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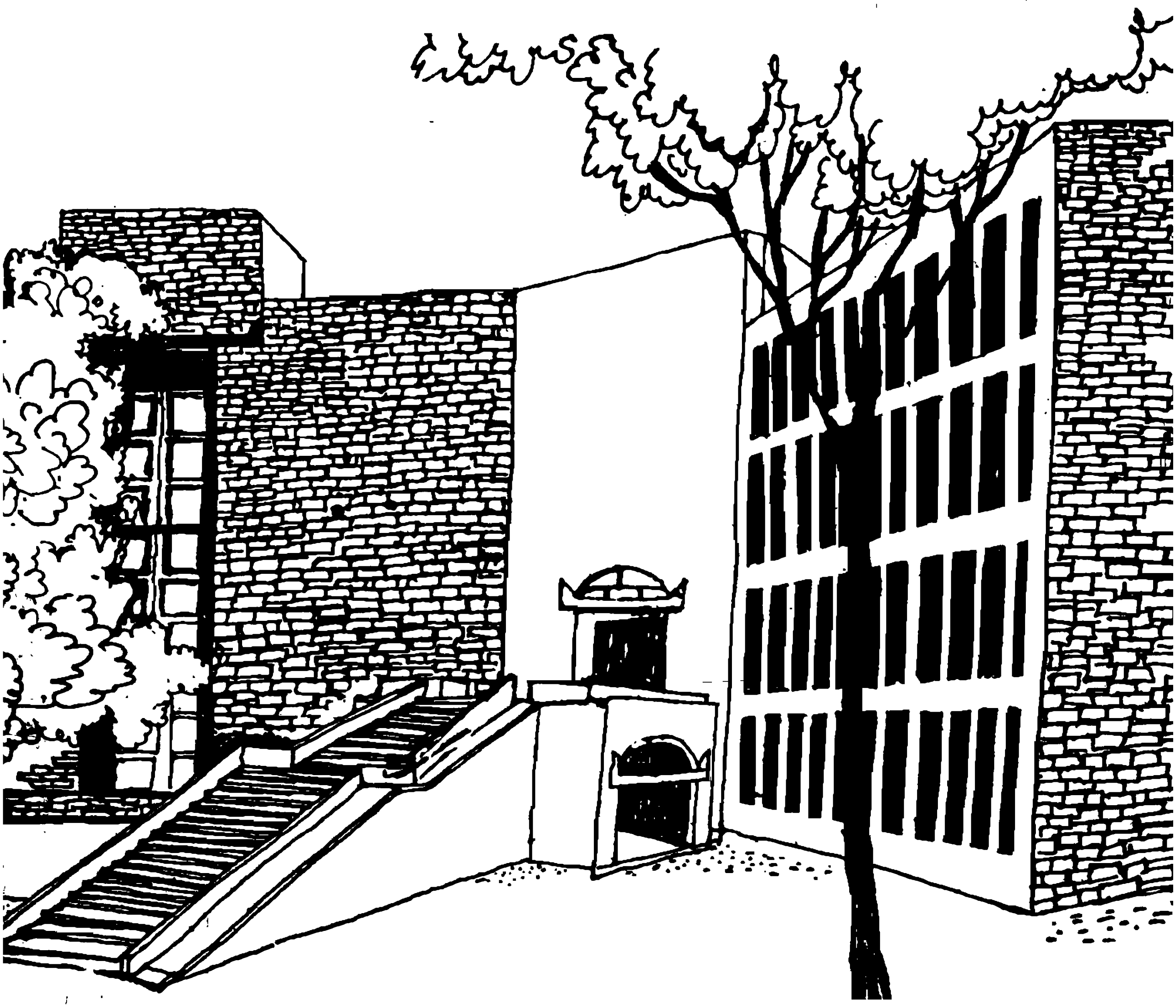

AXIOMATIC CHARACTERIZATIONS OF SOLUTIONS FOR RATIONING PROBELMS

By

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Abstract

Situations abound in the real world, where aggregate demand for a commodity exceeds aggregate supply. When such situations of excess demand occur, what is required is some kind of rationing. The literature on rationing problems has an interesting origin in the Babylonian Talmud.

The purpose of this paper is to characterize axiomatically and analyze some Talmudic solutions for rationing problems.

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1. Introduction:- Situations abound in the real world, where aggregate demand for a commodity exceeds total supply. In economics, the most common way in which such situations are seen to occur is when government intervenes by pegging the price of a commodity at a level below the market equilibrium price (i.e. the price at which quantity demanded is equal to quantity supplied). In management, the usual way in which such anomalies occur is in the context of supply chain management: there is a distributor of a commodity who is made available the total supplies by a producer; the distributor supplies the commodity to a finite number of retailers; if the orders placed by the retailers add up to a quantity greater than the supply available with the distributor, we are essentially facing a situation of excess demand once again. The excess demand problem in economics has been highlighted and surveyed lucidly, by Silvestre (1986). The excess demand problem in management is a part of a well established lore on frequent stock outs arising in distribution networks. In fact, the problem has such urgency, that computer games have been devised to highlight the merits of the problem. In the context of fish production in India for instance, Datta, Sinha and De [1996] forecast an excess demand of 79.39 tonnes with a 2% annual shift in demand and concommitant supply adjustment and an excess demand of 643.66 tonnes with a 5% annual shift in demand and supply being suitably adjusted. Both figures are for the year 2001.

When such situation of excess demand occur, what is required is some kind of rationing. The literature on rationing problems has an interesting origin in the Babylonian Talmud (: 2000 year old document, which forms the basis of Jewish civil, criminal and religious laws). There, considerable attention has been devoted to the study of a bankruptcy problem: a man dies leaving behind an estate, which is insufficient to meet all his debts. How should the estate be divided among the claimants? The obvious requirement is that the method of division be perceived as being fair.

Recent attempts at giving solutions to this old bankruptcy problem a game theoretic interpretation, can be traced to the paper by O'Neill [1982]. The study of a particular solution known as the contested garment solution received fresh analytical impetus in the work of Aumann and Maschler [1986].

In a paper by Curiel, Maschler and Tijs [1985] a solution, known as the adjusted proportional solution is proposed, as a method of allocation for rationing problems. The adjusted proportional solution, allows a simple modification, which we call the modified adjusted proportional solution. We show that the modified adjusted proportional solution agrees with the adjusted proportional solution for all two-agent rationing problems. Indeed, for all two-agent rationing problems, our solution is shown to coincide with the contested garment solution of Aumann and Maschler (1986).

In Moulin [1985, 1988] and Young [1987a, 1987b, 1988, 1993], the mathematical framework of bankruptcy or rationing problems is given the opposite interpretation of cost-sharing or taxation problems. Whereas in rationing problems we are interested in some measure of individual loss i.e. unsatisfied demands, in cost sharing the relevant index is net income that remains after taxation. Both these variables have identical mathematical form. However, in cost sharing if we are interested in maximizing the minimum net income, in rationing we would be interested in minimizing the maximum loss. We obtain a simple algorithm in this paper, which gives an explicit solution for the relevant min-max problem.

One of the most popular methods of allocating resources under rationing is the constrained equal awards method, also called the uniform rule by Benassy [1982]. This rule, gives each low demander what he/she demands; all high demanders are given an equal amount, which nevertheless exceeds what any low demander gets. Dagan [1996a] has a useful analysis of this rule. We provide an axiomatic characterization of the constrained equal awards solution using a kind of strategy proofness assumption and show that this rule is the only one to satisfy the desired axiom (along with another mild property). Results along similar lines for this and other solutions can be found in Dagan and Volij [1993].

