Working Paper

SHARE EQUIVALENT AND EQUITARIAN ALLOCATIONS
FOR PROBLEMS OF FAIR DIVISION

By

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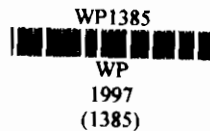
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Abstract

In this paper we begin with a given social endowment. A profile of shares (which could very well all be equal), is part of the environment. This is the planner's contribution to the economic environment, as conceived in this paper. First we formulate the concept of an envy free allocation as was done by Schmeidler and Vind [1972]. Then we propose the concept of a share equivalent allocation which is a generalization of the concept of an egalitarian equivalent allocation due to Pazner and Schmeidler [1978]. An allocation is share equivalent if every agent is indifferent between his allocation and what would result if an identical change in entitlements were affected for all the agents. (An agent's entitlement is his share of the social endowment in physical units.) If the identical change is a multiple of the social endowment vector, we say that the allocation is naturally share equivalent. We prove the existence of a naturally share equivalent allocation which is also Pareto efficient and prove that such allocations correspond to maximization of the minimum utility over all feasible allocations. For two agent economies we show that naturally share equivalent allocations are envy free and all envy free allocations are share equivalent.

In a final section to this paper we introduce the concept of an equitarian allocation (Young 1993). This is a feasible allocation such that each agent is indifferent between what he/she receives and a multiple of his entitlement, where the multiple is common to all the agents. The proof that a Pareto efficient equitarian allocation exists is similar to the proof which established the existence of a Pareto efficient and naturally share equivalent allocation. Hence we omit it. Subsequently, we observe that such allocations are the only ones which maximize the minimum utility over all feasible allocations, where the utility representations are once again suitably constructed.

Thus, we manage to generalize an existing notion of economic equity, by incorporating possible asymmetries that may need to arise for the sake of obtaining (final) distributive justice. As observed by Moulin (1995), problems of fair division arise perpetually in managerial contexts. With these results, perhaps a new insight would be gained in resolving such problems.

1. Introduction: Problems of fair division, are generally concerned with dividing a given bundle of goods amongst a finite number of agents. Thomson and Varian [1985], provide a rather exhaustive survey of the concepts and methods involved in solving such problems.

This field of study has in recent times found acceptability among management scientists. Many key decisions have significant consequences for a group of people, rather than a single individual. In some cases the decisions are arrived at by the group itself; in other cases, the intervention of a central decision maker may be necessary. Intervening on behalf of a group imposes additional concerns on the decision maker, one of which is fairness. These and related issues, have found expression in Fishburn and Sarin [1994] and Boiney [1995].

Historically, the genesis of the problem we study in this paper, is in Foley (1967), who first proposed the concept of envy free allocations. Of immediate interest to us is the paper by Pazner and Schmeidler (1978) where an alternative concept of distributional equity has been formulated. Implicit in the entire literature (: which may not be obvious at a first reading, but is sure to strike someone if he/she reads between the lines) is the special status that has been given

2

to equal division of resources, (whether directly or indirectly) in all the solution concepts. Of course, this does not imply any property rights over the social endowment; it merely assumes, that an agent can veto an allocation which fails to make him/her atleast as well off as he/she would be at equal division. On the face of it, equal division of resources seems extremely compelling. However, a moment's reflection reveals that equal share of resources is just as good or as bad as any other share of the resources. This is particularly true if the social endowment is an addition to a given distribution of resources (: perhaps as inheritances) to the agents in a society. The given distribution may be very unequal. To remedy this inequity, equal shares in the additional endowment may not be enough. We may need to given unequal shares (claims) to the social endowment in order to rectify this initial inequity. Thus any profile of shares may turn out to be ex post acceptable.

In this paper we begin with a given social endowment. A profile of shares (which could very well all be equal), is part of the environment. This is the planner's contribution to the economic environment, as conceived in this paper. First we formulate the concept of an envy free allocation as was done by Schmeidler and Vind [1972]. Then we propose the concept of a share equivalent allocation which is a generalization of the concept of an egalitarian equivalent allocation due to Pazner

3

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In a final section to this paper we introduce the concept of an equitarian allocation (Young 1993). This is a feasible allocation such that each agent is indifferent between what he/she receives and a multiple of his entitlement, where the multiple is common to all the agents. The proof that a Pareto efficient equitarian allocation exists is similar to the proof which established the existence of a Pareto efficient and naturally share equivalent allocation. Hence we omit it. Subsequently, we observe that such allocations are the only ones which maximize the minimum utility over all feasible allocations, where the utility representations are once again suitably constructed.

