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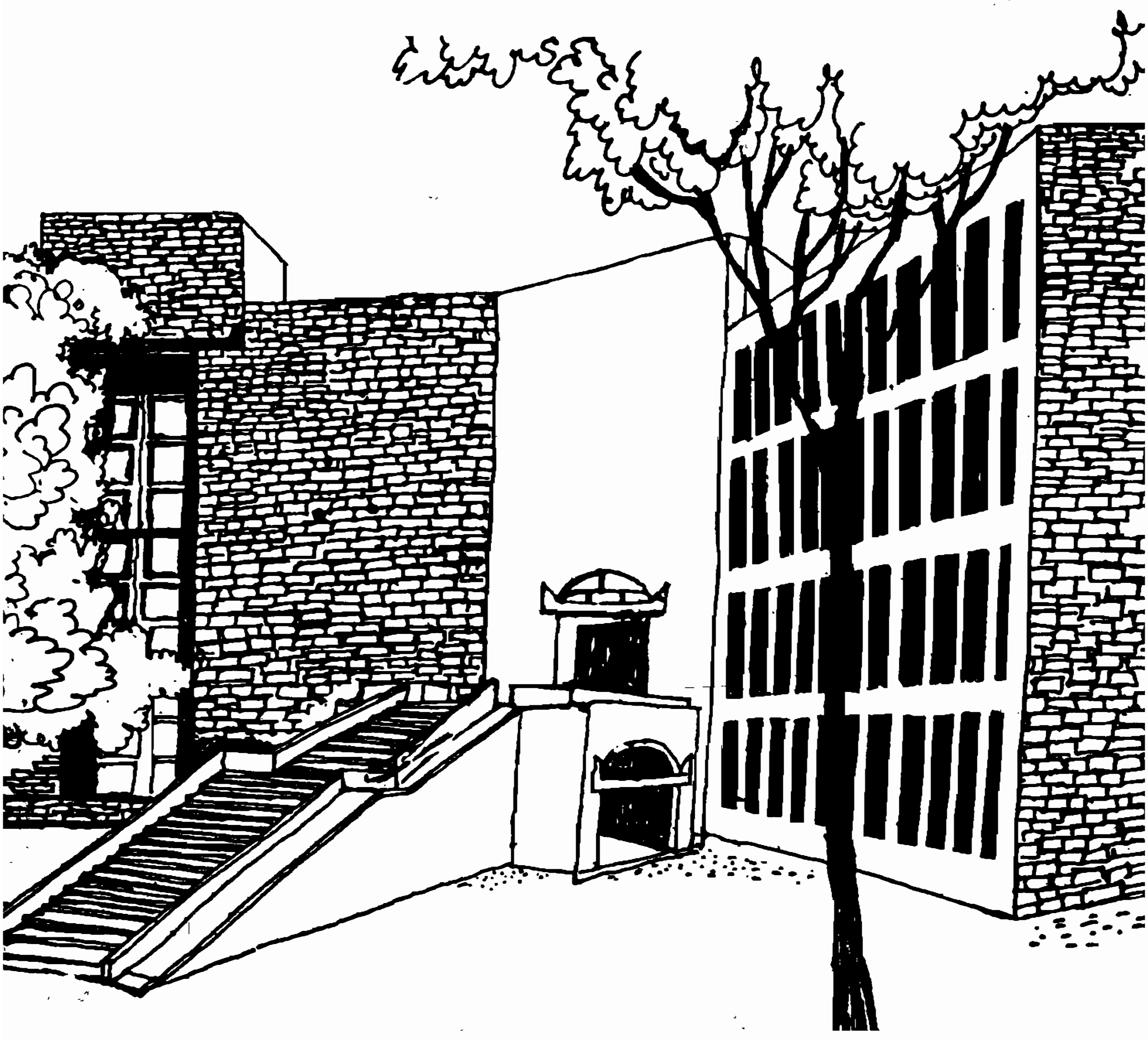

THE CONSTRAINED EQUAL AWARDS SOLUTION FOR CLAIMS PROBLEMS

By

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Abstract

In this paper we propose a variable population framework for the study of claims problems and obtain characterizations of the constrained equal award rule using the following properties: envy-freeness, individual rationality from equal division, resource monotonicity and bilateral consistency.