The above mentioned analysis takes place in a fixed population framework i.e. the agent set or the set of demanders is considered fixed. We subsequently move over to a variable population framework and invoke properties like population monotonicity and Consistency Population monotonicity says that with the arrival of a new agent, no existing agent can get more. Consistency says that if some agents leave with their share of the allocation, then the rule should give the earlier shares to the remaining agents, when what has to be allocated now is what remains after the departing agents have been given their shares. Our results are adaptations of results in Dagan [1996a] and Thomson [1995]. Their results were obtained for games of fair division with single peaked preferences. The basic difference between our framework and the literature on fair division with single peaked preferences are that our preferences have the diagrammatic representation of an isosceles triangle above the horizontal axis. Further, we restrict ourselves to only excess demand situations. With these restrictions, the proofs used by Dagan and Thomson fail to work, since they avail of the larger domain on which their solutions are defined.

Finally, we take up the case of the proportional solution and provide an axiomatic characterization of the same using a reduced game property and a property called restricted scale invariance for two agents. In the bargaining games context, reduced games properties have been discussed in Peters, Tijs and Zarzuelo [1994] and Lahiri [1996].

It may be of interest to note yet another sector of the Indian economy where excess demand leading to rationing of resources is a very common phenomena: the capital market. It has generally been observed, that initial public offerings of equity by firms are characterized by significant underpricing (:see Majmudar (1996) for a useful data base on the topic). This naturally leads to excess demand and a common method applied by firms in allocating shares is the proportional rule. However, as emphasized in the paper, the proportional rule is grossly manipulable. Individuals, have a tendency to overstate their true demands for the shares. A consequence of the analysis in this paper is the suggestion that there are other robust rationing rules, which can be applied for the same purpose.

2. <u>The Fixed Population Model</u>:- Consider a set of agents indexed by i=1, 2,...,n where n is a natural number greater than or equal to two. Let N = {1, 2, ..., n} denote the set of agents. A <u>rationing</u> (bankruptcy) <u>problem</u> is an ordered pair

$(d, S) \in \mathbb{R}^n_{++} \times \mathbb{R}_{+}$ such that $S < \sum_{i=1}^{n} d_i$.

Let B^N denote the set of all rationing problems(for N).

An <u>allocation</u> for $(d, S) \in B^N$ is a vector $x \in \mathbb{R}^n_+$ such

$x_i \leq d_i \, \forall i \in N$ and $\sum_{i \in N} x_i = S$.

A <u>solution</u> is a function $F: B^N \rightarrow \mathbb{R}^n_+$ such that F(d,S) in an allocation for (d,S) whenever $(d, S) \in B^N$.

Given $(d, S) \in B^N$, the effective demand vector (for (d,S)) denoted d^S is the vector whose i^{th} component

$d_i^s = \min\{ d_i , S \}$

Obviously, since S is what all there is for distribution any claim greater that S is as good as demanding the entire supply. Hence our definition of effective demand.

Given $(d, S) \in B^N$, the point of minimal expectation

$v^{(d,S)}$ (denoted merely by v whenever there is no scope for confusion) is the vector whose i^{th} coordinate v_i is equal to $\max\left\{0, S - \sum_{j\neq i} d_j\right\}$ i.e. what every one else willingly concedes to i.

<u>Observation 1</u>: $v_i \leq d_i \; \forall \; i \in N$

<u>Proof of observation</u>: Suppose $v_i > d_i$ for some $i \in N$

Clearly $d_i > 0 \rightarrow v_i = S - \sum_{j\neq i} d_j$

$\therefore S - \sum_{j\neq i} d_j > d_i$

$\rightarrow S > \sum_{j=1}^{n} d_j$ which is a contradiction. Hence the observation.

<u>Q.E.D.</u>

<u>Observation 2</u>:- Given $(d, S) \in B^N$ if x is any allocation for (d,S), then $x_i \geq v_i \; \forall \; i \in N$.

<u>Proof of observation</u>: Suppose $0 \leq x_i < v_i$ for some $i \in N$.

Then clearly $v_i = S - \sum_{j\neq i} d_j$.

$\therefore x_i < S - \sum_{j\neq i} d_j$.

$\therefore x_i + \sum_{j \neq i} d_j < S$

But $x_j \leq d_j \; \forall j \in N$

$\therefore S = \sum_j x_j \leq x_i + \sum_{j \neq i} d_j < S$ which is a contradiction.

This proves the observation.

Q.E.D.

Observations 3:

For all $(d, S) \in B^*$, $\forall i \in N$

$$v_i = \max\left\{0, S - \sum_{j \neq i} d_j^s\right\}.$$

Proof:- Let $i \in N$, $k \neq i$, $k \in N$.

If $d_k > S$ then $S - \sum_{j \neq i} d_j < S - d_k < 0$.

$\therefore v_i = 0$.

Since $d_k^* = S$ and $S - \sum_{j \neq i} d_j^* < S - d_k^* = S - S = 0$, $\max\left\{0, S - \sum_{j \neq i} d_j^*\right\} = 0$.

$$\therefore v_i = \max\left\{0, S - \sum_{j \neq i} d_j^*\right\}.$$

On the other hand if $d_k \leq S \; \forall k \in N$, $k \neq i$, then

$d_k^* = d_k \; \forall k \in N$, $k \neq i$, so that

$$\sum_{j \neq i} d_j^S = \sum_{j \neq i} d_j$$

This proves the observation in either case.

Q.E.D.

Observation 4:- Given $(d, S) \in B^N$, $\sum_{i \in N} v_i \leq S$

Proof of observation:- Let $x \in \mathbb{R}_+^n$ with $x_i = \frac{d_i}{\sum_{j \in N} d_j} S$.

It is easy to check that x is an allocation for (d,S). Thus the set of allocations for (d,S) is nonempty. Since $v_i \leq x_i \; \forall \, i \in N$ by observation 3, we have, $\sum_{i \in N} v_i \leq S$.

Q.E.D.

We now define the adjusted proportional solution

$AP : B^N \rightarrow \mathbb{R}_+^n$: Given $(d, S) \in B^N$, Let

$$d_i^* = \min \left\{ d_i - v_i, \, S - \sum_{j \in N} v_j \right\}$$

Then denoting $AP(d, S) = \bar{x}$, we get, $\bar{x}_i = v_i + \frac{d_i^*}{\sum_{j \in N} d_j^*} \left(S - \sum_{j \in N} v_j \right)$, $i \in N$.

In this paper we are concerned with a modified (version of the) adjusted proportional solution,

$MAP : B^{N} \to \mathbb{R}^{N}_{+}$, defined thus. Let $MAP(d, S) = \tilde{x}$. Then,

$$\tilde{x}_i = v_i + \frac{d_i^* - v_i}{\sum_{j \in N} (d_j^* - v_j)} \left(S - \sum_{j \in N} v_j \right) \forall\, i \in N.$$

This is precisely the solution that we discussed in the introduction. Unlike the adjusted proportional solution, the modified adjusted proportional solution satisfies Independence of Irrelevant Claims, a property which says that

$F(d, S) = F(d^*, S) \;\forall\, (d, S) \in B^{N}$. with $S > 0$.

3. The Two Agent Situation:- We are particularly interested (in this paper) in the modified adjusted proportional solution for two agent problems i.e. for the case n=2. Without loss of generality, and for greater ease of exposition, let us assume $d_1 \le d_2$ *whenever* $(d, S) \in B^{\{1,2\}}$ What does the adjusted proportional solution look like in this situation?

Theorem 1:- For n=2, MAP = AP. Hence AP satisfies Independence of Irrelevant Claims (since MAP does so always).

