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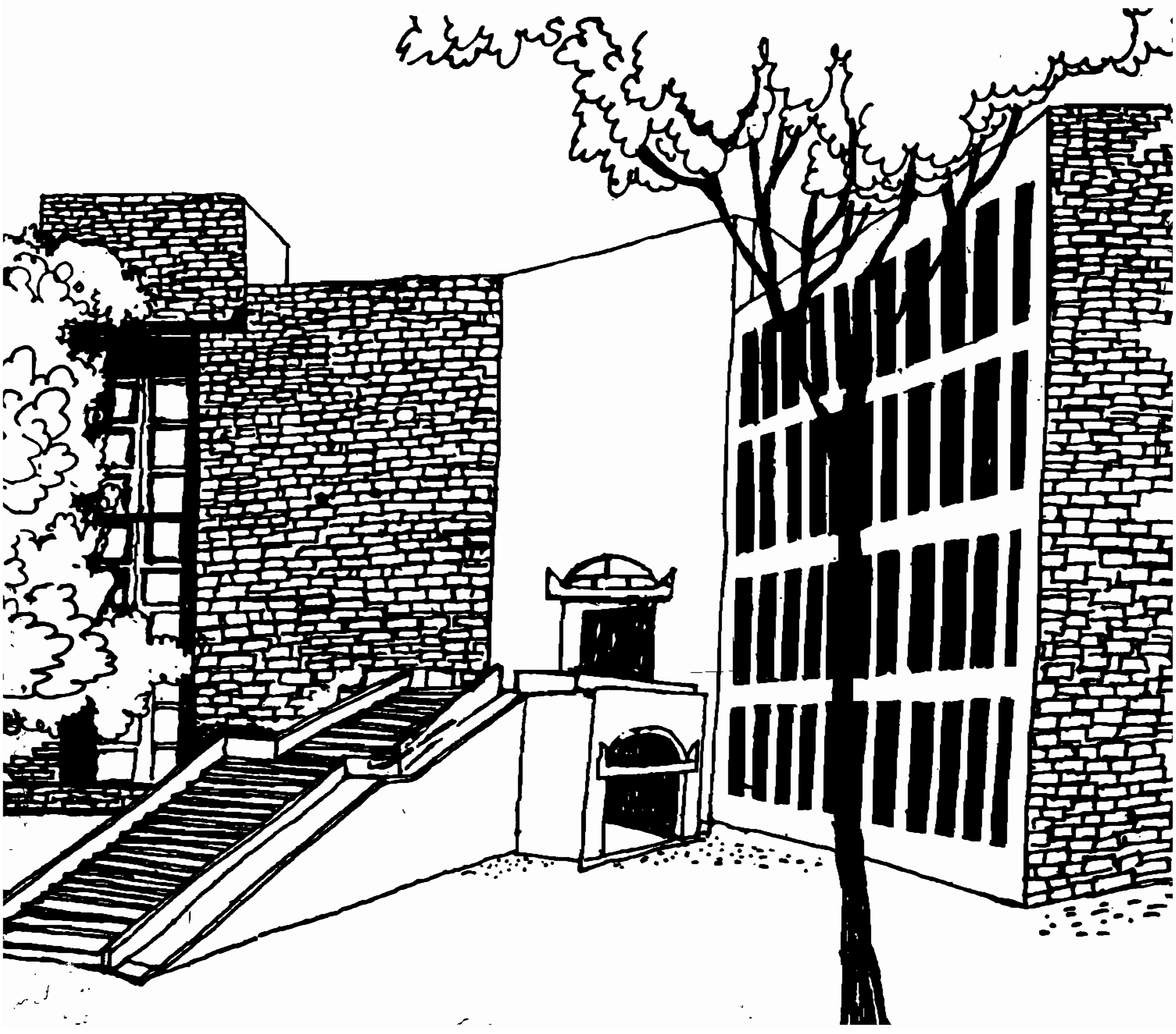

NON-RATIONALIZABILITY OF UTILITARIAN CONSISTENT CHOICE FUNCTIONS BY CONTINUOUS SOCIAL WELFARE ORDERING

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Abstract

In much of applied welfare economics, one finds the recurrent use of utilitarian objective functions, in arriving at social decisions. Apart from being completely insensitive to distributional issues, the utilitarian rule does not make single valued choices.

