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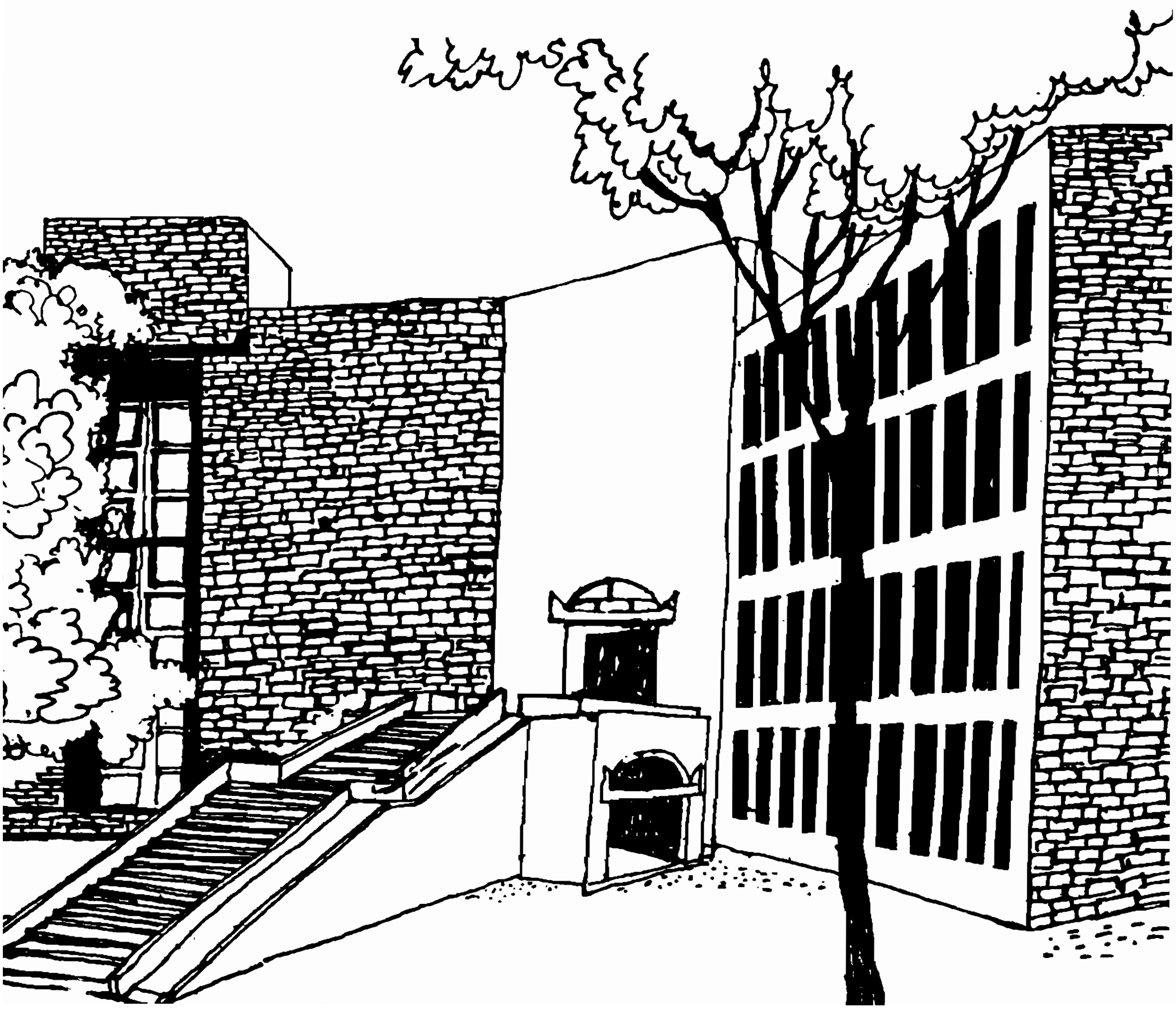

CHERNOFF AND A NEW CONGRUENCE AXIOM IMPLIES FULL RATIONALITY

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Abstract

Rationality in choice theory has been an abiding concern of decision theorists. A rationality postulate of considerable significance in the literature is the weak congruence axiom of Richter [1971] and Sen [1971]. It is well known that in discrete choice contexts of the classical type [i.e. all nonempty finite subsets of a given set comprise the set of choice problems], this axiom is equivalent to full rationality. The question is: whether a (considerable?) weakening of the weak congruence axiom would suffice to imply full rationality? This is the question we take up in this paper.

We propose a weaker new congruence axiom which along with the (mother of all axioms:) Chernoff Axiom implies full rationality. The two axioms are independent. We also study interesting properties of these axioms and their interconnections through examples.