Proof:- $v_i = \max\{0,\ S - d_j^S\},\ j \ne i$

$$= S - d_j^S$$

$$d_i^* = \min\{d_i - v_i,\ S - v_1 - v_2\}$$

$$= \min\{d_j - v_j,\ S - S + d_j^S - v_j\} = d_j^S - v_j.$$

Thus MAP (d,S) = AP (d,S).

Q.E.D.

We shall now explore the relation between v and d^S whenever $(d,S) \in B^{\{1,2\}}$. We have, $v_i = \max\{0, S - d_j^S\}$, $j \neq i$.

$$= S - d_j^S.$$

Further, $0 \le v_i \le d_i$.

Thus the vector where i gets v_i and j gets d_j^S is an allocation for (d,S). Given our earlier result that $v_i \le x_i$ whenever x is an allocation for (d,S), we get now that $v_i = \min\{x_i \,/\, x \text{ is an allocation for } (d,S)\}$ Further,

$d_i^S = \max\{x_i \,/\, x \text{ is an allocation for } (d,S)\}$. Thus d^S is the north - eastern extremity and v is the southwestern extremity of the rectangle, whose diagonal (which separates d^S and v) is precisely the set of all allocations for (d,S).

[Insert Figure-1 here].

Clearly, MAP (d,S) is the mid-point of the set of all allocations for (d,S). The two extreme points of the set of all allocations for (d,S) are (v_1, d_2^S) *and* (d_1^S, v_2). Thus we have the following theorem:

Theorem 2:- For n =2,

$$AP(d,S) = MAP(d,S) = \left(\frac{v_1 + d_1^*}{2}, \frac{v_2 + d_2^*}{2} \right)$$

The proof of this theorem follows essentially from Theorem 1 and the observation immediately before Theorem 2.

4. The Simple Geometry of The Modified Adjusted Proportional Solution:-

Let us revert to Figure 1 and show among other things that the rectangle B G C E must indeed be a square. This will lead to several equivalences.

First note that both angles ADO and angles DAO must be 45 degrees. Thus angles EBC and ECB are 45 degrees. Hence triangle EBC is an isosceles triangle. Thus the length of the side BE is equal to the length of the side EC. Thus the rectangle BGCE is indeed a square.

Hence GF must be perpendicular to the line AD. Hence F must be the point of least distance from G.

Further, it is not difficult to see that the angles FGC and BGF are both 45 degrees. Hence F is also the point of equal loss from G. Noting that F stands for MAP(d,S), we have the following theorem:

Theorem A: For n=2, given $(d, S) \in B^{(1,2)}$, $MAP(d, S)$ is the unique allocation which is at the point of least distance from d^s. Further, MAP (d,S) is also the unique allocation which equates losses among the two agents.

Given the above discussion it is easy to see that the co-ordinates of B are given by $(S - d_2^s, d_2^s)$ and the co-ordinates of C are given by $(d_1^s, S - d_1^s)$. Hence (and also from earlier discussions), $v = (S - d_2^s, S - d_1^s)$. By applying Theorem 2 in the paper we have the following result:

Theorem B:- For n = 2,

$$AP(d, S) = MAP(d, S) = \left(\frac{S + d_1^s - d_2^s}{2}, \frac{S + d_2^s - d_1^s}{2} \right).$$

This is precisely the Constested Garment Solution for the two agent case, discussed in Aumann and Maschler (1985).

5. The Quasi-Equal Loss Solution:-

Without loss of generality, we will assume that whenever we are given a rationing problem

$(d,S)\in\mathbb{R}^n_{++}x\mathbb{R}_+$ (*with* $\sum_{i=1}^{n} d_i > S$), *we have* $d_1 \leq d_2 \leq \ldots\ldots \leq d_n$. This does

not affect the ensuing analysis in any way; on the contrary it simplifies matters to a great extent.

Given (d,S) ϵ B^N let k(d,S) ϵ $\{1,\ldots,n\}$ be defined as follows:

$$k\ (d,S) = \min\ \{k/ \frac{1}{n-k+1}\left(\sum_{i=k}^{n} d_i - S\right) \leq d_k\}$$

Such a k(d,S) always exists.

We define the <u>quasi-equal loss solution</u> $Q : B^N \rightarrow \mathbb{R}^n_+$ as follows:

$Q_i\,(d,S) = 0\ \textit{if}\ i < k\ (d,S)$

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$$= d_i - \frac{1}{n-k(d,S)+1}\left(\sum_{j=k(d,S)}^{n} d_j - S\right)\forall i \geq k(d,S)$$

Basically, the quasi-equal loss solution operates by allocating nothing to those whose demands are very small and then allocating the total amount among the rest in such a way that the loss experienced by each agent in the latter group is equal. Indeed, individual loss in the latter group is

$\frac{1}{n-k(d,S)+1}\left(\sum_{j=k(d,S)}^{n} d_j - S\right)$. A point to be noted is that if

k(d,S) = 1 (i.e. the set of agents whose index comes before k(d,S) is empty) then we have the equal loss rule.

The above rule is an algorithm and as we shall see shortly, this algorithm is the unique solution of a well defined programming problem.

Theorem 3: Given $(d,S)\in B^N$, the unique solution to the programming problem

$$\min_{x_1,\dots,x_n}\left\{\max_{i=1,\dots,n}(d_i - x_i)\right\}\dots\dots\dots\dots\dots(1)$$

$$\text{s.t. } 0 \le x_i \le d_i \ \forall i\dots\dots\dots\dots\dots(2)$$

$$\sum_{i=1}^{n} x_i = S\dots\dots\dots\dots\dots(3)$$

is Q (d,S)

The proof proceeds by a sequence of lemmas:

Lemma 1: If $d_1 \ge \frac{1}{n}\left(\sum_{i=1}^{n} d_i - S\right)$, then the unique solution to (1) subject to (2) and (3) is given by Q(d,S).

Proof: Denote Q(d,S) by $\bar{X}$. Clearly $\bar{X}$ satisfies (2) and (3).

Now $\max_{i=1,\dots,n} \{d_i-\bar{x}_i\}=\frac{1}{n}(\sum_{i=1}^{n} d_i-S)$.

Towards a contradiction assume that there exists $x \in \mathbf{R}^n$ satisfying (2) and (3) such that $\max_{i=1,\dots,n} \{d_i-x_i\}<\frac{1}{n}(\sum_{i=1}^{n} d_i-S)$.

But then

$$\sum_{i=1}^{n} d_i-S=\sum_{i=1}^{n} d_i-\sum_{i=1} x_i<\sum_{i=1}^{n} d_i-S$$

which is a contradiction.

Thus suppose that there exists $x \in \mathbf{R}^n, x \neq \bar{x}$, with x satisfying (2) and (3) and $\max_{i=1,\dots,n} \{d_i-x_i\}=\frac{1}{n}(\sum_{i=1}^{n} d_i-S)$.

Since $x \neq \bar{x}$ there exists j such that $d_j-x_j>\frac{1}{n}\left(\sum_{i=1}^{n} d_i-S\right)$

contradicting $\max_{i=1,\dots,n} \{d_i-x_i\}=\frac{1}{n}(\sum_{i=1}^{n} d_i-S)$..

Hence the lemma.

Q.E.D.

Note the role played by $d_1 \geq \frac{1}{n}(\sum_{i=1}^{n} d_i - S)$, in the above is to ensure, $\bar{x}_i = d_i - \frac{1}{n}(\sum_{i=1}^{n} d_i - S) \geq 0 \forall i$

<u>Lemma 2</u>:- Suppose $d_1 < \frac{1}{n}(\sum_{i=1}^{n} d_i - S)$ and let x^* be a solution to (1) subject to (2) and (3). Then (a) $x_1^* = 0$ (b) $d_1 - x_1^* < \max_{i=1,\ldots,n} \{d_i - x_i^*\}$, (c) $\sum_{i=2}^{n} d_i > S$.

