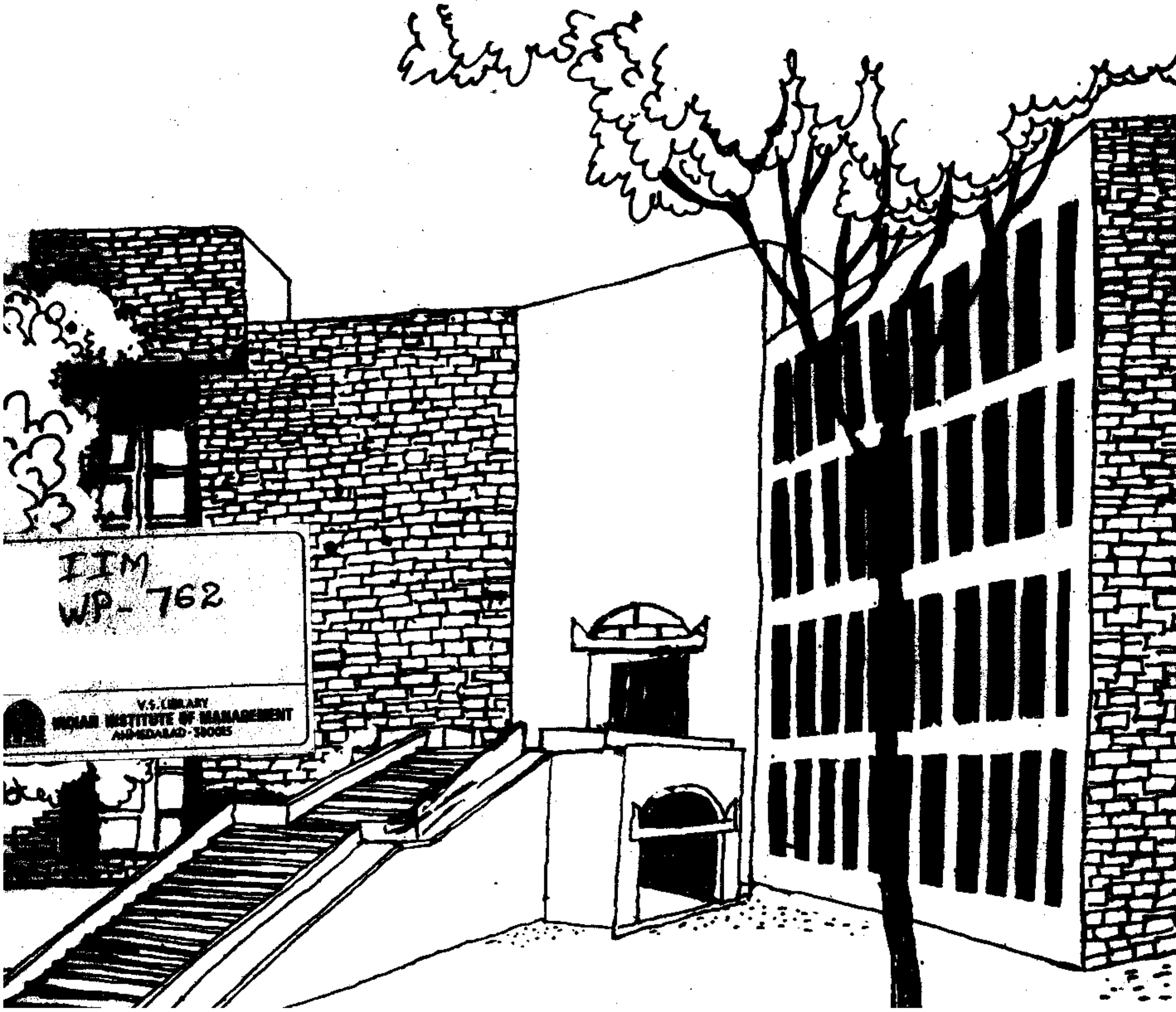
विद्याविनियोगाद्विकासः

IIM
AHMEDABAD

Working Paper



MULTI-ITEM (S, s) INVENTORY MODEL WITH POISSON DEMAND, GENERAL LEAD TIME AND ADJUSTABLE RECRDER TIME

By

S.K. Srinivasan

&

N. Ravichandran

W P No. 762
August, 1988

The main objective of the working paper series of the IIMA is to help faculty members to test out their research findings at the pre-publication stage.