1. Introduction:

In this paper we are concerned with the problem of allocating a single homogeneous divisible good amongst a finite number of agents, when their aggregate demand exceeds the total available supply. This is a problem which has precedents both in academics as well as in real world situations. The antecedents of the problem can be found in the bankruptcy literature studied in the Babylonian Talmud: the total estate is less than what has been willed away by a deceased ancestor to his several heirs. With this interpretation in mind, O'Neill [1982] and Aumann and Maschler [1985], provided rigorous game-theoretic analysis of some solutions to such problems. The papers by Curiel, Maschler and Tijs [1988], Dagan and Volij [1993], Dagan [1996] and Lahiri [1996a] are in the same tradition.

In the paper by Lahiri [1996a], an interpretation is proposed within the framework of supply-chain management. The problem there was to characterize the proportional solution using a reduced game property and the interpretation of the problem was thus: assume there is a distributor of a commodity who supplies to a finite number of retailers. Suppose the aggregate demand of the retailers is equal to the total amount available with the distributor. Then the problem is resolved very easily: give each retailer what he desires. However, if the aggregate demand exceeds the available supply, the distributor has to ration the retailers. How does he do it?

There is a major problem with this interpretation if we view our solutions as serious prescriptions in real world situations. If the retailers know that they are to be rationed (as it happens when there is chronic shortage) then they have strong incentives to inflate their demands for most allocation rules. The above scenario will deliver, only if shortages are not chronic, so that retailers run the risk of accumulating unwanted inventories by overstating their demands. In other words, orders should be placed with the distributors before the production is realised (as is usually the case) and there should be no inkling about what the final supply conveys: either enough or shortage. In such a situation, rationing rules will work.

Armed with this interpretation in Lahiri [1996b] we provide an algorithm for the min-max loss rule for claims problems, where loss is meausred by unsatisfied demands.

In Dagan [1996a], the constrained equal awards rule (to be defined later) has been shown to satisfy a number of desirable properties like independence of irrelevant claims, equal treatment and composition (a property probably due to Young [1988]). The framework of Dagan's paper is one of fixed population size. Lahiri [1996a] studies claims problem with a variable population as is done by so many others cited in the paper. In this paper we propose a variable population framework for the study of claims problems and obtain a characterization of the constrained equal award rule using the following properties: envy-freeness, individual rationality from equal division, resource monotonicity and bilateral consistency. The first property can be traced to Foley [1967]; the second and fourth to Dagan [1996b]; the third property is purely original.

In Dagan [1996b] axiomatic characterizations of the Constrained Equal Awards Solution (also called the Uniform Rule) is proposed without using any resource monotonicity assumption. However, the framework of that paper is resource allocation with single peaked preferences. In terms of preferences, our problems form a strict subset of the domain of single peaked preferences, allowing only those single peaked preferences, whose portion over the horizontal axis forms an isosceles triangle with the origin as a vertex. Thus results for single peaked preferences may fail to hold, since proofs on the larger domain of single peaked preferences normally avail of greater manouverability.

In this paper we show that if we add the assumption of resource monotonicity to each of Dagan's characterizations in Dagan [1996b], the results established there continue to hold on our domain. Since the proofs now are far from obvious, the subsequent discussion is necessary.

2. The model:

There is a population of "potential agents", indexed by elements in a set I. Let P denote the set of all non-empty finite subsets of I. Given MεP, let $\mathbb{R}_+^M$ *(respectively* $\mathbb{R}_{++}^M$) denote the set of all functions from M *to* $\mathbb{R}_+$ *(respectively* $\mathbb{R}_{++}$) . Here $\mathbb{R}_+$ is the set of all non-negative real numbers and $\mathbb{R}_{++}\equiv\mathbb{R}_+\setminus\{0\}$.

Given $M\varepsilon P$, a claims problem for M is an ordered pair $(c,E)\varepsilon\mathbb{R}_{++}^M x\mathbb{R}_+$ *such that* $\sum_{i\varepsilon M} c_i>E$.

Let $\mathcal{C}^M$ denote the set of all claims problems for M and $\mathcal{C}\equiv\bigcup_{M\varepsilon P}\mathcal{C}^M$. *Let* $X\equiv\bigcup_{M\varepsilon P}\mathbb{R}_+^M$.