<u>Proof</u> : We prove (b) first

Suppose $d_1 - x_1^* = \max_{i=1,\ldots,n} \{d_i - x_i^*\}$

Then $d_1 \geq d_1 - x_1^* \geq d_i - x_i^* \ \forall i$ implies $d_1 \geq \frac{1}{n}\left\{\sum_{i=1}^{n} d_i - S\right\}$ which is a contradiction.

This proves (b).

Given (b) we now prove (a).

Suppose $x_1^* > 0$. Clearly $x_1^* \leq d_1$. By (a) if $d_j - x_j^* = \max_{i=1,\dots,n} \{d_i - x_i^*\}$, then $j \neq 1$. Let $K = \left\{ j / d_j - x_j^* = \max_{i=1,\dots,n} \{d_i - x_i^*\} \right\}$

Clearly $K \neq \emptyset$ and $1 \notin K$. Let $\epsilon_1 > 0$ be such that

$d_j - x_j^* - \epsilon_1 / |K| > d_i - x_i^* \quad \forall \ j \in K, i \notin K.$

and $x_j^* + \epsilon_1 / |K| < d_j \quad \forall \ j \in K.$ Such an ϵ_1 clearly exists.

Let $\epsilon_2 > 0$ be such that $x_1^* - \epsilon_2 > 0$

and $d_1 - x_1^* + \epsilon_2 < d_j - x_j^* - \frac{\epsilon_1}{|K|} \quad \forall \ j \in K$

Let $\epsilon = \min \{\epsilon_1, \epsilon_2\} > 0$.

Define $x \in \mathbb{R}_+^n$ as follows :

$x_1 = x_1^* - \epsilon \qquad x_i = x_i^* \quad \forall i \notin K \cup \{1\}$

$x_i = x_i^* + \frac{\epsilon}{|K|} \quad \forall i \in K$

Clearly $\max_{i=1,\dots,n} \{d_i - x_i\} < \max_{i=1,\dots,n} \{d_i - x_i^*\}$ Contradicting

x^* solves (1) subject to (2) and (3).

Thus $x_1^* = 0.$

Finally we prove (c):

$$d_1 < \frac{1}{n}\left(\sum_{i=1}^{n} d_i - S\right)$$

$$\rightarrow n d_1 < \sum_{i=1}^{n} d_i - S$$

$$\rightarrow (n-1)\, d_1 < \sum_{i=2}^{n} d_i - S$$

Since k (d, S) > 1 and $d_1 > 0$, we get

$$\sum_{i=2}^{n} d_i - S > 0.$$

In the above lemma we made use of the fact that

$\max_{i=1,\dots,n} \{d_i - x_i\} > 0 \quad \forall x \in \mathbf{R}^*$ satisfying (2) and (3), in order to select an e_1. This is true; for if $\max_{i=1,\dots,n} \{d_i - x_i\} \leq 0,$ then

$\sum_{i=1}^{n} d_i - S \leq o$ which contradicts that

$$d_1 < \frac{1}{n}\left(\sum_{i=1}^{n} d_i - S\right).$$

Now we proceed to prove the main theorem.

Proof of theorem 3: If $d_1 \geq \frac{1}{n}\left(\sum_{i=1}^{n} d_i - S\right)$ then by Lemma 1 we get Q(d,S) is the unique solution to the programming problem (1) subject to (2) and (3).

If $d_1 < \frac{1}{n}\left(\sum_{i=1}^{n} d_i - S\right)$, then by lemma 2, if x^* is the solution to (1) subject to (2) and (3), then $x_1^* = 0$ *and* $(x_2^*, \ldots, x_n^*)$ *solves*

$$\min_{(x_2,\ldots,x_n)} \left\{ \max_{i=2,\ldots,n} \{d_i - x_i\} \right\} \quad s.t. \quad 0 \leq x_i \leq d_i,\ 1=2,\ldots,n$$

$$\sum_{i=2}^{n} x_i = S$$

We are now back to an (n-1) dimensional problem for which we either apply lemma 1 or lemma 2. Proceeding thus we get that Q (d,S) is the unique solution to the programming problem (1) subject to (2) and (3).

Q.E.D.

6. <u>The Constrained Equal Awards Solution</u>:-

The Constrained Equal Awards solution $CEA : B^N \rightarrow \mathbb{R}^n_+$

is defined as follows: $CEA(d,S)=x$ where $x_i=\min\{\lambda, d_i\}, i\in N$

and $\sum_{i=1}^{n} x_i = S$.

It is well known that for each $(d,S)\in B^N$, a unique $\lambda \geq 0$ exists which defines CEA(d,S).

We now state two properties which the constrained equal award solution satisfies.

<u>Equal Treatment (ET)</u>:- Given $(d,S)\in B^N, d_i=d_j \rightarrow F_j(d,S)=F_i(d,S)$.

Equal Treatment is standard and simple. It says, if two people make the same demands then they get identical awards. As a postulate of impartiality, nothing could be more meaningful.

Independence of Irrelevant Inflations (I^3):- Given $(d,S),(d',S)\in B^N$ if $d_i=d'_i \forall i\neq k, d_k\leq d'_k$ and $F_k(d,S)\leq d_k$ then $F_k(d,S) = F_k(d',S)$

Insensitivity to Irrelevant Inflations is a veiled strategy proofness type of condition which says that unilateral upward deviations do not affect the outcome, of the deviating agent provided one's demand is not met originally. It is not as mild a property as equal treatment; yet it provides the required force to characterize the CEA solution. It should be noted, that the solution for a deviating individual is insensitive to inflation of demand by the individual, if the award for the individual was originally less than what was originally demanded. This is the gist of the I^3 property. (I^3) along with (ET) does not appear to characterize the CEA solution uniquely. If we strengthen (ET) slightly to a Weak Monotonicity (WM) property, then (I^3) along with (WM) uniquely characterizes the CEA solution.

Weak Monotonicity (WM):- Given $(d,S)\in B$ if $d_i\leq d_j$ then $F_i(d,S)\leq F_j(d,S)$.

This property says that higher demanders do not get lesser amounts. It is easy to see that Weak Monotonicity implies Equal Treatment, though not conversely.

Theorem 4:- The only solution to satisfy WM and I^3 is CEA.

Proof:- It is easy to see that CEA satisfies these two properties. Hence suppose F is a solution which satisfies these two properties and towards a contradiction assume $F \neq CEA$. Thus there exists $(d,S) \in B^N$ such that $F(d,S) \neq CEA(d,S)$ Without loss of generality and in order to facilitate the proof assume $d_k \leq d_{k+1}$ $\forall k=1,\ldots,n-1$. Clearly there exists $i,j \in N, i<j$ such that $F_i(d,S)<d_i, F_j(d,S) \leq d_j$ and $F_i(d,S) \neq F_j(d,S)$. By $WM, F_i(d,S)<F_j(d,S)$. By WM once again we may assume, j = n and $i=\min\{k/F_k(d,S)<d_k\}$ By $WM, F_i(d,S)<F_n(d,S)$.

Define $d' \in \mathbb{R}^n_+$ as follows:

$$d'_k = d_k \ \forall k \neq i$$

$$d'_i = d_n$$

By $I^3, F_i(d',S)=F_i(d,S)$

By ET (which is implied by WM), $F_n(d',S)=F_i(d',S)$.

Thus $F_n(d',S)=F_i(d,S)<F_n(d,S)$.

Clearly there exists k such that i<k<n and $F_k(d',S)>F_k(d,S)$.

