,y,z}) = {x}, C({x,y,w}) = {x,w}, C({x,z,w}) = {x,w}. C({y,z,w}) = {y,w}. C satisfies CA, GC and NQTA. Note that C satisfies BSAA as well.

R = {(x,x), (y,y), (z,z), (w,w), (x,y), (y,z), (x,z), (x,w),
(w,x), (y,w), (w,y), (z,w), (w,z)} and,

C(S) = G(S, R) ∀ S ∈ [X]. R is a quasi-ordering. However, (z,w) ∈ R and (w,x) ∈ R. Yet (z,x) ∉ R. Hence R is not transitive. Thus, R is not an ordering. In view of Proposition 1, we may conclude that there does not exist any ordering on X, such that for every S in [X], C(S) is equal to the set of best in S with respect to the given ordering. Further, by appealing to Proposition 2 and Theorem 2, we may now assert the following :

Theorem 3: Given a choice function C on X, there exists a quasi-ordering R on X such that C(S) = G(S,R) ∀ S ∈ [X] if and only if any one of the following holds :

a) C satisfies CA, GC and NQTA;
b) C satisfies BSAA and NQTA.

Note:

Example 1 above gives an example of a choice function which does not satisfy either BSQTA or NQTA; Examples 2, 5, 6 above gives examples of choice functions which satisfy NQTA but not BSQTA. Thus, in view of Theorem 1 and Theorem 2 we may conclude the following.

Theorem 4 : BSQTA implies NQTA. However, the converse is not true.

5. Semitransitive Rationality:

A binary relation R on X is said to be semi-transitive if ∀ x, y, z. w ∈ X, (x,y) ∈ P (R) and (y, z) ∈ P(R) → (x, w) ∈ P(R) or (w,z) ∈ P(R).

If R is reflexive and complete, then R is semi-transitive if and only if ∀ x, y, z, w ∈ X, (x, y) ∈ P(R) and (y,z) ∈ P(R) → either (w, x) ∉ R or (z, w) ∉ R.

Proposition 4:

Let R on X be a binary relation which is reflexive, complete and semi-transitive. Then R is a quasi-order.

A binary relation R on X which is reflexive, complete and semi-transitive is called a semi-order (or a semi-ordering).

A set of necessary and sufficient conditions for a choice function C on X to satisfy C(S) = G(S, R) $\forall$ S ϵ [X], where R is a semi-order can be found in Fishburn [1975] and Gensemer [1991]. We present a different axiomatic characterization below.

A choice function C on X is said to satisfy the <u>New Semi Transitivity Axiom</u> (NSTA) if $\forall$ S, T ϵ [X] with T $\subset$ S \ C(S), [z ϵ T \ C(T) $\rightarrow$ [(z, x) $\notin$ R_c, whenever x ϵ C(S)]]

<u>Theorem 5</u>: Let C be a choice function on S such that C(S) - G(S, R) $\forall$ S ϵ [X], for some binary relation R on X. Then R is semi-transitive if and only if C satisfies NSTA.

Proof: Let R be a semi-ordering on X such that C(S) = G(S, R) $\forall$ S ϵ [X]. Let T $\subset$ S \ C(S) and z ϵ T \ C(T) where S, T ϵ [X]. Since R is of necessity quasi-transitive, there exists y ϵ C(T) : (y, z) ϵ P(R). Since y ϵ S \ C(S), there exists x ϵ C(S) such that (x, y) ϵ P(R). Let w ϵ C(S). Clearly, (x,w) $\notin$ P(R). Hence by semi-transitivity of R, (w, z) ϵ P(R). Thus, (z, w) $\notin$ R_c.

Now suppose C(S) = G(S, R) $\forall$ S ϵ [X] and suppose C satisfies NSTA. Let (x, y) ϵ P(R) and (y,z) ϵ P(R). Thus, {x} = C({x, y, z}) and {y} = C({y,z}). Now {y,z} $\subset$ (x, y, z) \ C ({x, y, z}) and z ϵ {y,z} \ C({y,z}). Thus by NSTA, (z, x) $\notin$ R_c. Thus z $\notin$ C({x,z}). Thus (x,z) ϵ P(R). Thus R is quasi-transitive. Now suppose w ϵ X and towards a contradiction suppose (x, w) $\notin$ P(R) & (w,z) $\notin$ P(R). If (z,w) ϵ P(R), then by quasi-transitivity of R, (x,w) ϵ P(R), which we have ruled out. If (y,w) ϵ P(R), then again (x,w) ϵ P(R) by quasi-transitivity of R which is not possible. If (w,y) ϵ P(R), then (w,z) ϵ P(R) by quasi-transitivity of R which we have ruled out. If (w,x) ϵ P(R), then (w,z) ϵ P(R), by quasi-transitivity of R which we have ruled out. Thus C({x,y,z,w}) = {x,w}. Now {y,z} $\subset$ {x,y,z,w} \ C ({x,y,z,w}) and z ϵ {y,z} \ C({y,z}). Thus by NSTA, (z,w) $\notin$ R_c. Hence (w, z) ϵ P(R) which is a contradiction. This proves the theorem.