The purpose of this paper is to show that choice functions which are utilitarian consistent (i.e. formed by selecting a point from the set of maximizers of a utilitarian objective function) cannot be rationalized by a continuous social welfare ordering. This would imply espousing kinds of objectives for capital budgeting problems other than the usual utilitarian one, if one desires to have a rational basis for investment planning decisions. A further result noted in the paper is that if a choice function is utilitarian consistent and symmetric then it cannot be rationalized by a social welfare function. This strengthens considerably the earlier result.

Introduction:- In much of applied welfare economics, one finds the recurrent use of utilitarian objective functions, in arriving at social decisions. Apart from being completely insensitive to distributional issues, the utilitarian rule poses yet another problem. Even for convex, comprehensive utility possibility sets, the utilitarian objective function fails to select a unique utility allocation vector. This, lack of predictive ability, seems to be a very disconcerting feature of the utilitarian rule.

Beginning with the seminal work of Nash [1950], a new approach describing solutions to choice problems, dawned. Here, a particular choice function (or solution) was postulated, and the endeavor was to find a set of axioms which uniquely characterized this solution. It was soon, observed, that since the maximizer of the sum of utilities in a utility possibilities set (or a choice problem) need not be unique, anything like an utilitarian solution could not be defined, unless we required a certain amount of strict convexity to be associated with a utility possibilities set. On such sub-domains, axiomatic characterizations of the utilitarian choice function is available [see for instance Myerson (1981)]. However, for a more general and the usual domain, the utilitarian choice function fails to be well-defined.

It is because of this, that in Lahiri [1996], a choice function called the additive choice function has been defined and axiomatically characterized. A significant feature of this choice function is that it does not satisfy the celebrated Nash's Independence of Irrelevant Alternatives Assumption (NIIA). Whether this is a desirable or undesirable feature of the choice function, only time can tell.

Our purpose in this paper is two-fold: first we define selections from the set of maximizers of the sum of utilities, which satisfy NIIA; this dispels, the impression, that the additive choice function embodies the generic characteristics of selections thus defined; second, we prove an impossibility result. The impossibility result we establish is that, there does not exist any selection from the set of maximizers of the sum of utilities, which can be rationalized by a continuous SWO. This would put to rest, all speculations about choices made by maximizing the sum of utilities, reflecting a rational basis to social decision making.

2. The Model and Some Solutions:- In this paper we consider two dimensional choice problems. Let $\mathbf{R}^2_+$ denote the non-negative orthant of two dimensional Euclidean space. A (two-dimensional) choice problem is a non-empty set S in $\mathbf{R}^2_+$ satisfying the following properties:

(i) S is compact, convex and comprehensive i.e.

$$0 \le x \le y \in S \rightarrow x \in S.$$

(ii) $\exists\, x \in S$ such that x >> 0.

(For x, y $\in$ $\mathbf{R}^2$, x = (x_1, x_2), y = (y_1, y_2), $x \geq y$ means $x_i \geq y_i$, i = 1, 2; $x > y$ means $x \geq y$ and $x \neq y$; $x >> y$ means $x_i > y_i$, i = 1, 2).

Let B be the set of all such choice problems. A choice function on B is a function $F : B \rightarrow \mathbf{R}^2_+$ such that

$F(S) \in S \ \forall S \in B$.

Given $S \in B$, let

$$u(s) \equiv \{ (x_1, x_2) / x_1 + x_2 \geq y_1 + y_2 ,\ \forall y = (y_1, y_2) \in S \}.$$

Given $S \in B$, let

$$a_1 (S) = \max \{ x_1 / (x_1, x_2) \in u(s) \quad \text{for some} \quad x_2 \geq 0 \}$$

$b_1(S) = \min \{ x_1 / (x_1, x_2) \in u(S)$ for some $x_2 \geq 0 \}$

Given $S \in B$, let $P(S) = \{ x \in S / y > x \rightarrow y \notin S \}$.

$P(S)$ is called the Pareto optimal set of S.
For $S \in B$, let $a_2(S)$, $b_2(S)$ be such that $(a_1(S), a_2(S))$, $(b_1(S), b_2(S)) \in P(S)$.