Given $(c,E)\varepsilon\mathcal{C}^M, M\varepsilon P$, an allocation for (c,E) is a vector $x\varepsilon\mathbb{R}_+^M$ such that $\sum_{i\varepsilon M} x_i=E$ *and* $x_i\leq c_i\forall i\varepsilon M$.

A solution is a function $F:\mathcal{C}\to X$ such that F(c,E) is an allocation for (c,E) whenever $(c,E)\varepsilon\mathcal{C}$

The constrained equal awards solution $CEA:\mathcal{C}\to X$ is defined as follows:
$\forall(c,E)\varepsilon\mathcal{C}^M, M\varepsilon P,\forall i\varepsilon M,\ CEA_i(c,E)=\min\{\lambda,c_i\}$ with $\lambda\geq0$ satisfying $\sum_{i\varepsilon M}\min\{\lambda,c_i\}=E$.

It is easy to show that such a λ always exists uniquely.

No-envy, property:- A solution F is said to satisfy the no-envy, property if $\forall M\varepsilon P,\forall(c,E)\varepsilon\mathcal{C}^M\forall i,j\varepsilon M, c_i-F_i(c,E)\leq|c_i-F_j(c,E)|$

The no-envy property is quite simple: between any two agents there should not arise a situation where any one's unfulfilled demands exceed the deviation of the other's from the first agent's claim i.e. no one's excess demand should be greater than either the excess supply or excess demand of the other from the one's point of view. If the situation were otherwise, then there would be an agent who would want someone else's allotment, since that would lead to a lower loss for him/her, where loss is measured in terms of deviation from announced demands.

Individual Rationality from equal division:- A solution F is said to satisfy individual rationality from equal division if $\forall M \varepsilon P, \forall (c,E) \varepsilon \mathcal{B}^M \ \forall i \varepsilon M,\ c_i - F_i(c,E) \leq |E/_{|M|} - c_i|$

Once again the meaning is clear: for every agent the excess demand should not exceed his deviation from equal division of resources.

The following theorem is immediate.

Theorem 1: (a) CEA satisfies the no-envy property
(b) CEA satisfies individual rationality from equal division.

Proof: Let $CEA(c,E) = x \varepsilon \mathbb{R}^M_+$ *for some* $M \varepsilon P, (c,E) \varepsilon \mathcal{B}^M$.

(a) Suppose towards a contradiction that there exists i, jεM with

$$c_i - x_i > |c_i - x_j|$$

Clearly $c_i \neq x_i$

$$\therefore 0 \leq x_i = \lambda < c_i$$

where $\sum_{k \varepsilon M} \min\ \{\lambda, c_k\} = E.$

Since $x_j \neq x_i$, *we have* $x_j \neq \lambda$.

Thus $x_j = c_j$

$$\therefore c_i - \lambda > |c_i - c_j| \text{ with } c_j < \lambda < c_i.$$

$$\therefore c_i - \lambda > c_i - c_j$$

$$\therefore \lambda < c_j$$ which is a contradiction.

This proves (a).

(b) Suppose towards a contradiction that there exists iεM with

$$c_i - x_i > |c_i - E/_{|M|}|.$$

Thus $x_i = \lambda$ *where* λ *is as in (a) and* $\lambda < c_i$.

$$\therefore c_i - \lambda > |\ c_i - E/_{|M|}\ |$$

Case 1: $\lambda < E/_{|M|}$.

$$\therefore E = \sum_{k \in M} x_k = \sum_{\lambda \le c_k} \lambda + \sum_{c_k < \lambda} c_k < |M| . \frac{E}{|M|} = E$$

which is a contradiction. Thus Case 1 cannot occur and we have

Case 2:- $$\lambda \geq \frac{E}{|M|}$$

$$\therefore c_i > \lambda \geq \frac{E}{|M|}$$

Hence $$c_i - \lambda > c_i - \frac{E}{|M|}$$

i.e. $$\lambda < \frac{E}{|M|}$$ which is again a contradiction.

This proves (b).

Q.E.D.

In the rest of the paper we invoke the following property:

Resource Monotonicity:- A solution F is said to satisfy resource monotonicity if $\forall M \varepsilon P$,

$(c,E) \varepsilon\, \mathcal{B}^M, (c,E') \varepsilon\, \mathcal{B}^M, E' \geq E$ *implies* $F(c,E') \geq F(c,E)$.

The meaning of resource monotonicity is simple and needs no further explanation.