In view of Proposition 4, Theorem 5 and Theorem 2, NSTA $\rightarrow$ NQTA. Example 8 shows that the converse is not necessarily true. In fact a combination of CA, GC and NQTA does not imply NSTA. The fact that CA and GC combined together does not imply NSTA or that BSAA does not imply NSTA follows from the examples that we have given above. Without being unnecessarily repetitive, we might mention that the logical independence of CA, GC and NSTA follows from Examples 1 to 7, since whenever the cardinality of X is three (or less) NSTA and NQTA are equivalent. Hence, we have the following theorem.

<u>Theorem 6</u>:

Let C be a choice function on X. Then there exists a semi-order R on X such that C(S) = G(S,R) $\forall$ S ϵ [X] if and only if any one of the following holds :

a) C satisfies CA, GC and NSTA.
b) C satisfies BSSA and NSTA.

6. Intervality and Rational Choice:

A binary relation R on X is said to satisfy intervality if $\forall$ x, y, z, w $\in$ X, (x, y) $\in$ P(R) & (z,w) $\in$ P(R) implies [either (z,y) $\in$ P(R) or (x,w) $\in$ P(R)]. By setting y = z we see easily, that a reflexive binary relation satisfying intervality, also satisfies quasi-transitivity. If R is reflexive and complete, then R satisfies intervality is equivalent to the following condition:

$\forall$ x, y, z, w $\in$ X, (x, y) $\in$ R & (z,w) $\in$ R implies [either (z,y) $\in$ R or (x,w) $\in$ R].

Proposition 5:

Let R be reflexive and complete. Then R satisfies intervality if and only if R satisfies the following condition :

Condition (*):

$\forall$ x, y, z, w $\in$ X, (x, y) $\in$ P(R) (y, z) $\in$ & R, (z, y) $\in$ R and (z,w) $\in$ R implies (x,w) $\in$ P(R)].

Proof: Let R be reflexive, complete and satisfy intervality and let x, y, z, w be as in Condition (*). Then since (z, y) $\notin$ P(R), by intervality (x, w) $\in$ P(R). Thus, R satisfies Condition (*).

Now let R be reflexive, complete and satisfy Condition (*). Let (x, y) $\in$ P(R) and (z, w) P(R). Suppose y = z. Then since R is reflexive, by condition (*), (x, w) $\in$ P(R). Suppose then that y $\neq$ z. If (z, y) $\in$ P(R) then we are done. Hence suppose, (z, y) $\notin$ P(R). If (y, z) $\in$ P(R), then since Condition (*) along with the reflexivity of R implies that R is quasi-transitive, we get (x, w) $\in$ P(R). Finally, if (y, z) $\in$ R & (z, y) $\in$ R, then by condition (*) we get (x, w) $\in$ P(R). Thus, R satisfies intervality. Q.E.D.

The following axiom can be found in Bandopadhyay and Sengupta [1991]: A choice function C on X is said to satisfy Bandopadhyay and Sengupta's Intervality Axiom (BSIA) if $\forall$ S $\in$ [X], S \ C (S) $\neq \emptyset$ implies that there exists x $\in$ C(S) such that (y, x) $\notin R_c$ whenever y $\in$ S \ C(S).

Proposition 6: A choice function C on X satisfies BSIA if and only if there exists a reflexive, complete binary relation R on X satisfying intervality such that C(S) = G (S, R) $\forall$ S $\in$ [X].

A reflexive, complete binary relation on X satisfying intervality is called an interval order.

In the literature on rational choice, we find the following two axioms as well:

A choice function C on X is said to satisfy Fishburn's Intervality Axiom (FIA) if $\forall$ S, T, $\in$ [X], C(S) $\cap$ [T \ C(T)] $\neq \emptyset$ implies [S \ C(S)] $\cap$ C(T) = $\emptyset$.