4

Thus, we manage to generalize an existing notion of economic equity, by incorporating possible asymmetries that may need to arise for the sake of obtaining (final) distributive justice. As observed by Moulin (1995), problems of fair division arise perpetually in managerial contexts. With these results, perhaps a new insight would be gained in resolving such problems.

2. <u>Model</u>:-   There are 1 commodities and n agents. Agent i, indexed by subscript i, is characterized by his consumption set $\mathbf{R}^1_+$ (: the non-negative orthant of 1 - dimensional Euclidean space) and a continuous utility function $u_i : \mathbf{R}^1_+ \to \mathbf{R}$ which is also assumed to be quasi - concave (i.e. $\forall x, y \in \mathbf{R}^1_+$ and $t \in [0,1]$,

$u_i (tx + (1-t)y) \geq \min \{ u_i (x), u_i (y) \})$ We also assume that $u_i : \mathbf{R}^1_+ \to \mathbf{R}$, $i = 1, \ldots, n$ are <u>weakly monotonic increasing</u> (i.e.

$x, y \in \mathbf{R}^1_+$, $x > y \to u_i (x) > u_i (y)$) and satisfies <u>at least one</u> of the two following properties either (i)

5

$\forall\, i,\ \forall\, x,\, y \in \mathbf{R}^l_+,\ x > y \to u_i(x) > u_i(y)$ i.e. $(u_i)^n_{i=1}$ are

strictly monotonic increasing;

or (ii) $\forall i,\ \forall x,\, y \in \mathbf{R}^l_+,\ x \in \mathbf{R}^l_{++},\ u_i(x) = u_i(y)$ implies $y \in \mathbf{R}^l_{++}$.

If $(u_i)^n_{i=1}$ satisfies (ii), $x \in \mathbf{R}^l_{++}$ and $y \in \mathbf{R}^l_+ \setminus \mathbf{R}^l_{++}$,   then

$u_i(x) > u_i(y).\ \forall\, i.$

The social endowment in the above scenario is a commodity

bundle $\omega \in \mathbf{R}^l_{++}$.

An <u>economy</u> is a list $e = ((u_i)^n_{i=1},\ \omega)$.   An <u>economy with</u>

<u>shares</u> is a pair (e, w) where e is an economy and $w \in \mathbf{R}^n_{++}$

with $\Sigma^n_{i=1}\, w_i = 1.$ w_i represents the share that agent i has

over the social endowment. $w_i\,\omega$ is the entitlement of agent

i in (e, w).

Given $e = ((u_i)_{i=1}^n, \omega)$ or (e, w) a feasible allocation is a

list $(x_i)_{i=1}^n \in (\mathbb{R}_+^l)^n$ such that $\Sigma_{i=1}^n x^i = \omega$. Let $A(e) \equiv A(e,w)$

denote the set of feasible allocations for e or (e,w).

Given e (or (e,w)) an allocation $(x_i)_{i=1}^n$ is said to be Pareto

efficient if it is feasible for e (or(e,w)) and there does not

exist any other allocation $(y_i)_{i=1}^n$ which is feasible for

e(or(e,w)) and such that $u_i(y_i) \geq u_i(x_i) \ \forall \ i=1,\ldots,n$ with atleast

one strict inequality.

Given (e, w) an allocation $(x_i)_{i=1}^n$ is said to be envy free if

$\forall \ i, j \in \{1,\ldots,n\}$ such that $w_i \omega + x_j - w_j \omega \in \mathbb{R}_+^l$, we have

$u_i(x_i) \geq u_i(w_i \omega + x_j - w_j \omega)$.

7

Given (e, w) an allocation $(x_i)_{i=1}^n$ is said to be fair if it is both envy free and Pareto efficient.

Given (e, w) an allocation $(x_i)_{i=1}^n$ is said to be share equivalent if $(x_i)^n \in A(e,w)$ and $u_i(x_i) = u_i(z_i) \ \forall i = 1, \ldots, n$, where $z_i = w_i \omega + \bar{z}$ for some $\bar{z} \in \mathbb{R}^1$ and for all $i = 1, \ldots, n$.