But k>i implies by $WM, F_k(d,S) \geq F_i(d,S) = F_n(d',S)$.

Thus $F_k(d',S) > F_n(d',S)$ which contradicts WM since k<n.

Q.E.D.

However for n = 2, (I^3) and (ET) uniquely characterizes the constrained equal award solution, as the following (which is a strengthening of the previous theorem) reveals.

Theorem 5: For n = 2, the only solution to satisfy (I^3) and ET is CEA.

Proof:- Suppose towards a contradiction, that there exists a rationing problem (d_1, d_2; S) and a solution F satisfying (I^3) and (ET) such that $F(d_1,d_2;S) \neq CEA(d_1,d_2;S)$. Let $(x_1,x_2) = F(d_1,d_2;S)$. Thus $x_1 \neq x_2$. There are two possible cases:

Case 1:- $x_1 < d_1$

Case 2:- $x_2 < x_1 = d_1$

Case 1:- If $x_1 < d_1$ where we have assumed without loss of generality $d_1 \leq d_2$, then by ET, we must have $d_1 < d_2$. Let $d_1' = d_2$.

By (I^3), $F_1(d_1, d_2; S) = x_1$.

By ET, $F_2(d_1, d_2; S) = x_1$.

$\therefore 2x_1 = S = x_1 + x_2$, *contradicting* $x_1 \neq x_2$.

<u>Case 2</u>:- $x_2 < x_1 = d_1$

$\therefore S = x_1 + x_2 < 2d_1$.

By ET, $F(d_1, d_1; S) = (\frac{S}{2}, \frac{S}{2})$

By ET, once again $d_1 < d_2$.

By I^3, $F_2(d_1, d_2; S) = \frac{S}{2}$. Thus $F_1(d_1\ d_2; S) = \frac{S}{2}$. Thus

$x_1 = x_2 = S$ contradicting $x_2 < x_1$.

This proves the theorem.

Q.E.D.

7. <u>The Variable Population Model</u>:-

There is a population of "potential agents", indexed by elements in a set I. Let P denote the set of all non-empty finite subsets of I. Given $M \epsilon P$, let $\mathbb{R}^M_+$ (*respectively* $\mathbb{R}^M_{++}$)

denote the set of all functions from M to $\mathbb{R}_+$ (respectively $\mathbb{R}_{++}$) . Here $\mathbb{R}_+$ is the set of all non-negative real numbers and $\mathbb{R}_{++}=\mathbb{R}_+\setminus\{0\}$.

Given $M\in P$, a rationing problem for M is an ordered pair $(d,S)\in\mathbb{R}^M_{++}\times\mathbb{R}_+$ such that $\sum_{i\in M} d_i>S$. .

Let B^M denote the set of all rationing problems for M and $B=\bigcup_{M\in P} B^M$. Let $X=\bigcup_{M\in P}\mathbb{R}^M_+$.

Given $(d,S)\in B^M, M\in P$, an allocation for (d,S) is a vector $x\in\mathbb{R}^M_+$ such that $\sum_{i\in M} x_i=S$ and $x_i\le d_i \forall i\in M$.

A <u>solution</u> is a function $F:B\to X$ such that F(d,S) is an allocation for (d,S) whenever $(d,S)\in B$.

The <u>constrained equal awards solution</u> $CEA:B\to X$ is defined as follows: $\forall(d,S)\in B^M, M\in P, \forall i\in M,\quad CEA_i(d,S)=\min\{\lambda,d_i\}$ with $\lambda\ge 0$ satisfying $\sum_{i\in M}\min\{\lambda,d_i\}=S$.

No-envy, property:- A solution F is said to satisfy the no-envy, property if $\forall M \in P, \forall (d,S) \in B^M \forall i,j \in M, d_i - F_i(d,S) \leq |d_i - F_j(d,S)|$

The no-envy property is quite simple: between any two agents there should not arise a situation where any one's unfulfilled demands exceed the deviation of the other's from the first agent's claim i.e. no one's excess demand should be greater than either the excess supply or excess demand of the other from the one's point of view. If the situation were otherwise, then there would be an agent who would want someone else's allotment, since that would lead to a lower loss for him/her, where loss is measured in terms of deviation from announced demands.

Individual Rationality from equal division:- A solution F is said to satisfy individual rationality from equal division if $\forall M \in P, \forall (d,S) \in B^M \, \forall i \in M, \quad d_i - F_i(d,S) \leq |S/_{|M|} - d_i|$

Once again the meaning is clear: for every agent the excess demand should not exceed his deviation from equal division of resources.

The following theorem is immediate.

Theorem 6: (a) CEA satisfies the no-envy property

(b) CEA satisfies individual rationality from equal division.

Proof: Let $CEA(d,S) = x \in \mathbb{R}^M_+$ for some $M \in P$, $(d,S) \in B^M$.

(a) Suppose towards a contradiction that there exists i, $j \in M$ with $d_i - x_i > |d_j - x_j|$

Clearly $d_i \neq x_i$

$\therefore 0 \leq x_i = \lambda < d_i$

where $\sum_{k \in M} \min\{\lambda, d_k\} = S$.

Since $x_j \neq x_i$, we have $x_j \neq \lambda$.

Thus $x_j = d_j$

$\therefore d_i - \lambda > |d_j - d_j|$ with $d_j < \lambda < d_i$.

$\therefore d_i - \lambda > d_i - d_j$

$\therefore \lambda < d_j$ which is a contradiction.

This proves (a).

(b) Suppose towards a contradiction that there exists $i \in M$ with

$d_i - x_i > |d_i - S/_{|M|}|$.

Thus $x_i = \lambda$ where λ is as in (a) and λ

$\therefore d_i - \lambda > |d_i - S/_{|M|}|$

Case 1: $\lambda < S/_{|M|}$.

$$\therefore S = \sum_{k \in M} x_k = \sum_{\lambda \le d_k} \lambda + \sum_{d_k < \lambda} d_k < |M| \cdot \frac{S}{|M|}$$

which is a contradiction. Thus Case 1 cannot occur and we have

Case 2:- $\lambda \ge \frac{S}{|M|}$

$$\therefore d_i > \lambda \ge \frac{S}{|M|}$$

$\therefore d_i - \lambda \le d_i - \frac{S}{|M|} = | d_i - \frac{S}{|M|} |$ which is again a contradiction.

This proves (b).

Q.E.D.

We now invoke the following property:

Resource Monotonicity:- A solution F is said to satisfy resource monotonicity if $\forall M \in P$,

$(d,S) \in B^M, (d,S') \in B^M, S' \ge S$ *implies* $F(d,S') \ge F(d,S)$.

The meaning of resource monotonicity is simple and needs no further explanation.

8. Axiomatic Characterizations of the CEA Solution In Terms of Consistency:

Claim 1:- Let $(d_i, d_j; S)$ be a two agent rationing problem. Suppose that solution F satisfies either no-envy or individual rationality from equal division. Suppose $d_i \leq d_j$ and

$(x_i, x_j) = F(d_i, d_j; S) \neq CEA(d_i, d_j; S)$. Then

$x_i < d_i, x_i \neq x_j$.

Proof:- Suppose not. Then the only another possibility is

$x_j < x_i < d_i \leq d_j$.

Since $d_j - x_j > d_j - x_i = | d_j - x_i |$, F violates no-envy

(:indeed j envies i).

Since $x_j < \frac{x_i + x_j}{2} < x_i < d_i \leq d_j$,

$d_j - x_j > d_j - \frac{x_i + x_j}{2} = | d_j - \frac{x_i + x_j}{2} |$. Thus F violates individual rationality from equal division.

Q.E.D.

Lemma 3:- If a solution F satisfies no-envy and resource monotonicity, then it coincides with CEA solution for all two agent problems.