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A choice function C on X is said to satisfy Aizerman and Aleskerov's Intervality Axiom (AAIA) if $\forall$ S, T $\in$ [X], $C(S) \subset C(C(S) \cup T)$ implies $C(T) \subset C(C(T) \cup S)$.

FIA is due to Fishburn [1971]. AAIA can be found in Aizerman and Aleskerov [1995].

Theorem 7: Let C be a choice function on X such that C(S) = G (S, R) $\forall$ S $\in$ [X], where R is a binary relation on X. Then R satisfies intervality, if and only if any one of the following conditions hold :

a) C satisfies FQTA and FIA

b) C satisfies AAIA

We now propose the following axiom :

A choice function C on X is said to satisfy the New Intervality Axiom (NIA) if $\forall$ S $\in$ [X], y, w $\in S \setminus C(S)$, y $\neq$ w, z $\in$ C(S), (y,z) $\in R_c$ implies w $\notin C(S \setminus \{x, z\})$.

Theorem 8: Let C be a choice function on X such that C(S) = G(S,R) $\forall$ S $\in$ [X], where R is a binary relation on X. Then R satisfies intervality if and only if C satisfies NQTA and NIA.

Proof : Let C(S) = G (S,R) $\forall$ S $\in$ [X] where R satisfies intervality. Thus, R is quasi-transitive and thus, C satisfies NQTA (by Theorem 2).

Now, let y, w $\in S \setminus C(S)$, y $\neq$ w, z $\in$ C(S), (y, z) $\in R_c$. Thus there exists x $\in$ C(S) : (x,w) $\in$ P(R). Suppose x $\neq$ z. Thus x $\in S \setminus \{z\}$ and hence w $\notin C(S \setminus \{z\})$. Hence suppose x = z. Now y $\in S \setminus C(S)$ implies that there exists v $\in$ C(S) such that (v, y) $\in$ P(R). Since (y,z) $\in R_c$, v $\neq$ z. Thus, (v,y) $\in$ P(R) & (z,w) $\in$ P(R) implies by intervality, that either (z,y) $\in$ P(R) or (v,w) $\in$ P(R). Since (y,z) $\in R_c$, we cannot have (z,y) $\in$ P(R). Thus, (v,w) $\in$ P(R). Since v $\neq$ z, v $\in S \setminus \{z\}$. Thus w $\notin C(S \setminus \{z\})$.

Now suppose C(S) = G (S, R) $\forall$ S $\in$ [X] and C satisfies NQTA and NIA. By Theorem 2, R is a quasi-transitive. Now suppose, (x,y) $\in$ P(R) & (z,w) $\in$ P(R). If (z,y) $\in$ P(R) we are done. Hence suppose, (z,y) $\notin$ P(R). If (y,z) $\in$ P(R). then by quasi-transitivity of R, (x,w) $\in$ P(R) and we are done. Thus, suppose (z,y) $\in$ R and (y,z) $\in$ R. If (z,x) $\in$ P(R), then (z,y) $\in$ P(R) by quasi-transitivity of R, which we have ruled out. If (x,z) $\in$ P(R), then (x,w) $\in$ P(R) by quasi-transitivity of R and we are done. Hence suppose (x,z) $\notin$ P(R). Thus (z,x) $\notin$ P(R) and (x,z) $\notin$ P(R). Similarly, (w,x) $\in$ P(R) would imply by quasi-transitivity that (z,y) $\in$ P(R) which we have ruled out.

Thus, C({x,y,z,w}) = {x,z} provided (x,w) $\notin$ P(R) and (z,y) $\notin$ P(R).

However, C({x,y,w}) = {x,w} if (x,w) $\notin$ P(R) and (z,y) $\notin$ P(R).

Let $S = \{x,y,z,w\}$. Now, $y, w \in S \setminus C(S)$, $y \neq w$, $z \in C(S)$ and $(y,z) \in R^c \subset R_c$. By NIA, $w \notin C(\{x,y,w\})$, contradicting what we have obtained above. Thus, R satisfies intervality.

Q.E.D.

Example 9: C satisfies CA, GC and NQTA but C does not satisfy NIA : Let $X = \{x,y,z,w\}$ and let $\Delta = \{a,a\} / a \in X\}$.

Let $R\ \Delta \cup \{(x.y), (z,w)\ (y,z), (z,y), (x,w), (w,x), (y,w)\ (w,y)\}$.

Let $C(S) = G(S,R)\ \forall S \in [X]$. C satisfies CA, GC and NQTA, but C does not satisfy NIA.