Lemma 1: Suppose $n = 2$ and (x_1, x_2) is envy free. Then (x_1, x_2) is share equivalent.

Proof: Consider $\{y_i \in \mathbb{R}^1 \ / \ u_i(w_i\omega + y_i) = u_i(x_i)\}$, $i = 1, 2$.

Suppose

$$u_j(w_j\omega + y_i) > u_j(x_j) \tag{1}$$

$$\forall \ y_i \in \mathbb{R}^1 \ with \ u_i(w_i\omega + y_i) = u_i(x_i) \tag{2}$$

8

Then since $x_1 - w_1 \omega$ satisfies (2), we have

$$u_j (w_j \omega + x_1 - w_1 \omega) > u_j (x_j)$$

Thus (x_1, x_2) is not envy free.

Hence

$$\{ y_1 \in \mathbf{R}^1 \; / \; u_1 (w_1 \omega + y_1) = u_1 (x_1) \} \cap \{ y_2 \in \mathbf{R}^1 \; / \; u_2 (w_2 \omega + y_2) = u_2 (x_2) \} \neq \phi$$

Let $\bar{z}$ belong to this non-empty intersection.

Thus (x_1, x_2) is naturally share equivalent.

Q.E.D.

3. Existence of Pareto Efficient and Share Equivalent
    Allocations:

    Theorem 1:- Given (e, w), there exists a Pareto-efficient and
    share equivalent allocation $(x_i)_{i=1}^{n}$.

9

Proof: Let C = { t ≥ 0 / there exists an allocation

$(x_i)_{i=1}^n \in A(e, w)$ such that

$u_i(x_i) \geq u_i(w_i \omega + t \omega) \forall i$ }

C is nonempty, since $0 \in C$. By our assumptions C is closed.

Put t = 1. By weak monotonicity, $1 \notin C$. Hence C is bounded

above. Let $\bar{t} = \{ t \in C \}, x$ and let $(x_i)_{i=1}^n$ be the corresponding

share equivalent allocation. We have to show $(x_i)_{i=1}^n$ is

Pareto efficient. Suppose $(x_i)_{i=1}^n$ is not Pareto efficient.

Then there exists $(y_i)_{i=1}^n \in A(e, w)$, such that

$u_i(y_i) \geq u_i(x_i) \forall i$, with atleast one strict inequality. Since

$z_i = w_i. \omega + \bar{t} \omega \in \mathbf{R}_{++}^l$, and $u_i(x_i) = u_i(z_i) \forall i$, two possibilities

arise:

10

either (a) $\forall i$, u_i satisfies (ii) in which case both

x_i and y_i ϵ $\mathbf{R}^l_{++}$.

or (b) $\forall i$, u_i satisfies (i) and both x_i and y_i ϵ $\mathbf{R}^l_+ \setminus \{0\}$

In either case, if $u_j(y_j) > u_j(x_j)$, there exists

$a \epsilon \mathbf{R}^l_+ \setminus \{0\}$ such that $u_j(y_j - a) > u_j(x_j)$ and

$u_i(y_i + \frac{a}{n-1}) > u_i(x_i) \forall i \neq j$. This we can conclude by continuity

and either weak or strong monotonicity of the utility functions.

Let $a_i = y_i + \frac{a}{n-1}$ if $i \neq j$ and $a_j = y_j - a$.

Then $(a_i)^n_{i=1} \epsilon A(e,w)$ and there exists $t > \bar{t}$ with $t \epsilon C$. This

11

contradicts the definition of $\bar{t}$.

It remains to show that $u_i(x_i) = u_i(w_i \omega + \bar{t} \omega) \; \forall i$.

Clearly $u_i(x_i) \geq u_i(w_i \omega + \bar{t} \omega) \; \forall i$.

Towards a contradiction assume $u_1(x_1) > u_1(w_1 \omega + \bar{t} \omega)$.

Then by an argument similar to above, there exists $a \epsilon \mathbf{R}_+^1 \setminus \{0\}$

such that $u_1(x_1 - a) > u_1(w_1 \omega + \bar{t} \omega)$ and

$$u_i(x_i + \frac{a}{n-1}) > u_i(w_i \omega + \bar{t} \omega) \; \forall i \neq 1.$$

This again leads to a contradiction of the definition of $\bar{t}$.

This proves the theorem.

Q.E.D.