Proof:- Towards a contradiction assume that there exists $\{i,j\} \in P$ and $(d_i, d_j, S) \in B^{\{i,j\}}$ such that

$F(d_i, d_j, S) \neq CEA\,(d_i, d_j, S)$ where we have that F satisfies no-envy and resource monotonicity. Without loss of generality assume $d_i \leq d_j$. Then if $(x_i, x_j) = F(d_i, d_j; S)$ we must have $x_i < d_i$, and $x_j \neq x_i$. If $x_j < x_i$ then $|d_j - x_j| > |d_j - x_i|$

Contradicting no-envy. Thus $x_j > x_i$.

If $x_i < x_j < d_i$, then $|d_i - x_i| > |d_i - x_j|$ contradicting no-envy.

Thus $x_i < d_i < x_j$. In fact we must have $x_i < d_i < 2d_i - x_i \leq x_j$ so that no-envy is satisfied. Thus $2d_i \leq x_i + x_j$.

Hence if $S < 2d_i, F(d_i, d_j; S) = CEA(d_i, d_j; S)$. By resource monotonicity, $F(d_i, d_j; S) = CEA(d_i, d_j; S)$ if $S \leq 2d_i$.

Thus for $S = 2d_i, F(d_i, d_j; S) = (d_i, d_i)$.

For $S>2d_i$, by monotonicity, $F_i(d_i,d_j;S)=d_i$. This contradicts $x_i<d_i$.

Hence $F(d_i,d_j;S) = CEA\,(d_i,d_j;S)$.

Q.E.D.

Lemma 4:- If a solution F satisfies individual rationality from equal division and resource monotonicity, then it must coincide with the Constrained Equal Awards Solution for all two agent problems.

Proof:- As in Lemma 1, let us assume that $(d_i, d_j ; S)$ is a claims problem and F satisfies the properties listed in Lemma 2. Suppose $F(d_i,d_j;S) = (x_i,x_j) \neq CEA(d_i,d_j;S)$.

Assuming without loss of generality $d_i \leq d_j$, we must have $x_i<d_i$, $x_i \neq x_j$.

Suppose $x_j < x_i < d_i \leq d_j$.

Then $d_j-\frac{x_i+x_j}{2}<d_j-x_j$ contradicting individual rationality from equal division. Thus $x_i<x_j$

If $x_i < x_j \le d_i \le d_j$, *then* $d_i - \frac{x_i + x_j}{2} < d_i - x_i$, once again contradicting individual rationality from equal division. Thus $d_i < x_j$

Suppose $\frac{x_i + x_j}{2} < d_i$.

Then $d_i - x_i \le d_i - \frac{x_i + x_j}{2}$

implies $x_i \ge \frac{x_i + x_j}{2}$ contradicting $x_j > x_i$.

Thus $x_i + x_j \ge 2d_i$

Hence for $S < 2d_i, F(d_i, d_j; S) = CEA(d_i, d_j; S)$

By resource monotonicity, $S > 2d_i$. implies $F_i(d_i, d_j; S) = d_i$ which contradicts $x_i < d_i$. Thus $F(d_i, d_j; S) = CEA(d_i, d_j; S)$.

Q.E.D.

Consistency: A solution F is said to satisfy consistency if $\forall M \in P, (d, S) \in B^M, x = F(d, S), \phi \neq N \subset M, \left(d_N, \sum_{i \in N} x_i \right) \in B^N$, implies

$$x_N = F\left(d_N, \sum_{i \in N} x_i\right)$$

Here $d_N = (d_i)_{i \in N}$ and $x_N = (x_i)_{i \in N}$.

Bilateral Consistency is simply the same property as above requiring in addition that N should be a set consisting of exactly two members.

Converse-Consistency: A solution F is said to satisfy converse-consistency if $\forall M \in P, (d,S) \in B^M, x$ is an allocation for (d,S) and $\forall \phi \neq N \subset M, N$ has exactly two members, $x_N = F\left(d_N, \sum_{i \in N} x_i\right)$, then $x = F(d,S)$.

The following lemma is easy to prove:

Lemma 5:- CEA satisfies consistency and converse-consistency.

We need one more lemma, before we can state the results that we promised in the introduction.

Lemma 6:- If F is a solution which satisfies bilateral consistency and agrees with CEA for all two agent problems, then F = CEA.

Proof:- Essentially the proof of Lemma 4 in Dagan (1996b).

We now have the following two major characterization theorems, by using the results obtained so far.

Theorem 6:- The unique solution on B to satisfy bilateral consistency, no-envy and resource monotonicity is CEA.

Theorem 7:- The unique solution on B to satisfy bilateral consistency, individual rationality from equal division and resource monotonicity is CEA.

9. Axiomatic Characterizations of the CEA Solution In Terms of Population Monotonicity:-

Let $\mathbf{N}$ be the set of natural numbers and let I = $\mathbf{N}$.

Resource Continuity: F is said to satisfy resource continuity if given M ϵ P (d,S) $\in B^M$ and $\epsilon > 0$, there exists $\delta > 0$

such that $|S'-S|<\delta,\ (d,S') \in B^M \rightarrow ||F(d,S)-F(d,S')||<\epsilon$ where the norm is simply the Euclidean norm.

Resource Continuity is really a mind regularity assumption.

Population Monotonicity: F is said to satisfy population monotonicity if $\forall\, Q \in P$ and

$k \in \mathbf{N}-Q,\ (d,S) \in b^Q,\ (d';S) \in b^{Q\cup\{k\}}$, if $d_i = d_i\ \forall\, i \in Q$, then

$F_i(d', S) \leq F_i(d, S) \forall i \in Q$.

Population monotonicity says that the arrival of a new agent, should not increase the wards for existing agents. This assumption seems quite reasonable.

Replication-Invariance: F is said to satisfy replication invariance if $\forall Q \in P$ and $k \in \mathbf{N}$, if $Q' \in P$ with $|Q'| = k\,|Q|$

and

$i \in Q$ implies (i, 1),..., (i, k) ϵ Q' such that for

$(d, S) \in B^Q$ and $(d', kS) \in B^{Q'}$, $d_{ij} = d_i, j=1,\ldots,k, i \in Q$, then x =

F (d, S)

implies $y_{(ij)} = x_i \; \forall\, i \in Q,\; j = 1,\ldots,k$, where y = F (d', kS)

$\epsilon\ \mathbf{R}_+^{Q'}$.

The meaning of replication invariance is quite simple: if a rationing problem is replicated k times (i.e.) the available supply is multiplied k times and corresponding to each original agent there are now k agents with the same demand) then each replica in the replicated problem gets what the original agent in the original problem got. This assumption seems harmless.

We now prove the main theorem of this section, which states that the only solution to satisfy no-envy, population monotonicity, resource continuity and replication invariance is the CEA solution.

Theorem 8:

The only solution to satisfy no-envy, population monotonicity, replication invariance and resource continuity is CEA.

Proof:

That CEA satisfies the above properties has been discussed earlier. Hence, let us establish the converse. Thus, suppose F is a solution which satisfies the desired properties and towards a contradiction assume that there exists $L \in P$, $(d, S) \in B^L$ such that $F(d,S) \neq CEA(d,S)$. Thus there exists $i, j \in L$ such that

$$x_i < d_i,\ x_i \neq x_j$$

where x = F (d, S).

By no-envy, we must have

$$x_i < d_i \leq 2d_i - x_i \leq x_j \leq d_j$$

If we keep the available supply fixed at S, and simply replicate each agent 'k' times, then by no-envy, each agent of the same type gets the same amount. By population monotonicity and no-envy, we must have

either $x_i^k \leq x_i < d_i \leq 2d_i - x_i \leq 2d_i - x_i^k \leq x_i^k \leq x_j \leq d_j$

or $x_j^k = x_j^k$

where x_i^k is the common amount that a type i agent gets in the replicated problem (where the supply remains) fixed.