As in Lahiri [1996] we define the additive choice $A : B \rightarrow \mathbf{R}^2_+$ is defined as follows:

$A(S) = \frac{1}{2} [a(S) + b(S)]$, where

$a(S) = (a_1(S), a_2(S)), b(S) = (b_1(S), b_2(S)) \; \forall S \in B.$

Apart from providing an axiomatic characterization of A, it was noticed in Lahiri [1996], that A does not satisfy the following property:

Nash's Independence of Irrelevant Alternatives (NIIA) Assumption: $F : B \rightarrow \mathbf{R}^2_+$ is said to satisfy NIIA if

$\forall S, T \in B, S \subset T, F(T) \in S \rightarrow F(S) = F(T).$

However, it will be noticed that the choice function 'a' (which may be called "additive favoring 1" choice function) and the choice function 'b' (which may be called "additive favoring 2" choice function) both satisfy NIIA. Yet, they do not satisfy the following fairly reasonable property:

Symmetry (SYM):- $F : B \rightarrow \mathbf{R}^2_+$ is said to satisfy SYM, if

given $S \in B$, it turns out that

$S' = \{(x_2, x_1) \ / \ (x_1, x_2) \in S\} = S$, then $F_1(S) = F_2(S)$.

We, therefore, define the following choice function $A^* : B \to \mathbf{R}^2_+$ which satisfies both NIIA and SYM:

Let $\Delta = \{ (x_1, x_2) \in \mathbf{R}^2_+ \ / \ x_1 = x_2 \}$.

Given $S \in B$, let $A^*(S) = \Delta \cap u(S)$ if $\Delta \cap u(S) \neq \phi$

$= b(S)$ if $x_1 > x_2 \ \forall \ (x_1, x_2) \in u(S)$

$= a(S)$ if $x_1 < x_2 \ \forall \ (x_1, x_2) \in u(S)$

It is easy to see that $A^* : B \to \mathbf{R}^2_+$ satisfies the following property as well:

<u>Pareto Optimality (PO)</u>: $F: B \to \mathbf{R}^2_+$ satisfies PO, if

$\forall \ S \in B, \ F(S) \in P(S)$

(Note: This is a property enjoyed by all choice functions $F: B \to \mathbf{R}^2_+$ such that $F(S) \in u(S) \ \forall \ S \in B$).

3. <u>Rationalizable Choice Functions</u>:- This section is based on Peters and Wakker (1991), Bossert (1994) and Bossert (1996).

Let R be a binary relation on $\mathbf{R}^2_+$ which is reflexive (i.e. $x R x \; \forall \; x \in \mathbf{R}^2_+$), transitive

(i.e. $xRy \wedge yRz \rightarrow xRz \; \forall \; x,y,z \in \mathbf{R}^2_+$) and total

(i.e. $x,y \in \mathbf{R}^2_+, \; x \neq y \rightarrow xRy \vee yRx$). Such an R is called an ordering.

An ordering R is said to be continuous, if

$\forall \; x \in \mathbf{R}^2_+, \; \{ y \in \mathbf{R}^2_+ / yRx \}$ and $\{ y \in \mathbf{R}^2_+ / xRy \}$, are closed.

Let $F: B \rightarrow \mathbf{R}^2_+$ F is said to be rationalizable by a continuous ordering R if

$\forall \; S \in B, \; \{ F(S) \} = \{ x \in S / xRy \; \forall \; y \in S \}$.

A choice function $F: B \rightarrow \mathbf{R}^2_+$ is said to be utilitarian consistent, if $F(S) \in u(S) \; \forall \; S \in B$.

<u>Theorem 1</u>:- Let $F: B \rightarrow \mathbf{R}^2_+$ be any utilitarian consistent solution. Then F is not rationalizable by any continuous ordering.

<u>Proof</u>:- Let $F: B \rightarrow \mathbf{R}^2_+$ be any utilitarian consistent solution. Towards a contradiction assume that F is

rationalizable by a continuous ordering R.

Let P be the asymmetric part of R.
Let S = cch { (d,0), (0,d) }, d > 0.