3. The Main Reults:- In this section we establish modifications of results appearing in Dagan (1996b) which are valid in our context.

Lemma 1:- If a solution F satisfies no-envy and resource monotonicity, then it coincides with CEA solution for all two agent problems.

Proof:- Towards a contradiction assume that there exists $\{i,j\} \varepsilon P$ *and* $(c_i,c_j,E) \varepsilon\, \mathcal{B}^{\{i,j\}}$ such that $F(c_i,c_j,E) \neq CEA(c_i,c_j,E)$ where we have that F satisfies no-envy and resource monotonicity. Without loss of generality assume $c_i \leq c_j$. *Then if* $(x_i,x_j) = F(c_i,c_j;E)$ we must have $x_i < c_i$, *and* $x_j \neq x_i$. If $x_j < x_i$ *then* $|c_j - x_j| > |c_j - x_i|$ Contradicting no-envy. Thus $x_j > x_i$.

If $x_i < x_j < c_i$, *then* $|c_i - x_i| > |c_i - x_j|$ contradicting no-envy.

Thus $x_i<c_i<x_j$. In fact we must have $x_i<c_i<2c_i-x_i\leq x_j$ so that no-envy is satisfied. Thus $2c_i\leq x_i+x_j$.

Hence if $E<2c_i,F(c_i,c_j;E)=CEA(c_i,c_j;E)$. By resource monotonicity, $F(c_i,c_j;E)=CEA(c_i,c_j;E)$ if $E\leq 2c_i$.

Thus for $E=2c_i,F(c_i,c_j;E)=(c_i,c_i)$.

For $E>2c_i$, by monotonicity, $F_i(c_i,c_j;E)=c_i$. This contradicts $x_i<c_i$.

Hence $F(c_i,c_j;E)=CEA(c_i,c_j;E)$.

Q.E.D.

Lemma 2:- If a solution F satisfies individual rationality from equal division and resource monotonicity, then it must coincide with the Constrained Equal Awards Solution for all two agent problems.

Proof:- As in Lemma 1, let us assume that (c_i, c_j ; E) is a claims problem and F satisfies the properties listed in Lemma 2. Suppose $F(c_i,c_j;E) = (x_i,x_j) \neq CEA(c_i,c_j;E)$.

Assuming without loss of generality $c_i\leq c_j$, we must have $x_i<c_i$, $x_i \neq x_j$.

Suppose $x_j<x_i<c_i\leq c_j$.

Then $c_j-\frac{x_i+x_j}{2}<c_j-x_j$ contradicting individual rationality from equal division. Thus $x_i<x_j$

If $x_i<x_j\leq c_i\leq c_j$, *then* $c_i-\frac{x_i+x_j}{2}<c_i-x_i$, once again contradicting individual rationality from equal division. Thus $c_i<x_j$

Suppose $\frac{x_i+x_j}{2}<c_i$.

Then $c_i-x_i\leq c_i-\frac{x_i+x_j}{2}$

implies $x_i\geq\frac{x_i+x_j}{2}$ contradicting $x_j>x_i$.

Thus $x_i+x_j\geq 2c_i$

Hence for $E<2c_i,F(c_i,c_j;E)=CEA(c_i,c_j;E)$

By resource monotonicity,

$$F(c_i,c_j;E) = CEA(c_i,c_j;E) = \left(\frac{E}{2},\frac{E}{2}\right) for\ E \le 2c_i.$$

By resource monotonicity, $E>2c_i$ implies $F_i(c_i,c_j;E) = c_i$ which contradicts $x_i<c_i$. *Thus* $F(c_i,c_j;E)=CEA(c_i,c_j;E)$.

Q.E.D.

4. Consistency and Related Results:-

Consistency: A solution F is said to satisfy consistency if

$$\forall M \epsilon P,(c,E)\epsilon \mathscr{G}^M, x=F(c,E), \phi \neq N \subset M, \left(c_N, \sum_{i \epsilon N} x_i\right) \epsilon \mathscr{G}^N, \text{ implies } x_N = F\left(c_N, \sum_{i \epsilon N} x_i\right)$$

Here $c_N=(c_i)_{i\epsilon N}$ and $x_N = (x_i)_{i\epsilon N}$.

Bilateral Consistency is simply the same property as above requiring in addition that N should be a set consisting of exactly two members.