If (1) holds $\forall k$, then

$$kx_j^k \geq k\ (2d_i - x_i) > S$$

for k ϵ **N** sufficiently large.

Hence for a sufficiently large replication, (2) holds.

Since i and j ϵ L were arbitrarily chosen, we get that there exists k^* ϵ **N**, such that if each agent is replicated k^* times and the supply is held fixed at S, then F(d', S) = CEA (d', S') were d' is as defined in the statement of the replication invariance property.
However, by replication invariance,

$F_{(i,l)}\ (d', k^*S) = F_i\ (d, S)\ \forall\ i \in L,\ l = 1, \ldots k^*$ where (i,l) is the l^{th} agent of type i (i.e. the l^{th} replica of agent i in the original problem).

Thus, there exists i, j ϵ L such that

$$x_i < d_i \leq 2d_i - x_i \leq x_j \leq d_j$$

and $x_i^{k^*} = x_j^{k^*} < d_i$

As the total resources are increased form S to k^*S, the individual awards of type i and type j agents change from $x_i^{k^*}$ to x_i and $x_j^{k^*}$ to x_j respectively. By resource continuity, there exists $S' > S$, $S' < k^*S$ such that if y_i is what a type i agent gets at S' and y_j, is what a type j agent gets as S', then $y_i < y_j < d_i$

Thus no-envy is easily seen to be violated; infact, i envies j.

This contradiction establishes the theorem.

Q.E.D.

10. <u>Axiomatic Characterizations Of The Proportional Solutions in terms of A Reduced Game Property</u>:

The <u>proportional</u> solution $\overline{P} : B \rightarrow X$ is defined thus:

$\forall L \in \overline{P}, \forall (d,S) \in B^L, \overline{P}(d,S) = \theta(d,S)\, d,$ *where* $\theta(d,S) > 0$ is chosen to satisfy $\sum_{i \in L} \theta(d,S)\, d_i = S$

Clearly, $\theta(d,S) < 1,$ *since* $\sum_{i \in L} d_i > S$

Thus $\overline{P}_i(d,S) < d_i \; \forall i \in L$

We are interested in the following property:

<u>Reduced Game Property (RGP)</u>:

Given $M \in P, |Q| \geq 2$ and $(d,S) \in B^M,$ *let* $x = F(d,S)$. Let $\phi \neq L \subset M$ and $y = F(d_L, S)$, *where* $d_L = (d_i)_{i \in L}$. Let $S' = \sum_{i \in L} x_i$. Then

$$F_L(d,S) = \frac{S'}{S} F(d_L, S).$$

A weaker version of the above property known as the Weak Reduced Game Property, is simply the same statement with cardinality of L equal to two i.e. $|L| = 2$. It is easy to see that RGP implies Weak RGP.

In the rest of the paper we prove that the proportional solution satisfies RGP and that the only solution to satisfy Weak RGP is the proportional solution provided the solution agrees with the proportional solution for all two agent problems.

<u>Theorem 10</u>: $\bar{P}$ satisfies RGP

<u>Proof</u>: Let

$M \in P$, $|Q| \geq 2$, $\phi \neq L \subset M$, $(d,S) \in B^M$ and $x = P(d,S)$. Let $y = P(d_L, S)$.

Thus $x = \theta\,(d,S)\; d$ and $y = \theta\,(d_L,S)\; d_L$, $S' = \theta\,(d,S)\; \sum_{i \in L} d_i$

$$\therefore x_L = \theta(d,S)\, d_L = \frac{S'}{\Sigma_{i \in L} d_i} d_L$$

$$= \frac{S'}{S} \; \frac{S}{\sum_{i \in L} d_L} \; d_L$$

$$= \frac{S'}{S}\; \theta\;(d_L,S)\; d_L$$

$$= \frac{S'}{S}\; y$$

Since $\sum_{i\in L} y_i = \theta\ (d_L, S) \sum_{i\in L} d_i = S$

$\rightarrow \theta\ (d_L, S) = S / \sum_{i\in L} d_i$

Theorem 11: Suppose $F: B \rightarrow X$ is such that $\forall\, M \in P$ *with* $|M| = 2$ and all $(d, S) \in B^M$ *we have* $F(d, S) = \overline{P}(d, S)$. If F satisfies Weak RGP, then $F = \overline{P}$ *on* B.

Proof: Let $M \in P$. *If* $|M| = 1$ or 2 there is nothing to prove. Hence assume $|M| > 2$. Without loss of generality assume $M = \{1, \ldots, n\}$ *for some* $n \in \mathbf{N}$ *with* $n > 2$.

Let $(d, S) \in B^M$ and X = F(d, S). We have to show that $X = \overline{P}(d; S)$. Consider $i \in M$, $i \neq 1$. By the hypothesis of the theorem, $F_1(d_1, d_i; S) = \frac{d_1}{d_1 + d_i} S$.

By Weak RGP, $X_1 = \frac{X_1 + X_i}{S} \frac{d_1}{d_1 + d_i} S$

i.e. $X_1(d_1 + d_i) = d_1(X_1 + X_i)$ i.e. $X_1\, d_i = d_1\, X_i$

$$\therefore X_1 \sum_{i \neq 1} d_i = d_1 \sum_{i \neq 1} x_i$$

Adding $d_1 x_1$ to both sides, we get

$$X_1 \sum_{i \in M} d_i = d_1 \sum_{i \in M} x_i = d_1 S.$$

$$\therefore X_1 = \frac{d_1}{\sum_{i \in M} d_i} S$$

Since instead of 1 we could have chosen any $j \in M$ and obtained,

$$X_j = \frac{d_j}{\sum_{i \in M} d_i} S,$$

We get that $F(d,S) = \bar{P}(d,S)$.

This proves the theorem.

<u>Q.E.D.</u>

This theorem essentially defines the proportional solution uniquely on the class of all claims problems, modula the restriction that it is already known that for all two dimensional problem it has the explicit functional representation of the proportional solution. Hence the only problem is to characterise the proportional solution for two dimensional problems.

It may be argued that for several types of problems, notably the kind envisaged by the supply chain management problem, the proportional rule is the natural one to apply for two agents (i.e. two retailer) situations. Indeed, if the distributor is impartial as far as retailers go, then what could be more natural than dividing an amount between them in proportion to their demands (which in effect is a proxy for the segmented market demands). However, this reasoning is a justification for applying the proportional rule not merely in the two agent situation, but for situations consisting of any finite number of agents. Thus, inspite of the fact that the given reasoning is very convincing, from the standpoint of the present paper it is insufficient, since it is not amenable to any analytical expression other than the direct one. Put simply, in this paper we want to derive the proportional solution, not define it. Thus we suggest the following property:

Restricted Scale Invariance for Two Agents:

$\forall i, j \in N, i \neq j, \forall (d, S), (d; S) \in B^{\{i,j\}}$ if $d_i + d_j = d'_i + d'_j$,

then $F_i(d'; S) = \frac{d'_i}{d_i} F_i(d, S)$

and $F_j(d'; S) = \frac{d'_j}{d_j} F_j(d; S)$.

The property Restricted Scale Invariance for Two Agents is fairly strong; it says that given two hypothetical situations where two retailers place different demands with the distributor, if it turns out that the aggregate demand remains the same, then for each retailer, the ratio of the awards should be equal to the ratio of the demands. Observe, this covers the situation where the retailers swap their demands i.e. a simple permutation. It is instructive to note that in the sequel no additional symmetry assumption is required to characterise the proportional solution for two agent problems.