Suppose F(S) >> 0. Let $\overline{x} = F(S)$ *with* $\overline{x}_1 > 0$, $\overline{x}_2 > 0$.

$\therefore \overline{x}\ P\ (d, 0)$ *and* $\overline{x}\ P\ (0, d)$.

By continuity of R, y P (d, 0) and y P (0,d) ∀ y in a sufficiently small neighborhood N of $\overline{x}$.

Choose ϵ>0 so small that, if T = cch {(d-ϵ, 0), (0, d)}, then $T \cap N \neq \phi$. Since F(T) ϵ u(T), F(T) = (0,d). But,

$T \cap N \neq \phi \rightarrow y \in T$ such that y P (0, d) which is a contradiction.

Hence, F(S) = (0, d) or (d, 0).

Suppose without loss of generality F (S) = (d, 0).
Then (d, 0) P (0, d).
By continuity of R, (d - ϵ, 0) P (0, d) for ϵ > 0 sufficiently small.

Since, if T = cch { (d - ϵ, 0), (0, d) }, F (T) ϵ u (T) implies F (T) = (0, d), we get that (d - ϵ, 0) P F (T) which is a contradiction.

Thus F is not rationalizable by any continuous ordering.

Q.E.D.

Note:- In the above proof cch refers to comprehensive, convex hull.

Remark 1:- Thus A^* is not rationalizable by any continuous ordering. Contrast this with the result in Peters and Wakker (1991), which says that if a choice function satisfies PO, CONT and NIIA, it is rationalizable by an upper semicontinuous ordering.

Remark 2:- The above theorem is easily seen to be valid for utilitarian compatible choice functions defined on the space of n-dimensional choice problems i.e. collection of nonempty, compact, convex, comprehensive subsets of $\mathbf{R}^n_+$ (: the non-negative orthant of n-dimensional Euclidean space) each of which admits a strictly positive vector.

Remark 3:- In view of our proof of the main theorem and Remark 1, the following observation is easily seen to be valid: Let $h: \mathbf{R}^n_{++} \times \mathbf{R}_{++} \to \mathbf{R}^n_+$ be a function which is homogeneous of degree one and such that $\forall (p,w) \in \mathbf{R}^n_{++} \times \mathbf{R}_{++}$, $\sum_{i=1}^{n} p_i\, h_i\,(p,w) \leq w$. Further suppose that $\forall\ (p,w) \in \mathbf{R}^n_{++} \times \mathbf{R}_{++}$ and $\forall\ x \in \mathbf{R}^n_+$ with $\sum_{i=1}^{n} p_i x_i \leq w$, $\sum_{i=1}^{n} h_i(p,w) \geq \sum_{i=1}^{n} x_i$. Then there does not exist any total, reflexive, transitive and continuous binary relation R on $\mathbf{R}^n_+$ such that $\forall (p,w) \in \mathbf{R}^n_{++} x \mathbf{R}_{++}, \{h(p,w)\} = \{ x \in \mathbf{R}^n_+ / \sum_{i=1}^{n} p_i x_i \leq w$ and $xRy\ \forall\ y \in \mathbf{R}^n_+$ with $\sum_{i=1}^{n} p_i y_i \leq w\}$.

Remark 4:- Let $\mathbf{R}^n_{++} = \{x \in \mathbf{R}^n_+ / x_i > 0 \ \forall \ i=1,\ldots,n\}$. In view of Remark 1, a slight modification of our proof of the main theorem yields the result that if F is a choice function for n-dimensional choice problems which is weighted utilitarian consistent, then there does not exist any total, reflexive, transitive and continuous binary relation R on $\mathbf{R}^n_+$, which rationalizes F.

Here, F is weighted utilitarian consistent with weights $w \in \mathbf{R}^n_{++}$

if $F(S) \in u^w(s) \equiv \{x \in S / \sum_{i=1}^{n} w_i x_i \geq \sum_{i=1}^{n} w_i y_i \ \forall \ y=(y_1,\ldots,y_n) \in S\}$ for all n-dimensional choice problems.

Infact, we can prove a slightly stronger theorem than the one proved above.