INDIAN INSTITUTE OF MANAGEMENT
AHMEDABAD-380015
INDIA

Multi-Item (S, s) Inventory Model with Piosson Demand, General Lead Time and Adjustable Reorder Time

S.K. Srinivasan* and N. Ravichandran**

ABSTRACT

Stochastic Multiple Item Inventory System of (S, s) type is studied with general lead time distribution and Poisson demand. The demands occur according to Poisson process for one unit of a specific item at a time. The demand types are characterised by a discrete distribution $\{\alpha_i\}$ $\alpha_i \geq 0 : \Sigma \alpha_i = 1$. An order is placed when the inventory level drops to s_i for any of the types in the inventory. The quantity reordered is subject to review at the epoch of replenishment so as to level up the inventory of all the items to S_i. Shortages are assumed to be lost. An explicit expression is provided for the marginal distributions of the Invetntory level process. Using this, the expected total cost of the inventory system is obtained explicitl The results for the special care of instantaneous lead time are deduced from the earlier results.

---

* Department of Mathematics, Indian Institute of Technology, Madras - 600 036.

** Indian Institute of Management, Ahmedabad - 380 015.

INTRODUCTION:

The object of this contribution is to provide an analysis of (S, s) inventory systems with multiple items when the demands occur according to a stationary Poisson process and the lead time is a non-negative random variable with arbitrary distribution. Shortages, if any are assumed to be lost. To render the analysis simple, we consider a model in which the reorder quantity is adjustable on arrival. The reorder quantity is so decided at a replenishment epoch that inventory level of all the items is raised to fill the capacity of the warehouse. In a model of this type it is possible to have a reorder policy in such a manner that reorders can be placed as and when the inventory level of any of the items drops to a specified lower limit s_i. However, we shall make a simplyfying assumption so that at most one order is pending at any time. We assume that reorder is placed for all the items whenever anyone of the items drops to the inventory level s_i. No further reorders will be placed until the pending order materialises. Since multi-inventory systems with positive lead time have not been studied in great depth it is considered worthwhile to deal with a simple model of this type in the first instance.

Inventory systems of multiple items have been the subject matter for many investigators in the past. Such studies vary from simple extensions of EoQ analysis to sophisticated stochastic models. Miller (1974) considered a single period model with stochastic demand, and determined the ordering policy for a multiple item model to minimize a function of the items back ordered with constraints on the operational budget. Agarwal (1984) considered a multi-item inventory model. He determined the optimal ordering cycle by grouping some of the items which are similar in nature. The order for these items are coordinated.

Silver (1974) and Thomstone and et.al (1975) consider multiple item inventory systems. The models considered by them are known as coordinated inventory control models. These models have three level parameters instead of two in the classical (S, s) model. These parameters are called order level, can order level and order upto level. When the inventory of an item drops down below the order level an order is placed for this item by definition. In addition, this order includes all items whose level is less than the can order level in order to economise on the ordering cost. Numerical procedures are established to determine the optimal levels. The most general model attempted in this direction corresponds to compound Poisson demand and deterministic lead time.

Muckatadt (1979) reports the application of a multi-item, multi-location multi-Echelon system applicable in defence and suggests a procedure to determine the optimal ordering policy and stock at various locations.

To the best of knowledge of the authors, it appears no serious attempt has been made sofar to analyse Stochastic multi-item inventory systems with realistic assumptions like positive lead time etc. This contribution is a beginning in this direction. Thus, in the paper we provide an analysis of the statistical characteristics of the inventory level process and obtain an explicit expression for the expected value of the steady state cost function. We have preferred to deal with the case of shortages since much reported research in the literature deal with either complete or partial backlogging. However, the assumption of readjustable reorder size by definition is capable of describing both the lost sales and backorder cases (Srinivasan 1988).