1. Introduction: Rationality in choice theory has been an abiding concern of decision theorists. A rationality postulate of considerable significance in the literature is the weak congruence axiom of Richter [1971] and Sen [1971]. It is well known that in discrete choice contexts of the classical type [i.e. all nonempty finite subsets of a given set comprise the set of choice problems], this axiom is equivalent to full rationality. The question is: whether a (considerable?) weakening of the weak congruence axiom would suffice to imply full rationality? This is the question we take up in this paper.

We propose a weaker new congruence axiom which along with the (mother of all axioms:) Chernoff Axiom implies full rationality. The two axioms are independent. We also study interesting properties of these axioms and their interconnections through examples.

2. An Overview of Full Rationality: In this section we closely follow Suzumura (1983).

Let X be a nonempty universal set and let Σ denote the set of all non-empty finite subsets of X. A choice function (on Σ) is a correspondence $C:\Sigma \rightarrow\rightarrow X$ such that

$\phi \neq C(S) \subset S \ \forall \ S \in \Sigma.$

Given a choice function C, we define binary relations:

$$R_c = \bigcup_{S \in \Sigma} [C(S) \times S]$$

$$R^c = \bigcup_{x, y \in X} [C(\{x, y\}) \times \{x, y\}]$$

$$R_c^* = \bigcup_{S \in \Sigma} [C(S) \times (S \setminus C(S))].$$

Given a binary relation R on X, let T(R) denote its transtive closure:

$(x, y) \in T(R) \leftrightarrow \exists \ t \in \mathbf{N}$ and $x = x^0, x^1, \ldots, x^t = y$ such that

$(x^{\tau-1}, x^{\tau}) \in R \ \forall \ \tau = 1, \ldots, t.$

A choice function C is said to satisfy the Weak Axiom of Revealed Preference (WA) if $(x, y) \in R_c \rightarrow (y, x) \notin R_c^*$.

A choice function C is said to satisfy the Weak Congruence Axiom (WCA) [of Richter (1971) and Sen (1971)] if

$(x,y) \in R_c,\ x \in S, y \in C(S) \rightarrow x \in C(S)$.

A choice function C is said to satisfy the [Richter (1971), Sen (1971)] Strong Congruence Axiom (SCA) if

$(x,y) \in T(R_c), x \in S, y \in C(S) \rightarrow x \in C(S)$.

A choice function C is said to satisfy the Strong Axiom of Revealed Preference (SA) if $(x,y) \in T(R_c^*) \rightarrow (y,x) \notin R_c$.

A choice function C is said to satisfy the Houthakker Axiom (HA) if $(x,y) \in T(R_c) \rightarrow (y,x) \notin R_c^*$.

A choice function C is said to satisfy Arrow's Axiom (AA) [Arrow, 1959] if $S, T \in \Sigma, S \subset T$ and

$\phi \neq C(T) \cap S \rightarrow C(S) = C(T) \cap S.$

A binary relation R on X is said to be

(a) reflexive if $(x,x) \in R \ \forall\ x \in X$

(b) complete if $x \neq y, x, y \in X \rightarrow (x,y) \in R \vee (y,x) \in R$

(c) transitive if $\forall x, y, z \epsilon X, (x,y) \epsilon R \wedge (y,z) \epsilon R \rightarrow (x,z) \epsilon R$

Given a binary relation R on X and $S \epsilon \Sigma$ let

$G(S,R) = \{x \epsilon S / (x,y) \epsilon R \ \forall y \epsilon S\}$.

A choice function C is said to be full rational (FR) if there exists a reflexive, complete and transitive binary relation R on X:

$C(S) = G(S,R) \ \forall S \epsilon \Sigma$.

The following diagram is obtained from received theory:

$$\begin{array}{lllll} WA\ (\leftrightarrow WCA) & \leftarrow SA & \leftarrow HA & (\leftrightarrow SCA) \\ \downarrow & & & \uparrow \\ AA & \leftrightarrow & & FR \end{array}$$

The mother of all choice axioms is the following:

A choice function C is said to satisfy the Chernoff Axiom (CA) if $\forall S, T \epsilon \Sigma, S \subset T \rightarrow C(T) \cap S \subset C(S)$.

Clearly CA is weaker than AA. Let us provide an example to show that CA does not imply rationality, let alone full rationality.

Example 1:- $X = \{x, y, z\}$, $C(X) = \{x\}$, $C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{a\}) = \{a\}$ $\forall a \in X$.

Clearly C satisfies CA.