Given a choice function $F : B \to \mathbf{R}^2_+$, say that it is rationalized by a social welfare function $V : \mathbf{R}^2_+ \to \mathbf{R}$ if $\forall \ S \in B$,

$\{F(S)\} = \{x \in S / V(x) \geq V(y) \ \forall \ y \in S\}$.

Note: No assumptions are being made with regard to the continuity of V.

Theorem 2:- If $F : B \to \mathbf{R}^2_+$ is utilitarian consistent and symmetric then it cannot be rationalized by any social welfare function $V : \mathbf{R}_+^2 \to \mathbf{R}$.

Proof:- Towards a contradiction assume that $F : B \to \mathbf{R}_+^2$ is utilitarian consistent and symmetric which is also rationalized by a social welfare function $V : \mathbf{R}_+^2 \to \mathbf{R}$. Let $0 < d' < d$.

Let $T_d = cch\{(0,d), (d,0)\}$; then $F(T_d) = (\frac{d}{2}, \frac{d}{2})$.

$T_{d'} = cch\{(0,d'), (d',0)\}$ implies $F(T_{d'}) = (\frac{d'}{2}, \frac{d'}{2})$.

Both the above follow from symmetry of F. Thus

$V\left(\frac{d}{2}, \frac{d}{2}\right) > \max\{V(d,0), V(0,d)\}$.

Let $T = cch\left\{(0,d), \left(\frac{dd'}{2d-d'}, 0\right)\right\}$

$F(T) = (0,d)$ implies

$$V(0,d) > V\left(\frac{d'}{2}, \frac{d'}{2}\right) \; since \left(\frac{d'}{2}, \frac{d'}{2}\right) \in T$$

Let $T' = cch\left\{ (d,0)\,,\, \left(0, \frac{dd'}{2d-d'}\right) \right\}.$

We get in a similar fashion.

$$V\,(d,0) > V\left(\frac{d'}{2}, \frac{d'}{2}\right).$$

Thus $V\left(\frac{d}{2},\frac{d}{2}\right) > \max\{\, V(d,0)\,,\; V(0,d)\,\}.$

$$\geq \min\{\, V(d,0)\,,\; V(0,d)\,\}.$$

$$> V\left(\frac{d}{2},\frac{d'}{2}\right) > \max\{\, V(d',0)\,,\; V(0,d')\,\}.$$

Let r(d) be a rational number between $V\left(\frac{d}{2}, \frac{d}{2}\right)$ and

max {V(d,0), V(0,d)}, d > 0.

Thus r is a function from $\mathbf{R}_{++}$ to the rationals which is strictly increasing and hence one-to-one. But this is impossible. Hence the theorem.

Q.E.D.

The implications of this theorem, are rather powerful as we shall soon observe.

Remark 5:- The solution A* is symmetric and utilitarian consistent. Hence, by Theorem 2, it is not rationalizable, by any social welfare function.

Call a choice function $F : B \to \mathbf{R}^2_+$ continuous if wherever $\{S^n\}_{n \in \mathbf{N}}$ is a sequence of choice problems if B, converging in the Hausdorff topology to S $\in$ B, then $\lim_{n \to \infty} F(S^n) = F(S)$. In such a situation F is said to satisfy Continuity (CONT).

It is easy to see that A* does not satisfy CONT.

In Peters and Wakker (1991), we have a result of fundamental importance: A sufficient condition for $F : B \to \mathbf{R}^2_+$ to be rationalizable by a social welfare function is that F satisfies PO, NIIA and CONT.

A* satisfies PO, NIIA but not CONT. Further as observed in Remark 5, A* is not rationalizable by any social welfare function. Thus continuity seems to be quite necessary for the characterization theorem in Peters and Wakker (1991). In fact one cannot drop continuity as a requirement in the stated result.

4. Conclusion:- The above conclusion regarding non-rationalizability of utilitarian consistent solutions is significant for the modest literature on capital budgeting problems, where linear objective functions are used to evaluate inter-sectoral returns arising out of allocations of investments to the different sectors (see Baumol and Quandt [1965]; Levary [1996]). Since the constraint sets for such problems in the space of sectoral returns, can have linear Pareto optimal segments (infact they may even be convex polytopes), criteria other than the weighted sum of returns are called for in investment planning problems as well.

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