Converse-Consistency: A solution F is said to satisfy converse-consistency if $\forall\ M\epsilon P,(c,E)\epsilon\mathscr{G}^M, x$ is an allocation for (c,E) and $\forall\ \phi\neq N\subset M, N$ has exactly two members, $x_N=F\left(c_N,\sum_{i\epsilon N}x_i\right)$, *then* $x=F(c,E)$.

The following lemma is easy to prove:

Lemma 3:- CEA satisfies consistency and converse-consistency.

We need one more lemma, before we can state the results that we promised in the introduction.

Lemma 4:- If F is a solution which satisfies bilateral consistency and agrees with CEA for all two agent problems, then F = CEA.

Proof:- Essentially the proof of Lemma 4 in Dagan (1996b).

We now have the following two major characterization theorems, by using the results obtained so far.

Theorem 2:- The unique solution on $\mathscr{G}$ to satisfy bilateral consistency, no-envy and resource monotonicity is CEA.

Theorem 3:- The unique solution on $\mathscr{G}$ to satisfy bilateral consistency, individual rationality from equal division and resource monotonicity is CEA.

5. Conclusion:- The results contained in Theorems 2 and 3 are new to the best of our knowledge and establishes the contrained equal awards rule as the unique solution to satisfy rather mild yet meaningful properties. This should enhance its value in real life situations.

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Appendix

In this appendix we take a relook at the axiomatic characterization of the Constrained Equal Awards Solution due to Dagan (1996) and modify it to fit into our framework. The basic difference between our approach and that of Dagan (1996), is that Dagan allows claims to be equal to zero whereas our claims are all positive. Since a person with zero claims automatically gets nothing, such agents should not affect the allocation for agents with positive claims. With this change in definition, the Composition axiom due to Dagan, now needs to be embedded in a variable agent framework as we have done and the content of the said property needs to be modified somewhat.

Independence of Irrelevant Claims (IIC):- A solution F satisfies independence of irrelevant claims if

$\forall M \varepsilon P, \forall (c,E) \varepsilon B^M, F(c,E) = F(c^B,E)$ where

$c_i^B = \min\{c_i,E\}\ \forall i \varepsilon M$ and

$c^B = (c_i^B)_{i\varepsilon M}$.

Lemma 1:- CEA satisfies IIC.

Proof:- Let $M \varepsilon P$ and $(c,E) \varepsilon B^M$. Let x = CEA (c,E) where $x_i = \min\{\lambda, c_i\} \forall i \varepsilon M$.

Case 1:- $\lambda \leq E$.

Then $\min\{\lambda, c_i\} = \min\{\lambda, c_i, E\}$

$= \min\{\lambda, \min\{c_i, E\}\}$

$= \min\{\lambda, c_i^B\} \quad \forall i \varepsilon E.$

Case 2:- $\lambda > E$.

Suppose $\min\{\lambda, c_i\} \neq c_i$ for some $i \varepsilon M$.

Thus $c_i > \lambda > E$.

Since $x_i = \lambda < c_i, E = \sum_{j\varepsilon M} x_j \geq x_i = \lambda > E$ which is a contradiction.

Thus $\min\{\lambda, c_i\} = c_i \quad \forall i$

Hence $x_i = c_i \quad \forall i$

$\therefore E = \sum_{i\varepsilon M} c_i > E$ which is a contradiction.

Thus Case 2 cannot occur.

This proves Lemma 1.

Q.E.D.

Composition(Comp.):- A solution F satisfies composition if

$\forall M\varepsilon P, \forall (c,E')\varepsilon B^M$ and for all

$0 \leq E' \leq E, F(c,E') = x$ and

$K = \{i\varepsilon M / x_i < c_i\}$ then

$F_i(c,E) = F_i(c,E') \forall i\varepsilon M \setminus K$

$= F_i(c_K - x_K, E - E') + x_i \forall i\varepsilon K.$ where

$x_k = (x_i)_{i\varepsilon k} \quad and \quad c_k = (c_i)_{i\varepsilon k}$

Lemma 2:- CEA satisfies Comp.