Towards a contradiction assume that there exists a binary relation R on X : $C(S) = G(S\ R)$ $\forall S \in \Sigma$.

Thus $C(\{y, z\}) = \{y\} \rightarrow (y, z) \in R$ and

$C(\{x, y\}) = \{x, y\} \rightarrow (y, x) \in R$

Since $C(\{y\}) = \{y\} \rightarrow (y, y) \in R$, we have

$(y,a) \in R\ \forall\ a \in X$. Thus $y \in G(X,R)$. However, $y \notin C(X)$.

The following observation is immediate:

Given a choice function C if there exists a binary relation R on X such that $C(S) = G(S,R)\ \forall S \in \Sigma$ then $R = R^c$

It is easy to see that if C satisfies CA then $C(S) \subseteq G(S, R^c) \forall S \in \Sigma$.

3. A New Congruence Axiom:- In this section we propose and derive consequences of a new congruence axiom.

A choice function C is said to satisfy New Congruence Axiom (NCA) if $(x,y) \in R^c, x \in S, y \in C(S) \rightarrow x \in C(S)$.

Since $R^c \subset R_c$ we must have WCA → NC.

However, the converse need not be true:

Example 2:- X = {x, y, z, w}; C(X) = {x},
C({x, y, z}) = {x, y}, C({x, y, w}) = {x},
C({y, z, w}) = {w}, C({x, z, w}) = {x}, C({x, y}) = {x},
C({x, z}) = {x}, C({y, z}) = {y}, C({w, z}) = {w},
C({x, w}) = {x}, C({y, w}) = {w}, C({a}) = {a} $\forall a \in X$.

Here C satisfies NCA; however C does not satisfy WCA. This is because (y, x) $\in R_c, x \in C(X)$ and yet $y \notin C(X)$.

It should be noted that CA and NCA are independent.

Example 3:- $X = \{x, y, z\}$; $C(X) = \{x\}$, $C(\{x, y\}) = \{x,y\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, $C(\{a\}) = \{a\}$ $\forall a \in X$. C satisfies CA as is easily verified. However, C does not satisfy $NCA: (y,x) \in R^c, x \in C(X)$ but $y \notin C(X)$.

Example 4:- $X = \{x, y, z\}$; $C(X) = \{x, y\}$, $C(\{x, y\}) = \{y\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, $C(\{a\}) = \{a\}$ $\forall a \in X$. C satisfies NCA as is easily verified.

However, C does not satisfy $CA: x \in C(X)$ but $x \notin C(\{x,y\})$.

Theorem 1:- A choice function C is full rational if and only if C satisfies CA and NCA.

Proof:- It is easy to see that if C is fully rational then C satisfies CA and NCA.

Now suppose C satisfies CA and NCA, By CA, $C(S) \subset G(S,R^c)$ $\forall$ $S \in \Sigma$.

Now suppose $x \in G(S, R^c)$. Thus $x \in S$ and $(x,y) \in R^c \ \forall y \in S$. Thus $(x,y) \in R^c \ \forall \ y \in C(S)$. By NCA, $x \in C(S)$.

Thus $G(S, R^c) \subset C(S)$.

Hence $C(S) = G(S, R^c)$.

Remains to show that R^c is transitive (:since it is always reflexive and complete). Let $(x,y) \in R^c$ *and* $(y,z) \in R^c$. Consider S = {x, y, z}. If $x \in C(S)$, then $x \in C(\{x,z\})$ by CA and hence $(x,z) \in R^c$. If $y \in C(S)$, then $(x,y) \in R^c$ and NCA implies $x \in C(S)$. Thus $(x,z) \in R^c$. If $z \in C(S)$, then $(y,z) \in R^c$ and NCA implies $y \in C(S)$. But then, $y \in C(S)$ *and* $(x,y) \in R^c$ implies $x \in C(S)$. Hence $(x,z) \in R^c$. Thus R^c is transitive. This proves the theorem.

Q. E. D.

The following axiom is of considerable interest in the literature on rational choice theory:

A choice function C is said to satisfy generalized Condorcet property (GC) if $G(S, R^c) \subset C(S) \forall S \in \Sigma$.

It is well known that GC along with CA implies acyclic rationality, which is really the minimal rationality requirement one can require for a choice function. On the other hand NCA along with CA implies full rationality, which in some senses is the maximal rationality requirement one can require for a choice function. It is not very difficulty to see that NCA implies GC.

Remark 1:- In Aizerman and Aleskerov (1995) it is asserted that any choice function can be expressed as a finite union of choice functions satisfying G.C. This is true only if we allow choice functions to be empty valued. Otherwise it is not true.