THE PRESENT MODEL:

We consider a system with m distinct types of inventory. The maximum level (S_i) and the reorder level (s_i), for any item i is assumed to be distinct with $(S_i - s_i) > s_i$. Demands to the system occur according to a Stationary Poisson process. Only one unit is demanded at every demand epoch. However, any type of item can be demanded at a demand epoch. The specification of the demand type requirement is done by a discrete distribution $\{a_i\}$, $a_i \geq 0$ $\sum_{i=1}^{m} a_i = 1$. We assume that at the time origin the warehouse is filled to its capacity. As time progresses the stock level starts depleting. Let $\tau_1, \tau_2, \ldots \tau_m$ be the epochs at which the stock-level reaches $s_1, s_2, \ldots s_m$ for all items respectively. Let $\tau = \min_j (\tau_j)$.

A reorder is placed at the epoch τ, which materialises, after a random lead time. During the lead time, it is possible for some other item j ($j \neq i$), the inventory level may reach the reorder level. However, no order shall be placed in response to the current inventory situation. At the time of replenishment, the inventory position for all the items is reviewed and appropriate replenishment is made so as to bring the inventory level to S_i for $1 \leq i \leq m$. The demands that occur when the stock level is zero is assumed to be lost. The inventory level at any time is represented by a process $\{L(t), \ t \geq 0\}$ which takes values on m-dimensional vectors. For the system under consideration both the reorder points and replenishment points are uniquely defined and are regenerative epochs (See Srinivasan (1976)) for the stochastic process $\{L(t), \ t \geq 0\}$. We shall use this property extensively in our analysis.

NOTATIONS:

λ - Demand rate

λ_i - Demand rate for the ith type; $\lambda_i = \lambda a_i$

a event : event corresponding to demands

a(i)event: event corresponding to demand for the ith type.

b event : event corresponding to replenishment epochs.

r(i) event: event corresponding to reorders induced by the ith type $(1 \leq i \leq m)$

X_b : Random variable representing the time interval between two successive b-events.

$X_{br(i)}$: Random variable representing the time interval between a b-event and the first reorder event (which happens to be ith type).

L(t) : Level process (vector valued) representing inventory position at any time t.

b(t) : pdf of the lead time with mean $\frac{1}{\mu}$.

$f_X(t)$: pdf of a random variable X; X = b, br(i) etc.

* : Convolution.

f(s) : Laplace transform of f(t).

$p_i(k, t)$: Poisson probability of k events with parameter λ_i.

$P_i(k, t)$: $\sum_{j=0}^{k-1} p_i(j, t)$, Cumulative Poisson with maximum of (k-1) point events.

$\bar{P}_i(k, t)$: $1 - P_i(k, t)$

$N_j(t)$: Counting process associated with events of type j in the interval of length t, $j = a, a(i), b, r, r(i$

MAIN RESULTS:

Our primary objective is to obtain the expected cost in observing the specified (see model assumption) ordering policy in the long run. This is achieved by studying the stationary distribution of the inventory level process $\{L(t), t \geq 0\}$. As pointed out earlier $\{L(t), t \geq 0\}$ is a vector valued process. Considerable simplification results in the analysis of the system of margÂinal distribution of the process is of interest. Infact, it turns out that the marginal distribution of the inventory level is sufficient to study the cost expression and the associated optimization problems.

We start with a few important observations

<u>Lemma 1:</u>

1. The marginal demand process corresponding to demands of type k is a Poisson process with parameter λ_k (= $\lambda \alpha_k$).