Lemma 7: Let $|I| \geq 2$. *Suppose* $F: B \to X$ satisfy Restricted Scale Invariance for Two Agents. Then

$\forall M \in P$ *with* $|M| = 2$, $\forall (d, S) \in B^M$, $F(d, S) = \overline{P}(d, S)$

Proof of Lemma 7: Let $M = \{i, j\}$, $i, j \in P$ and suppose

$(x_i, x_j) = (F_i(d_i, d_j; S), F_j(d_i, d_j, S))$ where $(d_i, d_j; S) \in B^M$.

By Restricted Scale Invariance for Two Agents, there exists functions $f_i : \mathbf{R}^2_{++} \to \mathbf{R}_{++}$ and $f_j : \mathbf{R}^2_{++} \to \mathbf{R}_{++}$ such that

$x_i = d_i f_i(d_i + d_j, S) \quad x_j = d_j f_j(d_i + d_j, S)$

Let d = $d_i + d_j$. *Then* $d_i f_i(d, S) + d_j f_j(d, S) = S$

$\forall d_i, d_j > o$ *such that* $d_i + d_j = d$. *Let* $d_i = d_j = d/2$

$\therefore f_i(d, S) + f_i(d, S) = 2S/d$ Suppose towards a contradiction that for some

$0 < S < d$, $f_i(d, S) = \theta \frac{S}{d} > S/d$ (: *the case where* $f_i(d, S) = \theta \frac{S}{d} < \frac{S}{d}$ *is similar since in that case* $f_j(d, S) > S/d$).

$$\therefore d_i \theta \frac{S}{d} + d_j (2 - \theta) \frac{S}{d} = S, \; \forall d_i, d_j > 0$$

with $d_i + d_j = d$.

$\therefore \theta d_i + (2 - \theta) d_j = d_i + d_j$ *i.e.* $(\theta - 1) d_i = (\theta - 1) c_j$

$\to d_i = d_j$ *since* $\theta \neq 1$.

But this is a contradiction since $f_i(d, S)$ is independent of

d_i *and* d_j. *Thus* $f_i(d,S) = \frac{S}{d} = f_j(d,S)$.

Q.E.D.

As a corollary to Theorem 10, 11 and Lemma 7 we have the following:

Corollary 1: The only solution on B to satisfy RGP, and Restricted Scale Invariance for Two Agents is $\bar{P}$.

Corollary 2: The only solution on b to satisfy Weak RGP, and Restricted Scale Invariance for Two Agents is P.

In the introduction we have referred to the last condition as relatively mild. The assumption of Restricted Scale Invariance for Two Agents is mild when viewed as a requirement applicable only for two dimensional problems, whereas our claims problems can entertain arbitrary finite number of agents.

The entire situation in this section, when adapted to the supply chain management framework is riddled with the possibility of retailers misrepresenting their demands since they operate in a situation of rationing. The possibility of the distributor knowing the true demands, though realistic, is contrary to the spirit of decentralization in which this paper has been conceived. Thus retailers do benefit by inflating their demands.

Let us assume that the retailers inflate their demand uniformly and multiplicatively i.e. in each succeeding period the previous demand is multiplied by a constant say $x > 1$, x being the same for all retailers. In this case the proportional rule remains intact and inviolable.

On the other hand if in each succeeding period they inflate their demand uniformly but additively i.e. by adding $x > 0$, then the proportional rule converges to the rule which allocates the good equally among the retailers.

There are a host of other possibilities open which leads to a distortion in the proportional rule and which may be amenable to a separate analysis. We leave such an analysis as an open problem for the interested reader.

It should be pointed out at this juncture, that the possibility of misrepresenting demands may defeat the purpose of rationing when there are chronic shortages. However, if shortages are unforeseen (which is tantamount to the retailers being unaware of the true supply), then the effectiveness of the proportional solution carries through in letter and spirit.

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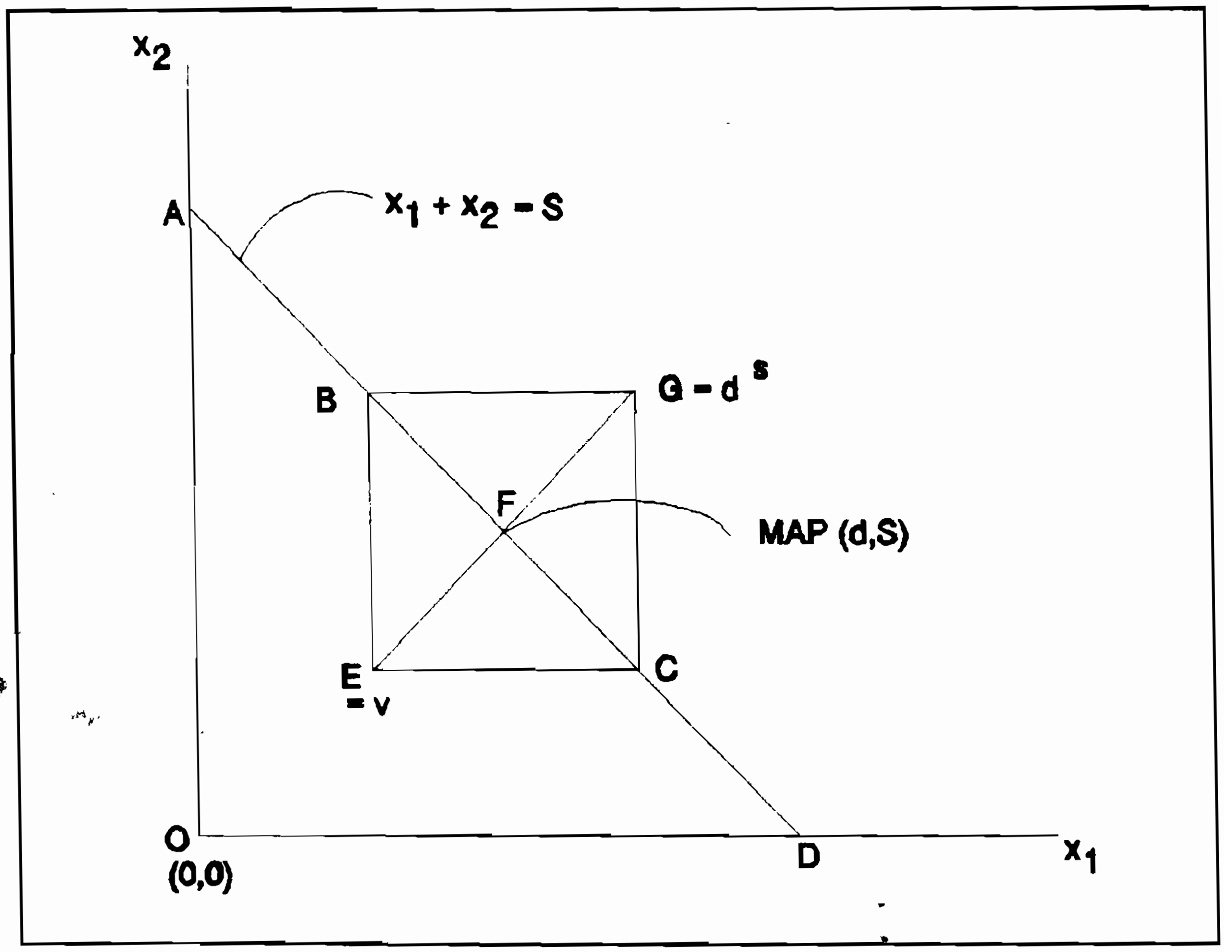

Figure-1