In view of Theorem 8 and Example 9, we may state the following:

Theorem 9: Let C be a choice function on X. Then $C(S) = G(S,R)\ \forall S \in [X]$, where R is an interval order on X, if and only if, C satisfies CA, GC, NQTA and NIA.

7. Functional Acyclicity:- A choice correspondence C is said to satisfy Functional Acyclicity (FA), if given any collection of sets $S_1,\ldots, S_r$ in [X], $C(S_{t+1}) \cap (S_t \setminus C(S_t)) \neq \phi\ \forall t = 1,\ldots, r\text{-}1$ implies $C(S_1) \cap (S_r \setminus C(S_r)) = \phi$.

Observe that FA implies FIA.

A choice correspondence C is said to be interval rationalizable if there exists functions $h: X \to \Re$ and $\epsilon : [X] \to \Re$ such that $C(S) = \{ x \in S / h(x) \geq h(y) + \epsilon (S)\ \forall y \in S\}$.

A choice correspondence C is said to be threshold rationalizable if there exists functions $h: X \to \Re$ and $V : [X] \to \Re$ such that $C(S) = \{ x \in S / h(x) \geq V(S)\}$.

Theorem 10 :- A choice correspondence is interval rationalizable if and only if it is threshold rationalizable.

Proof:- Theorem 3.15 in Aizerman and Aleskerov [1995].

Theorem 11 :- A choice correspondence C is said to be threshold rationalizable if and only if it satisfies functional acyclicity.

Proof: Let C be a choice correspondence which is threshold rationalizable. Thus there exists $h: X \to \Re$ and $V : [X] \to \Re$ such that
$C(S) = \{x \in S / h(x) \geq V (S)\}\ \forall S \in [X]$.
Towards a contradiction assume that $C(S_{t+1}) \cap (S_t \setminus C(S_t)) \neq \phi\ \forall t = 1,\ldots, r\text{-}1$
and $C(S_1) \cap (S_r \setminus C(S_r)) \neq \phi$

Let $x_{t+1} \in C(S_{t+1}) \cap (S_t \setminus C(S_t)) \neq \phi \; \forall \, t = 1, \ldots, r-1$
and $x_1 \in C(S_1) \cap (S_r \setminus C(S_r))$

Thus $h(x_t) \geq V(S_t) \; \forall \, t = 1, \ldots, r$
further $h(x_{t+1}) < V(S_t) \; \forall \, t = 1, \ldots, r-1$
and $h(x_1) < V(S_r)$

Thus from the weak inequalities we get, $\sum_{t=1}^{r} h(x_t) \geq \sum_{t=1}^{r} V(S_t)$

and from the strict inequality we get, $\sum_{t=1}^{r} h(x_t) < \sum_{t=1}^{r} V(S_t)$

This contradiction implies that C must satisfy functional acyclicity.

Let $R_c^* = \bigcup_{S \in [X]} [C(S) \; x \; (S \setminus C(S)]$ *and*

$$T(R_c^*) = \{(x,y) \in X \; x \; X \; / \; \exists \; t \in \mathbb{N} \text{ and } (x^k)_{k=1}^{t} \text{ all } \in X \text{ with } x^1 = x, \; x^t = y \text{ and } (x^k, x^{k+1}) \in R_c^* \; \forall \, k \in \{1, \ldots, t-1\}\}$$

$T(R_c^*)$ is transitive. Further, $(x, y) \in T(R_c^*)$ implies $(y,x) \notin T(R_c^*)$, by Functional Acyclicity . Let $R = \Delta \cup T(R_c^*)$ where $\Delta = \{(x,x) \; / \; x \in X\}$. By Szpilrajn's extension theorem (see Fishburn [1970]) there exists the a function $h : X \rightarrow \mathbf{R}$ such that $(x, y) \in T(R_c^*)$ implies $h(x) > h(y)$.

Given $S \in [X]$, let $V(S) = \min \{h(y) \; / \; y \in C(S)\}$.

Clearly, $x \in C(S) \rightarrow h(x) \geq V(S)$ and $x \in S$

Now, suppose $x \in S$, $h(x) \geq V(S)$ and towards a contradiction assume $x \notin C(S)$. Let $y \in C(S)$ with $h(y) = V(S)$. Thus, $(y, x) \in R_c^*$. Thus by the above $h(y) > h(x)$ which contradicts what we obtained above. Thus $x \in S$, $f(x) \geq V(S) \rightarrow x \in C(S)$.

........*Q.E.D.*

Coupled with Theorem 10, we have thus proved:

<u>Theorem 12</u> :- A choice correspondence is interval rationalizable if and only if it satisfies functional acyclicity.

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