Remark: A cursory book at the proof of Theorem 1, will show that we can essentially prove the following stronger assertion:

<u>Theorem</u>: Given (e, w) and $\bar{z} \epsilon \mathbf{R}^{1}_{+}$, there exists

$(x_i)^n_{i=1} \epsilon A(e, w)$ and $\bar{t} \geq 0$ such that

(i) $(x_i)^n_{i=1}$ is Pareto efficient for (e, w)

(ii) $u_i(x_i) = u_i(w_i \omega + \bar{t} \bar{z}) \ \forall i = 1, \ldots, n$.

Given (e, w), an allocation $(x_i)^n_{i=1}$ is said to be naturally share equivalent if $(x_i)^n_{i=1} \epsilon A(e, w)$ and

$u_i(x_i) = u_i(w_i \omega + t \omega) \ \forall \ i = 1, \ldots, n$ and for some $t \geq 0$. We have shown above that for every (e, w) there exists a naturally share equivalent allocation which is at the same time Pareto efficient.

Since each u_i is at least weakly increasing, if

$(x_i)^n_{i=1}$ and $(y_i)^n_{i=1}$ are both naturally share equivalent and

13

Pareto efficient, then $u_i(x_i) = u_i(y_i), i = 1, \ldots, n$. This is so

because $\omega \in \mathbf{R}_{++}^l$.

It is easily verified that if each u_i is strictly

quasi-concave, then there is a unique naturally share

equivalent and Pareto efficient allocation.

Given $i \in \{1, \ldots, n\}$, let $v_i : \mathbf{R}_+^l \to \mathbf{R}$ be defined as follows:

$$v_i(x) = s,$$

where $u_i(x) = u_i(w_i \omega + s\omega)$

Given $x \in \mathbf{R}_+^l$, such an s exists uniquely under the

assumptions we have invoked.

Theorem 2:- Let $(x_i)_{i=1}^n$ be naturally share equivalent and Pareto

14

efficient. Then $(x_i)_{i=1}^n$ solves

$$\min_{i=1,\ldots,n} \{v_i(y_i)\} \to \max$$

s.t. $(Y_i)_{i=1}^n \in A (e, w)$

Proof:- We have seen that $\min_{i=1,\ldots,n} \{v_i(x_i)\} = \bar{t}$ where $\bar{t}$ is as

defined in Theorem 1. Let $t > \bar{t}$ and suppose $\min_{i=1,\ldots,n} \{v_i(y_i)\} = t$.

Then $u_i(y_i) \geq u_i(w_i\omega + t\omega) \ \forall \ i=1,\ldots,n$.

Since $t > \bar{t}$, this contradicts definition of $\bar{t}$.

Q. E. D.

In fact if we study the proof of Theorem 1 closely, we can

easily conclude that if $(z_i)_{i=1}^n$ solves the above programming

15

problem then it must be naturally share equivalent and Pareto efficient.

Finally, we have the following theorem of considerable interest:

Theorem 3:- Let n = 2 and (x_1, x_2) be naturally share equivalent and Pareto efficient. Then (x_1, x_2) is fair.

Proof:- We only need to show that (x_1, x_2) is envy-free. Let

$x_i = w_i \omega + z_i, i = 1, 2$ and suppose $u_i(w_i \omega + z_i) = u_i(w_i \omega + \overline{t} \omega), \overline{t} \geq 0$.

$\geq u_i(w_i \omega)$

If $z_1 = z_2 = 0$, then clearly (x_1, x_2) is envy free. Thus suppose

$z_1 \neq 0, z_2 \neq 0$ and towards a contradiction assume,

$u_1(w_1 \omega + z_2) > u_1(w_1 \omega + z_1)$.

Then $u_1(w_1\omega) = u_1(w_1\omega + \dfrac{z_1+z_2}{2}) > u_1(w_1\omega+z_1) \geq u_1(w_1\omega)$

(because $z_1+z_2=0$) : see appendix for reasons behind strict

inequality. This contradiction establishes the theorem.

Q. E. D.

4. **Equitarian Allocations**:- Given (e, w), say that an allocation $(x_i)_{i=1}^n$

is **equitarian** if there exists $\bar{t} \geq 0$ such that

$u_i(x_i) = u_i(\bar{t}w_i\ \omega)\ \forall\ i$.

The proof of the following theorem is similar to the proof of Theorem 1.

Theorem 4:- Given (e, w), there exists a Pareto efficient and equitarian allocation.