Example 5:- Let X = {x, y, z}; Let C({x, y}) = {x},

$C(\{x, z\}) = \{x\}$, $C(\{x, y, z\}) = \{y\}$, and $C(S) \neq \Phi$ for all other $S \epsilon \Sigma$. Towards a contradiction suppose there exists $C_1, \ldots, C_p$ satisfying GC such that

$$C(S) = \bigcup_{i=1}^{p} C_i(S) \;\forall\; S \epsilon \Sigma.$$

Clearly $C_i(\{x, y\}) = \{x\} \forall i$

and $C_i(\{x, z\}) = \{x\} \forall i$.

By GC, $x \epsilon C_i(X) \forall i$ and thus $x \epsilon C(X)$, contradicting our definition of C.

Remark 2:- It is interesting to note that in general a choice function satisfying CA need not be expressible as a finite union of choice functions satisfying AA.

Example 6:- Let $X = \{x, y, z\}$; $C(X) = \{y\}$,

$C(\{x, y\}) = \{x, y\}$, $C(\{x, z\}) = \{z\}$, $C(\{y, z\}) = \{y\}$, $C(\{a\}) = \{a\}$ $\forall a \in X$.

Suppose towards a contradiction that

$C(S) = \bigcup_{i=1}^{p} C_i(S)$ $\forall S \in \Sigma$. where $C_1, \ldots, C_p$ satisfies AA.

Thus $C_i(X) = \{y\}$ $\forall i$

and $C_j(\{x, y\}) \ni x$ for some j.

Now $y \in C_j(X) \cap (\{x, y\})$.

Thus by AA, $C_j(\{x, y\}) = C_j(X) \cap \{x, y\} = \{y\}$ which is contradiction.

Remark 3:- If a choice function C satisfies CA and GC then there exists a function $\phi : X x X \to \mathbf{R}$ such that

$C(S) = \{x \in X / \phi(x,y) \geq 0 \ \forall \ y \in S\}, \ \forall S \in \Sigma$

For, let $\phi(x,y) = 1$ if $\{x\} = C(\{x, y\})$

$\phi(x,y) = 0$ if $\{x, y\} = C(\{x, y\})$

$\phi(x,y) = -1$ if $\{y\} = C(\{x, y\})$

Let $\bar{C}(S) = \{x \in X / \phi(x,y) \geq 0 \forall y \in S\} \forall S \in \Sigma$.

Let $x \in \bar{C}(S)$. Thus $x \in C(\{x,y\}) \forall y \in S$.

By GC, $x \in C(S)$.

Now, let $x \in C(S)$. By CA, $x \in C(\{x,y\}) \forall y \in S$.

Thus $\phi(x,y) \geq 0 \forall y \in S$. Thus $x \in \bar{C}(S)$.

Thus $C(S) = \tilde{C}(S)\ \forall S \epsilon \Sigma$.

Remark 4:- We have already noted that AA → CA. It is easy to see that AA → NCA: let $(x, y) \in R^c$, $x \in S$, $y \in C(S)$. Thus $C(S) \cap \{x, y\} \neq \phi$

By AA, $C\{\{x, y\} = C(S) \cap \{x, y\}$
Thus $x \in C(S)$.

On the other hand CA and NCA → AA. It is enough to show that $S, T \epsilon \Sigma, S \subset T \text{ and } C(T) \cap S \neq \phi$ implies

$C(S) \subset C(T) \cap S$ ($: C(T) \cap S \subset C(S)$ follows from CA). Let $x \epsilon C(S)$ and towards a contradiction assume $x \notin C(T)$.

Let $y \epsilon C(T) \cap S$. Thus by CA, $C(S) \cap \{x, y\} \subset C(\{x, y\})$ and so $x \in C(\{x, y\})$. Thus $(x, y) \in R^c$. Since $x \in S \subset T$ and $y \in C(T)$, by NCA, $x \in C(T)$ which is a contradiction. Thus
$C(S) \subset C(T) \cap S$.

In view of Remark 4 and Theorem 1, we have the following diagram:

$$\begin{array}{lcl} \mathrm{WA}\ (\leftrightarrow \mathrm{WCA}) & \leftarrow \mathrm{SA} \leftarrow & \mathrm{HA}\ (\leftrightarrow \mathrm{SCA}) \\ \downarrow & & \uparrow \\ \mathrm{AA} & \leftrightarrow \quad \mathrm{CA+NCA} \quad \leftrightarrow & \mathrm{FR} \end{array}$$

The direction of the arrow simply indicates the implications which are relatively easy to establish. Actually each entry is equivalent to any other on Σ.