2. The marginal counting processes induced by the demands for specified types are statistically independent.

<u>Proof:</u> Let $f_{X_{a(i)}}$ be the pdf of the time interval between two successive demands of type i. In our notation, we denote the demand events as a(i). Then,

$$f_{X_{a(i)}}(t) = \lambda e^{-\lambda t} \alpha_i + \sum_{n=1}^{\infty} [\lambda e^{-\lambda t} \bar{\alpha}_i]^{(n)} * \lambda e^{-\lambda t} \alpha_i \quad \text{.. (1)}$$

The Laplace transform of the pdf of $X_{a(i)}$ is equivalent to,

$$f_{X_{a(i)}}(s) = \frac{\lambda \alpha_i}{s + \lambda} + \sum_{n=1}^{\infty} \left[\frac{\bar{\alpha}_i \lambda}{s + \lambda} \right]^n \frac{\lambda \alpha_i}{s + \lambda}$$

$$= \frac{\lambda \alpha_i}{s + \lambda \alpha_i}$$

On inversion,

$$f_{X_{a(i)}}(t) = \lambda \alpha_i \, e^{-\lambda \alpha_i t} \quad \text{... (2)}$$

Observation (2) is equivalent to the validity of the following for the case m = 2,

$$\Pr\{N_1(t) = k_1;\ N_2(t) = k_2\} = \Pr\{N_1(t) = k_1\} \cdot \Pr\{N_2(t) = k_2\}$$

Observing that the LHS of the above is

$$\frac{e^{-\lambda t}\,(\lambda t)^{k_1+k_2}}{(k_1 + k_2)!} \binom{k_1 + k_2}{k_1} (a_1)^{k_1} (a_2)^{k_2} \qquad \ldots. (3)$$

On simplification, this results

$$e^{-(\lambda a_1)t}\,\frac{(\lambda a_1)^{k_1}}{k_1\,!} \quad e^{-(\lambda a_2)t}\,\frac{(\lambda a_2)^{k_2}}{k_2\,!} \qquad \ldots. (4)$$

Hence, observation (2).

It is interesting to note the simplicity in the marginal process due to Poisson demand. In the case when the demand process is a general renewal process, while the marginal process (or type k demands) remains as renewal process, the statistical independddnce of the associated counting process does not appear to be valid.

The stochastic behaviour of the inventory process $\{I(t),\ t \geq 0\}$ is influenced by the demands to the system and the joint replenishment to the system. Accordingly these two point event play a vital role in the analysis of $\{L(t), \ t \geq 0\}$. These are the epochs corresponding to reorder points and the replenishment epochs. The reorder epochs are induced by a demand which reduces the inventory level to s_i for some i. When a reorder is produced by the demand of type i, the inventory level of the other types j ($\neq$ i) is at a level $> s_j$ by definition. Thus, the reorder points are unique and well-defined.

The Lead time is a non-negative random variable: As the reorder size is adjustable on replenishment the inventory level immediately after replenishment increases to the maximum capacity. Hence, for the evaluation of the distributional characteristics of the process $\{L(t), \ t \geq 0\}$ it is needless to keep trade of the inventory level process during the lead time. As we shall see later, that this situation is far more simpler than the situation that arises in the context of fixed order size. (Srinivasan, 1979).

We also observe that the time intervals between successive replenishment events form a renewal process and the reorder epochs induce an MRP. Further, it is evident that two replenishment events are intercepted by exactly one reorder event and vice-versa. Hence, the rate of occurrence of replenishment events and reorder events shall be identical in the stationary case. This observation will be useful in obtaining the cost expression.

We now obtain the pdf $f_{X_b}(t)$ of the time interval characterising replenishment epochs. The random variable representing the time interval between successive pdfs can be written as the sum of two independent random variables:-

(a) The time interval between a b-event and the following reorder event r(i).

(b) The Lead time.

Hence, to determine $f_{X_b}(t)$ we need to obtain $f_{X_{br(i)}}(t)$. The next Lemma obtains this.