Given (e, w) where $e = (u_i)_{i=1}^n$, let $v_i : \mathbf{R}_+^l \to \mathbf{R}_+$ be

defined as follows:

17

$\forall \ x_i \ \in \ \mathbf{R}_+^{\textit{l}}, \ v_i(x_i) \ = \ s \ \sim \ u_i \ (x_i) \ = \ u_i \ (s \ w_i \ \omega).$

It is easy to check that v_i as above is well defined. We now

have the following theorem:

<u>Theorem 5</u>:- Given $(e, \ w)$, $(\overline{x}_i)_{i=1}^n$ is Pareto efficient and

equitarian if any only if, it solves

$$\min_{i=1,\ldots,n} \ [v_i(x_i)] \ \rightarrow \ \max$$

s.t. $(x_i)_{i=1}^n \ \in \ A(e,w).$

<u>Proof</u>: Very similar to the proof of Theorem 2. The remark
immediately after the proof of Theorem 2 is also relevant.

<u>Q.E.D.</u>

18

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Appendix

Theorem: Suppose $u: \mathbf{R}_+^l \to \mathbf{R}$ satisfies the conditions assumed in the

main text. Then it is semi-strictly quasi-concave i.e.

$\forall x, y \in \mathbf{R}_+^l$ with $u(x) \neq u(y)$ $\forall$ $t \in (0,1), u(tx+(1-t)y) > \min\{u(x), u(y)\}$.

Proof: Let $x, y \in \mathbf{R}_+^l$ with u (x) > u (y) and towards a contradiction

assume that there exists $\alpha \in (0,1)$ *with* $u(\alpha x+(1-\alpha)y) = u(y)$.

Clearly neither x >> y nor y >> x (: by weak monotonicity; for in

that case $\alpha x+(1-\alpha)y > y$, leading to a contradiction).

Case 1:- x >> 0.

Then choose $\bar{x} \in \mathbf{R}_+^l$ such that $x > \bar{x}$ *and* $u(\bar{x}) > u(y)$. This is clearly

possible by continuity. Clearly $\alpha x+(1-\alpha)y > \alpha \bar{x}+(1-\alpha)y$

By weak monotonicity

$u(y) = u(\alpha x + (1-\alpha)y) > u(\alpha \bar{x} + (1-\alpha)y)$ contradicting u is quasi-concave.

<u>Case 2</u>:- $x \epsilon \mathbf{R}^1_+ \setminus \mathbf{R}^1_{++}$ but u is strictly monotonic. Clearly x > 0.

Then choose $\bar{x} \epsilon \mathbf{R}^1_+$ such that $x > \bar{x} > 0$ and $u(\bar{x}) > u(y)$. This is

clearly possible by continuity. Clearly $\alpha x + (1-\alpha)y > \alpha \bar{x} + (1-\alpha)y$.

By strict monotonicity

$u(y) = u(\alpha x + (1-\alpha)y) > u(\alpha \bar{x} + (1-\alpha)y)$ contradicting u is quasi-concave.

<u>Case 3</u>:- $x \epsilon \mathbf{R}^1_+ \setminus \mathbf{R}^1_{++}$ but $a \epsilon \mathbf{R}^1_{++}$, $b \epsilon \mathbf{R}^1_+ \setminus \mathbf{R}^1_{++}$ implies u(a) > u(b).

In this case clearly $y \epsilon \mathbf{R}^1_+ \setminus \mathbf{R}^1_{++}$. Suppose u(tx + (1-t)y) > u(y) for

some $t \epsilon (0,1)$. By weak monotonicity there exists

$\bar{y} > tx + (1-t)y$ with $u(\bar{y}) > u(tx + (1-t)y)$. By continuity there exists

$\alpha \epsilon (0,1)$ such   that $u(\alpha \bar{y} + (1-\alpha)y) = u(tx + (1-t)y)$.   But

$\alpha \overline{y} + (1-\alpha) y \in \mathbf{R}^1_{..}$ and $tx + (1-t) y \in \mathbf{R}^1_{.} \setminus \mathbf{R}^1_{..}$ which   leads   to   a

contradiction.

Hence $u(tx + (1-t) y) = u(y) \, \forall t \in (0,1)$.

Thus $u(x) = u(y)$ by continuity.

Hence Case 3 is not possible.

This proves the theorem.