Proof:- Let $(c,E')\varepsilon B^M, M\varepsilon P \ and \ x = CEA(c,E').$

Let $K = \{i\varepsilon M / x_i < c_i\} \ and \ E > E'$

Let $y = F(c_k - x_k, E - E').$

$\therefore \sum_{i\varepsilon M \setminus K} x_i + \sum_{i\varepsilon K} (x_i + y_i) = E.$

Suppose $x_i = \min\{\lambda, c_i\} \ \forall i\varepsilon M$

and $y_i = \min\{\lambda', c_i - x_i\} \ \forall i\varepsilon K.$

Note:- $\lambda \geq c_i \forall i\varepsilon M \setminus K.$

Case 1: $x_i + y_i < c_i$ for some $i\varepsilon K.$

$\therefore x_i < c_i \rightarrow x_i = \lambda < c_i.$

$y_i < c_i - x_i \rightarrow y_i = \lambda' < c_i - x_i.$

$\therefore x_i + y_i = \lambda + \lambda'$

i.e. $x_i + y_i < c_i \rightarrow x_i + y_i = \lambda + \lambda'$

Case2 : $x_i + y_i > c_i$ for some $i\varepsilon K.$

This case is not possible since $y_i \leq c_i - x_i \ \forall i\varepsilon K.$

Case 3:- $x_i + y_i = c_i \ for \ some \ i\varepsilon K.$

$y_i = c_i - x_i \rightarrow \lambda' \geq c_i - x_i$

Now $x_i \leq \lambda \rightarrow \lambda' \geq c_i - \lambda.$

or $\lambda' + \lambda \geq c_i.$

$\therefore x_i + y_i = \min\{\lambda + \lambda', c_i\} \forall i\varepsilon K$

Since $x_i = \min\{\lambda + \lambda', c_i\} = c_i \forall i\varepsilon M \setminus K,$

we get that CEA satisfies comp.

Q. E. D.

Our composition property differs from that of Dagan (1996), since our's is embedded in a variable agent framework. This is necessary, for if in the first round of fair division some agent gets what he claims, then by our definition of a claims problem his claims cannot be entertained for the second round of fair division, since his net claim in the second round is zero. Note, that we require all claims to be strictly positive.

Equal Treatment (ET):

$$\forall M \varepsilon P, \forall (c,E) \varepsilon B^M, c_i = c_j \rightarrow F_i(c,E) = F_j(c,E), \textit{ whenever } i,j \varepsilon M.$$

It is easy to check from the definition of the constrained equal awards solution that it satisfies (ET).

Theorem 1:- The only solution on B to satisfy IIC, Comp. and ET is CEA.

Proof: We have already established that CEA satisfies the above properties. Hence let us assume that F is a solution on B which satisfies the above properties, and let $(c,E) \varepsilon B^M, M \varepsilon P$. Without loss of generality assume $M = \{1,2,\ldots,m\}$ i.e. the first m natural numbers for some $m \varepsilon \mathbf{N}$, and also that $0 < c_1 \leq c_2 \leq \ldots \leq c_m$. This will only facilitate the proof.

Step 1: Suppose $E \leq mc_1$.

Let $E_1 = c_1$

By IIC and ET,

$$F(c,E) = \left(\frac{E}{m},\ldots,\frac{E}{m}\right) \quad \text{if} \quad 0 \leq E \leq E_1$$

$$= CEA(c,E) \textit{ if } 0 \leq E \leq E_1$$

Now let $E_2 = 2c_1 - \frac{c_1}{m}$

If $E_1 \leq E \leq E_2$, *then* $E - E_1 \leq c_1 - \frac{c_1}{m} \leq c_i - F_i(c,E_1) \forall i \varepsilon M$

Let $x^1 = F(c,E_1) = CEA(c,E_1)$

By IIC and ET,

$$F(c - x^1, E - E_1) = \left(\frac{E-E_1}{m},\ldots,\frac{E-E_1}{m}\right) \textit{ if } E_1 \leq E \leq E_2$$

$$= CEA(c - x^1, E - E_1)$$

By Comp.