<u>Lemma 2:</u> The pdf of the random variable representing the time interval between a b-event and the epoch corresponding to a (first) reorder point is given by

$$f_{X_{br(i)}}(t) = p_i(S_i - s_i - 1, t)\lambda_i \prod_{j \neq i} P_j(S_j - s_j, t) \qquad \ldots (5)$$

Expression (5) is obtained by considering the following:-

For a reorder event of type i to occur at t, the $(S_i - s_i)$th demand of the ith type should occur in (t, t+dt) and the demands for the remaining types should be less than $S_j - s_j$. Using independence and Poisson nature of various demand events expression (5) is obtained.

We now summarize our observations as:

<u>Theorem 1:</u>

The epochs $\{t_i\}$ corresponding to the occurrence of b-events form a renewal process with pdf

$$f_{X_b}(t) = \sum_{i=1}^{m} f_{X_{br(i)}}(t) * b(t) \qquad \text{.... (6)}$$

The pdf of the time interval between a reorder of type i and that of type j is obtained as

$$f_{X_{r(i)r(j)}}(t) = b(t) * f_{X_{br(j)}}(t) \qquad \text{.... (7)}$$

The mean value of time between two successive b-events is given by

$$E[X_b] = \int_0^{\infty} \prod_{j=1}^{m} P_i(S_i - s_i, t)\,dt + \frac{1}{\mu} \qquad \text{.... (8)}$$

For the special case of m = 2,

$$E[X_b] = \frac{1}{\lambda} \sum_{k=0}^{S_1-s_1-1} \alpha_1^k \sum_{j=0}^{S_2-s_2-1} \alpha_2^j \binom{j+k}{k} + \frac{1}{\mu} \qquad \text{.. (9)}$$

As observed earlier $\{L(t),\ t \geq 0\}$ is a vector valued process with continuous time parameter. Hence, a complete characterisation of the process requirees the evaluation of the multinomial distribution of the type $Pr\{L(t) = \underline{n}\}$ where $\underline{n} = (n_1, n_2, \ldots\ldots n_m)$ is a vector of dimension n representing the inventory level with specification on the type of different items. In this contribution our requirement is not on the absolute characterisation and discussion of the process $\{L(t),\ t \geq 0\}$. All we need for further discussion is the marginal distribution of the inventory level. Let $L_i(t)$ be the inventory level of type i at time t. We need to find the distribution of $L_i(t)$. The methods and procedure we employ to obtain marginal distribution can be used to derive the joint distribution with very little effect.

In addition to the several pdfs obtained earlier, we define the following functions to obtain the marginal inventory distribution.

$$\psi_i(n,t) = \lim_{\Delta \to 0} Pr\{L_i(t)=n\ \ N_b(t)=0 \mid N_b(o)-N_b(-\Delta) = 1\},$$

$$0 \leq n \leq S_i;\quad 1 \leq i \leq m$$

$$\bar{\Lambda}_i(n,t) = \lim_{\Delta_1,\Delta \to 0} Pr\{L_i(t)=n \mid N_b(o)-N_b(-\Delta)=1\},\quad 0\leq n\leq S_i,\quad 1 \leq i \leq m$$

$$f_j(m_i,t) = \lim_{\Delta_1,\Delta \to 0} Pr\{ L_i(t) = m_i,\ N_{r(j)}(t+\Delta) - N_{r(j)}(t) = 2 \mid N_b(o) - N_b(-\Delta_1) = 1 \},$$

$$i \neq j; \qquad 1 \leq i \leq m$$

$$s_j < m_j \leq S_j; \qquad 1 \leq j \leq m$$

The function $\psi_i(n,t)$ function represents the probability that the inventory (marginal) level of the ith type is n; there is a replenishment at the time origin and no-replenishment in (o,t). The $\bar{\pi}_i(n,t)$ represent a similar marginal distribution with no-restriction on the number of replenishments. The $f_j(m_i,t)$ represents the pdf of the random variable representing the time between a b-event and the following r(j) event with the restriction that the level of the ith type $(i \neq j)$ is m_i. These functions are sufficient to derive the main results:

<u>Theorem 2:</u> The functions $\psi_i(n,t)$ describing the marginal level process $L_i(t)$ are given by,

$$\psi_i(n,t) = p_i(S_i-n,t) + \{ \sum_{\substack{j \neq 1 \\ j \neq i}}^{m} f_{X_{br(j)}}(t) * \bar{B}(t) \} \ p_i(S_i-n,\ t)$$

$$1 \leq i \leq m; \qquad s_i < n \leq S_i \qquad \ldots (10)$$

$$\Psi_i(n,t) = f_{X_{br(i)}}(t) * [\,\bar{B}(t)\; p_i(s_i-n,t)\,] + \sum_{m_i=s_i+1}^{S_i} \sum_{\substack{j=1 \\ j\neq i}}^{m} f_j(m_i,t) * [\,\bar{B}(t) p_i(m-n,t)\,]$$

$$1 \le i \le m; \quad o < n \le s_i \qquad \ldots. \ (11)$$

$$\Psi_i(o,t) = f_{X_{br(i)}}(t) * [\,\bar{B}(t)\; \bar{P}_i(s_i,t)\,] + \sum_{m_i=s_i+1}^{S_i} \sum_{\substack{j=1 \\ j\neq i}}^{m} f_j(m_i,t) * [\,\bar{B}(t)\; \bar{P}_i(m_i,t)\,],$$

$$1 \le i \le m \qquad \ldots \ (12)$$

The derivation of (10) is based on the classification on the occurrence of a reorder in (o,t) or otherwise. When a reorder is placed in order that no b-event occurs in the interval under consideration the order placed in (o,t) should not materialise before t.

The derivation of (11) is reasoned as follows: The required probability is possible only when there is a reorder. This reorder may be due to the ith event or any other event j different from i. Again in the time duration under consideration there can be no replenishment. Derivation of (12) is similar to that of (11).

Using expressions (10) - (12), the stationary distribution of the marginal level process is obtained by renewal theoretic arguments and Key renewal theorem Cox (1962). We state this as,

<u>Theorem 3:</u>

The marginal level process $\{L_i(t),\ t \geq 0\}$ has a probability distribution

$$\bar{\Lambda}_i(n,t) = \Psi_i(n,t) + \sum_{n=1}^{\infty} f_b^{(n)}(t) * \Psi_i(n,t) \qquad \ldots (13)$$

$$1 \leq i \leq m; \quad o \leq n \leq S_i$$

and the stationary distribution

$$\bar{\Lambda}_i(n) = \frac{1}{E\,[X_b]} \int_o^{\infty} \bar{\Lambda}_i(n,u)\,du$$

Theorems (1) - (3) provide all the results necessary to write the expected cost expression.

Let $C_{\bar{\Lambda}_i}$ be unit shortage cost (of the item i)

K - fixed ordering cost.

u_i - unit variable cost

C_{si} - storage cost (of type i) per unit per unit time.

Then the expected cost expression per unit time is

$$\frac{1}{E[X_b]}\left[K + \sum_{i=1}^{m} u_i (S_i - s_i)\right] + \sum_{i=1}^{m} \sum_{n=0}^{S_i} n C_{si} \pi_i(n) + \sum_{i=1}^{m} \pi_i(o)\, C_{\pi_i} \qquad \ldots\ (15)$$

The expected cost expression consists of three distinct components. The first component is contributed by the reorders. We have introduced a fixed cost of order and a variable component for every item. Normally, the items replenished at a reorder epoch is bounded by $S_i - s_i$. We have taken the quantity as equivalent to $S_i - s_i$. (in this sense this component is a higher estimate). Thus the ordering (total) cost is multiplied by the frequency of replenishment epochs to determine the contribution in the total cost expression corresponding to ordering.

The second component consists of the average inventory of the ith type. The capital blocked on this is obtained by multiplying the average inventory by the unit cost of the items. Summation over all items determine the contribution corresponding to the storage cost.