$$F(c,E) = F(c,E_1) + F(c - x^1, E - E_1)$$

$$= CEA(c,E_1) + CEA(c - x^1, E - E_1)$$

$$= CEA(c,E) \quad \textit{if } E_1 \leq E \leq E_2$$

Let $x^2=F(c,E_2)=CEA(c,E_2)=\left(\frac{E_2}{m},\ldots,\frac{E_2}{m}\right)$

Let $E_3=3c_1-\frac{3c_1}{m}+\frac{c_1}{m^2}$

For $E_2\leq E\leq E_3, E-E_2\leq E_3-E_2=c_1-\frac{2c_1}{m}+\frac{c_1}{m^2}$

$$=c_1-\frac{c_1}{m}\left(2-\frac{1}{m}\right)$$

$$=c_1-\frac{E_2}{m}$$

By repeating the argument in the previous step, we get

$CEA(c,E)=F(c,E)$ *if* $0\leq E\leq E_3$

In particular $CEA(c,E)=\left(\frac{E}{m},\ldots,\frac{E}{m}\right)$

We carry on thus at each stage, defining E_k *from* E_{k-1} *as follows*:

$$E_k=E_{k-1}-\frac{E_{k-1}}{m}+c_1=E_{k-1}\left(\frac{m-1}{m}\right)+c_1$$

Suppose $E_{k-1}\leq mc_1$. Then $E_k\leq mc_1$.

Since $E_1=c_1\leq mc_1$ we get $E_k\leq mc_1 \forall k\in\mathbf{N}$.

Further $E_{k-1}>E_{k-2}$ *implies*

$$E_k-E_{k-1}=\frac{m-1}{m}(E_{k-1}-E_{k-2})>0$$

Since $E_2>E_1$, *we get* $E_k>E_{k-1} \forall k\in\mathbf{N}$.

Thus $\underset{k\to\infty}{lt}\ E_k$ exists and it is easy to see from the difference equation defining E_k that

$$\underset{k\to\infty}{lt}\ E_k=mc_1.$$

Thus $\forall 0\leq E<mc_1$,

$$F(c,E)=CEA(c,E)$$

and this result is obtained by replicating the argument that we made for E_1, E_2 *and* E_3. Since Comp. in particular implies that $F(c,E)\geq F(c,E')$ whenever $E\geq E'$ *and* $(c,E),(c,E')\in B$, we get $F(c,E)=CEA(c,E)\forall 0\leq E\leq mc_1$: in fact

$$F(c,mc_1)\geq CEA(c,mc_1);\ since\ mc_1=\sum_{i\in M}F_i(c,mc_1)$$

$= \sum_{i \in M} CEA_i(c, mc_1), F(c, mc_1) = CEA(c, mc_1).$

Step 2:- $mc_1 \leq E \leq c_1 + (m-1)c_2$

By Comp. $F_1(c,E) = c_1 = CEA_1(c,E)$ $\forall$ $mc_1 \leq E$.

Without loss of generality suppose $c_2 > c_1$.

Then we repeat the arguments in step 2 for $E - mc_1$ instead of $E, M \setminus \{1\}$ instead of M and $c_2 - c_1, \ldots, c_m - c_1$ instead of $c_1, c_2, \ldots, c_m$ to obtain $F(c,E) = CEA(c,E)$ for $0 \leq E \leq c_1 + (m-1)c_2$. (In the final step, we appeal to Comp. once again with mc_1 being allotted in the first round of fair division and $E - mc_1$ being allotted in the second round).

In general if $\{i \in M / c_i = c_1\} = K$ *and* $|K|$ is the cardinality of K, we first show that $F(c_{M \setminus K} - c_i e_{M \setminus K}, E - mc_1) = CEA(c_{M \setminus K} - c_1 e_{M \setminus K}, E - mc_1)$ $\forall mc_1 \leq E \leq c_1 \; |K| + c_2(m - |K|)$ and then appeal to Comp. to conclude, $F(c,E) = CEA$ $\forall$ $0 \leq E \leq c_1 |K| + c_2(m - |K|)$.

In the above $e_{M \setminus K}$ is the vector in $\mathbb{R}_+^{M \setminus K}$ with all coordinates equal to 1 and $C_{M \setminus K} = (c_i)_{i \in M \setminus K}$

We carry on with the above line of argument till we find an integer $l \in \mathbf{N}$ such that

$c_1 + c_2 + \ldots + (m - l + 2)c_{l-1} \leq E < c_1 + c_2 + \ldots + c_{l-1} + (m - l + 1)c_l$

Such an $l \leq m$ surely exists since $E < \sum_{i \in M} c_i$

Thus we obtain $F(c,E) = CEA(c,E), \forall$ $(c,E) \in B$.

<u>Q.E.D.</u>