The final component represents the cost of shortages. We have provided a flexibility in the expression to accommodate different shortage cost for different types. The shortages for the ith type of items occur according to a rate λ_i $(= \lambda \alpha_i)$ as when the corresponding marginal distribution of the inventory level is zero. Hence, the total shortage cost is obtained a sum of the product of rate of occurrence of demands for type i events and the stationary marginal probability for inventory level zero.

As all the quantities used in (15) are numerically computable, a simple search procedure shall determine the optimal parameters to minimize cost.

SPECIAL CASE:

We now present the modifications in the $\psi_i(n,t)$ functions for the special case of instantaneous lead-time. It is immediate that,

$$\psi_i(n,t) = 0 \qquad 0 \le n \le s_i; \quad 1 \le i \le m$$

$$= p_i(S_i-n,t) \prod_{j=1\,:\,j \ne i}^{m} P_j(S_j-s_j,t) \qquad s_i < n \le S_i; \quad 1 \le i \le m \qquad \text{.. (16)}$$

and the stationary distribution is given by

$$\bar{\Pi}_i(n) = \frac{1}{E[X_b]} \int_0^\infty \psi_i(n,u)\,du, \qquad s_i < n \le S_i$$

$$= 0 \qquad \text{Otherwise} \qquad \text{.. (17)}$$

The value of $E[X_b]$ used in (17) is determined by the first term of expression (8).

For the special case of $m = 2$,

$$E[X_b] = \frac{1}{\lambda} \sum_{k=0}^{S_1-s_1-1} \alpha_1^k \sum_{j=0}^{S_2-s_2-1} \alpha_2^j \binom{k+j}{j}$$

and

$$\int_0^\infty \psi_1(n,t)\,dt = \frac{1}{\lambda}\, \alpha_1^{S_1-n} \sum_{j=0}^{S_2-s_2-1} \alpha_2^j \binom{S_1-n+j}{j}$$

Hence,

$$\pi_1(n) = \frac{\alpha_1^{S_1-n} \displaystyle\sum_{j=0}^{S_2-s_2-1} \alpha_2^j \binom{S_1-n+j}{j}}{\displaystyle\sum_{k=0}^{S_1-s_1-1} \alpha_1^k \sum_{j=0}^{S_2-s_2-1} \alpha_2^j \binom{k+j}{j}} ; \quad s_1 < n \leq S_1$$

It is interesting to note that the Stationary distribution is not uniform as in the case of single item inventory model (Sivazlian, 1975).

REFERENCES

1. Agarwal, V. (1984): Coordinated order cycles under joint replenishment of multi-item inventories. Nav. Log. Res. Qtly. PP. 131-136.

2. D.R. Cox (1962): Renewal theory, Methuen, London.

3. A. Federgven (1984): Coordinated replenishment in a multi-item inventory system with compound poission demands, Management Science, pp. 344-357.

4. B.L. Miller (1971): A multi-item inventory model with joint back-order criterion, Oper. Res. pp. 1467-1476.

5. J.A. Muckstadt (1979): A Three Echelon, Multi-item model for recoverable items, Nav. Log. Res. Quarterly, pp. 199-222.

6. E.A. Silver (1974): A Control system for Co-ordinated inventory replenishment, IJPR, pp. 647-671.

7. S.K. Srinivasan (1976): Stochastic point process Griffin, London.

8. S.K. Srinivasan (1979): General Analysis of (S, s) inventory systems with random lead times and unit demands. J. Math. Phy. Sciences, Vol. 13, No. 2, pp. 107-129.

9. S.K. Srinivasan (1988): Analysis of (s, S) inventory systems with general lead time and demand distribution- Optimization (in print).

10. B.D. Sivazlian (1975): Continuous review (s, S) inventory system with arbitrary inter-arrival distribution between unit demands,Oper. Res., Vol. 22, pp. 65-71.

11. T.M. Thomstone and E.A. Silver (1975): A Coordinated inventory control system. IJPR, pp. 581-602.

-x-x-x-